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ON THE INVOLUTION AND EVOLUTION OF QUATERNIONS.

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THE subject-matter of quaternions is really nothing more nor less than that of substitutions of the second order, such as occur in the familiar theory of quadratic forms. A linear substitution of the second order is in essence identical with a square matrix of the second order, the law of multiplication between one such matrix and another being understood to be the same as that of the composition of one substitution with another, and therefore depending on the order of the factors; but as regards the multiplication of three or more matrices, subject to the same associative law as in ordinary algebraical multiplication.

Every matrix of the second order may be regarded as representing a quaternion, and *vice versâ*; in fact if, using i to denote $\sqrt{(-1)}$, we write a matrix m of the second order under the form

$$\begin{matrix} a+bi, & c+di, \\ -c+di, & a-bi, \end{matrix}$$

we have by definition,

$$m = a\alpha + b\beta + c\gamma + d\delta,$$

where $\alpha = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \quad \beta = \begin{matrix} i & 0 \\ 0 & -i \end{matrix}, \quad \gamma = \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}, \quad \delta = \begin{matrix} 0 & i \\ i & 0 \end{matrix}.$

Now $\alpha^2 = \alpha, \quad \beta^2 = \gamma^2 = \delta^2 = -\alpha,$

$$\alpha\beta = \beta\alpha = \beta, \quad \alpha\gamma = \gamma\alpha = \gamma, \quad \alpha\delta = \delta\alpha = \delta,$$

$$\beta\gamma = -\gamma\beta = \alpha, \quad \gamma\delta = -\delta\gamma = \beta, \quad \delta\beta = -\beta\delta = \gamma;$$

so that we may for $\alpha, \beta, \gamma, \delta$, substitute $1, h, k, l$, four symbols subject to the same laws of self-operation and mutual interaction as unity and the three Hamiltonian symbols. Now I have given the universal formula for expressing any given function of a matrix of *any* order as a rational function of that matrix and its latent roots; and consequently the qth power or root of any quadratic matrix, and therefore of any quaternion, is known. As far as I am informed, only the square root of a quaternion has been given in the text-books on quaternions, notably by Hamilton in his Lectures on Quaternions.

The latent roots of m are the roots of the quadratic equation

$$\lambda^2 - 2a\lambda + a^2 + b^2 + c^2 + d^2 = 0.$$

The general formula

$$\phi m = \Sigma\phi\lambda_1 \frac{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_i)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_i)};$$

where i is the order of the matrix m, when $i=2$ and $\phi m = m^{\frac{1}{q}}$, becomes

$$m^{\frac{1}{q}} = \frac{\lambda_1^{\frac{1}{q}} - \lambda_2^{\frac{1}{q}}}{\lambda_1 - \lambda_2} m - \frac{\lambda_2\lambda_1^{\frac{1}{q}} - \lambda_1\lambda_2^{\frac{1}{q}}}{\lambda_1 - \lambda_2},$$

where λ_1, λ_2 are the roots of the above equation. If μ is the modulus of the quaternion, namely is $\sqrt{(a^2+b^2+c^2+d^2)}$, and $\mu\cos\theta = a$, the latent roots λ_1, λ_2 assume the form

$$\mu(\cos\theta \pm i\sin\theta).$$

When the modulus is zero the two latent roots are equal to one another, and to a, the scalar of the quaternion; so that in this case the ordinary theory of vanishing fractions shows that

$$m^{\frac{1}{q}} = a^{\frac{1}{q}}\left(\frac{m}{a} + \frac{q-1}{q}\right).$$

In the general case there are q^2 roots of the qth order to a quaternion. Calling $\frac{\pi}{q} = \omega$, and writing $m^{\frac{1}{q}} = Am + B$,

$$A = \frac{\mu^{\frac{1}{q}}}{\mu} \frac{\cos\left(\frac{\theta}{q} + 2k\omega\right) + i\sin\left(\frac{\theta}{q} + 2k\omega\right) - \cos\left(\frac{\theta}{q} + 2k'\omega\right) + i\sin\left(\frac{\theta}{q} + 2k'\omega\right)}{2i\sin\theta},$$

$$B = -\mu^{\frac{1}{q}} \frac{\cos\left(\frac{q-1}{q}\theta + 2k\omega\right) + i\sin\left(\frac{q-1}{q}\theta + 2k\omega\right) - \cos\left(\frac{q-1}{q}\theta + 2k'\omega\right) + i\sin\left(\frac{q-1}{q}\theta + 2k'\omega\right)}{2i\sin\theta}.$$

For the q system of values $k = k' = 1, 2, 3 \dots q$, the coefficients A and B will be real, for the other $q^2 - q$ systems of values imaginary; so that there are q quaternion-proper qth roots of a quaternion-proper in Hamilton's sense, and $q^2 - q$ of the sort which, by a most regrettable piece of nomenclature, he terms bi-quaternions. The real or proper-quaternion values of $m^{\frac{1}{q}}$ are

$$\frac{\mu^{\frac{1}{q}}}{\sin\theta}\left\{\sin\left(\frac{\theta}{q} + 2k\omega\right)\frac{m}{\mu} + \sin\left(\frac{q-1}{q}\theta + 2k\omega\right)\right\},$$

$\mu^{\frac{1}{q}}$ meaning *the* or (when there is an alternative) *either* real value of the qth root of the modulus.

In the qth root (or power) of a quaternion m, the form $Am + B$ shows that the vector-part remains constant to an ordinary algebraical factor *près*; and we know *à priori* from the geometrical point of view that this ought to be the case. When the vector disappears a porism starts into being; and besides the values of the roots given by the general formula, there are others involving arbitrary parameters. Babbage's famous investigation of the form of the homographic function of $\frac{px+q}{rx+s}$ of x, which has a periodicity of any given degree q, is in fact (surprising as such a statement would have appeared to Babbage and Hamilton) one and the same thing as to find the qth root of unity under the form of a quaternion!

It is but justice to the eminent President of the British Association to draw attention to the fact that the substance of the results here set forth (although arrived at from an independent and more elevated order of ideas) may be regarded as a statement (reduced to the explicit and most simple form) of results capable of being extracted from his memoir on the Theory of Matrices, *Phil. Trans.* Vol. CXLVIII. (1858) (*vide* pp. 32—34, arts. 44—49).