99.

ON SUCCESSIVE INVOLUTES TO CIRCLES.—SECOND NOTE*.

[*Philosophical Magazine*, XXXVI. (1868), pp. 459—466.]

SINCE the appearance of the former Note on this subject, I have enjoyed the inestimable advantage of securing the cooperation of my all-accomplished and omni-capable friend Mr Spottiswoode, to whose kindness and skill my readers are indebted for the beautiful figures given in the following pages, which I shall proceed briefly to describe, and which, as far as I can learn, offer the first examples of the actual visible representation of any derived involutes of the circle beyond those of the first order. I propose, for want of a better word, provisionally to give the name of Cyclodes (suggested by Professor Cayley) to these spirals. They may be considered a genus of a more general class of spirals which I propose to name algebraical spirals, defined by the condition that the perpendicular on the tangent from a certain fixed point (which may be termed its pole) is a rational algebraical function of the angle of contingence; so that a cyclode may be said to be an *integral* algebraical spiral, that is one in which the perpendicular on the tangent becomes a rational integral function of the angle of contingence.

I find in a certain question, presently to be alluded to, the theory of the class so indisputably bound up with that of the genus, as to persuade me of the importance of the theory of the former being gone into by someone who has leisure for the investigation, and of the desirableness of an organic description being discovered or devised for the rational fractional case. The peculiar feature of the cyclode class is the absence of points of inflection, real or imaginary. The cusps of cyclodes are strictly analogous to the asymptotes in algebraical curves, like them entering and disappearing in pairs, creating

* The thought foreshadowed in the concluding paragraph of the former note leads to the following theorem.

Let f, ϕ, ψ be quantics in α, β; F the unicursal function obtained by elimination of α, β between

$$x=f, \quad y=\phi, \quad z=\psi;$$

$\Delta_x . F$ the discriminant of F regarded as a quantic in x and 1; $J(\phi, \psi)$ the Jacobian of ϕ, ψ; R the result of eliminating ϕ, ψ between

$$y=\phi, \quad z=\psi, \quad J(\phi, \psi)=0;$$

Q the product of all the homogeneous linear functions of y, z which vanish at the double points of F; then I say (and the proof is all but self-evident) $\Delta_x . F=R . Q^2$.

partial interruptions of continuity, and thus separating the curve into distinct branches*. In the same way as the order of an algebraical curve is deter-

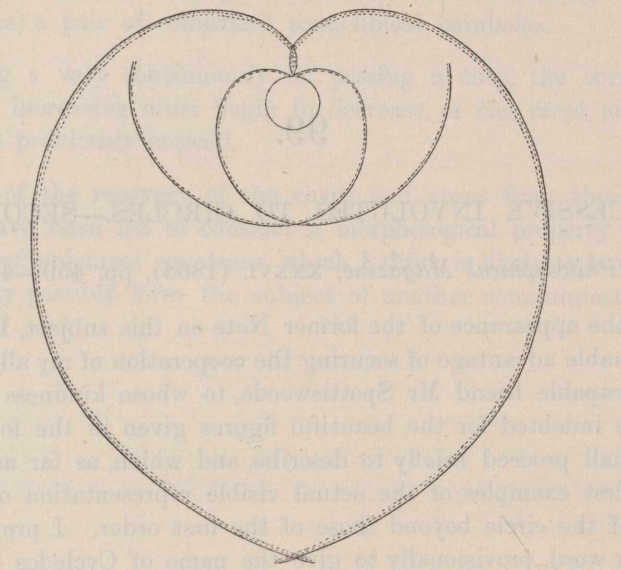

Fig. 1.

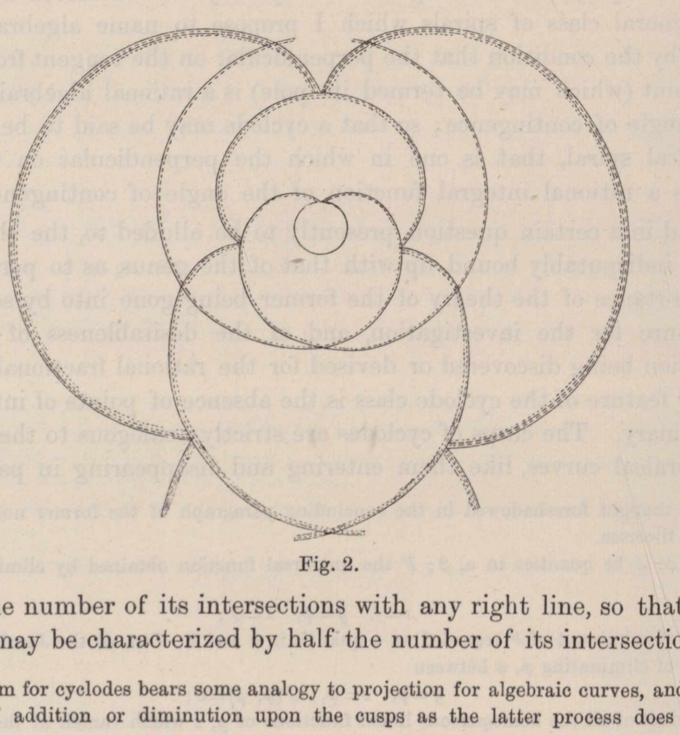

Fig. 2.

mined by the number of its intersections with any right line, so that of any such spiral may be characterized by half the number of its intersections with

* Parallelism for cyclodes bears some analogy to projection for algebraic curves, and operates in the way of addition or diminution upon the cusps as the latter process does upon the asymptotes.

any circle having its centre at the pole. When the rational fraction which expresses the value of the perpendicular is of the degree m in the numerator and n in the denominator, the order will thus become the *dominant* of the two quantities $m+n$, $2n$.

Figs. 1, 2, 4, 5, exhibit examples of cyclodes of the first, second, and third orders, distinguished respectively, where required, by the number

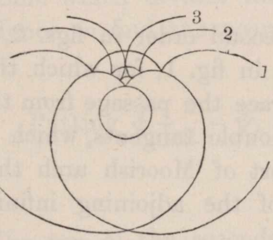

Fig. 3.

1. Triple tangent.
2. Double tangent passes through Cusp.
3. Double tangents coincide.

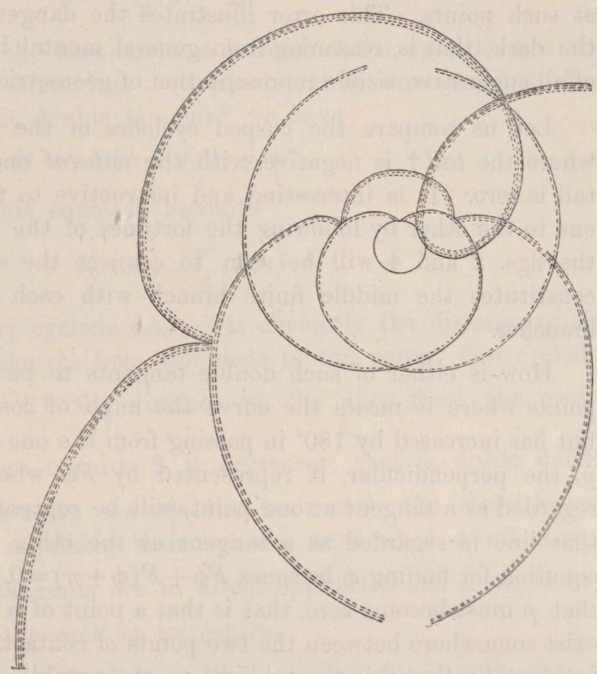

Fig. 4.

of accompanying dotted lines*. Let us consider more closely those of the second order, which separate themselves into two classes, the cusped and

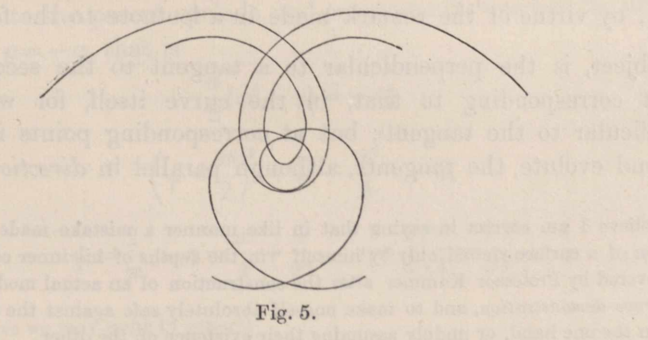

Fig. 5.

* In figs. 5 and 3, which refer to cyclodes of the second order exclusively, it has not been thought necessary to adjoin the dotted lines.

uncusped. The cusped class are the analogues of the hyperbola, the uncusped class of the ellipse, and the very remarkable secondary cyclode whose tail (to use the late Dr Whewell's expression) is zero, and which may be termed the natural one of the order, is the analogue of the parabola. In the former Note I spoke of "points of retrocession"; instead of points of retrocession, I propose to call these "points of radiation" or "radiant-points"; the intervention of the cusps prevents the happening of the supposed "retrocession" at such points. This error illustrates the danger of, so to say, fighting in the dark, that is, reasoning from general mental impressions in the absence of all suggestive visible representation of geometrical forms*.

Let us compare the cusped cyclodes of the second order in figs. 2, 4, where the *tail*† is negative, with the *natural* one in fig. 1, for which this tail is zero. It is interesting and instructive to trace the passage from the one to the other by following the fortunes of the double tangents, which in the figs. 2 and 4 will be seen to connect the sort of Moorish arch that constitutes the middle finite branch with each of the adjoining infinite branches.

How is either of such double tangents to be determined? At the two points where it meets the curve the angle of contingence is not the same, but has increased by 180° in passing from the one to the other. Accordingly p, the perpendicular, if represented by $F\phi$ when the double tangent is regarded as a tangent at one point, will be represented by $-F(\phi+\pi)$ when that line is regarded as a tangent at the other point of contact, and the equation for finding ϕ becomes $F\phi+F(\phi+\pi)=0$. Thus we see incidentally that p must become zero, that is that a point of a radiation must necessarily exist somewhere between the two points of contact. And here I may remark incidentally that this throws light on the notable equation, applicable to any curves whatever,

$$\frac{ds}{d\phi}=p+\frac{d^2p}{d\phi^2};$$

for $\frac{d^2p}{d\phi^2}$, by virtue of the remark made in a footnote to the former paper on this subject, is the perpendicular to a tangent to the second evolute at a point corresponding to that, in the curve itself, for which p is the perpendicular to the tangent; but at corresponding points in a curve and its second evolute, the tangents, although parallel in *direction*, are opposite

* I believe I am correct in saying that in like manner a mistake made by Steiner in his description of a surface viewed only by himself "in the depths of his inner consciousness," was first discovered by Professor Kummer after the construction of an actual model. So impossible is it to *prove demonstration*, and to make oneself absolutely safe against the fallacy of ignoring entities on the one hand, or unduly assuming their existence on the other.

† In general the *tail* is the distance of the cusp of the first involute from the corresponding points of the involutes successively engendered therefrom.

in *flow*. Hence $p + \frac{d^2p}{d\phi^2}$ $\left(\text{and not } p - \frac{d^2p}{d\phi^2}\right)$ is the distance between these two tangents; and it is obvious that such distance is identical with the radius of curvature corresponding to the perpendicular p; so that, viewed in this light, the differential equation above written is reduced to a *truism*. Returning to our cyclode of the second order, we may write its equation under the form

$$p = \frac{a}{1 \cdot 2}(\phi^2 - \gamma),$$

where a is the radius of the base-circle, ϕ is zero at the apse, that is the point which divides the curve into two equal and symmetrical branches. Hence to find the nearest double tangent* we have

$$\phi^2 - \gamma + (\phi + \pi)^2 - \gamma = 0.$$

Putting $\phi + \frac{\pi}{2} = \psi$, this equation becomes

$$\psi^2 = \gamma - \frac{\pi^2}{4}.$$

The *tail* of the secondary cyclode (say τ) is obviously the distance of the apse (in respect of the centre) from the node of the parent first cyclode; and its length is $\frac{a}{2}(\gamma - 2)$, the distance of the apse from the centre being $\frac{a}{2}\gamma$. As γ increases towards 2, ψ decreases; that is, either double tangent tends more and more towards the horizontal; the Moorish arch therefore sinks, and the adjoining haunches rise until when $\phi = 0$, that is $\gamma = \frac{\pi^2}{2}$, the two double tangents are in direct opposition and merge into a triple tangent touching the arch at its centre.

As γ continues to decrease by ϕ becoming negative, the central arc sinks below the level of the adjoining branches, and the double tangents slope more and more towards its extremities, until at length they pass through the cusps; when this takes place the tangent to the second cyclode becomes perpendicular to the parent cyclode, and consequently touches the originating circle so that $p = -a$, that is

$$\frac{a}{2}(\phi^2 - \gamma) = -a,$$

that is $$\left(\psi - \frac{\pi}{2}\right)^2 - \psi^2 - \frac{\pi^2}{4} = -2,$$

that is $$\psi = \frac{2}{\pi}, \quad \gamma = \frac{\pi^2}{4} + \frac{4}{\pi^2}, \quad \tau = \frac{a}{2}\left(\frac{\pi}{2} - \frac{2}{\pi}\right)^2.$$

* For of course we may write in general

$$(\phi^2 - \gamma) + [\{\phi + (2i+1)\pi\}^2 - \gamma] = 0,$$

i being any integer, and ϕ will give the direction of a double tangent.

As γ goes on decreasing, the double tangents quit the Moorish arch altogether and connect the two infinite branches, which turn their protuberances towards each other more and more, until finally they touch and the double tangents coincide. This happens when $\phi = -\frac{\pi}{2}$, that is $\gamma = \frac{\pi^2}{4}$.

As the tail goes on still to decrease, the double tangents become imaginary, the infinite branches intersect and cut out a lune, one extremity of which, the two cusps of the cyclode under consideration, and the cusp of the parent cyclode, together form a quadrangle, which continually contracts its dimension until finally it vanishes with the tail and the central arc, and the four points merge into the remarkable *round* point indicated in fig. 1, corresponding to the parabolic or transition case between the cusped and uncusped species. This paradoxical point is a mere creature of the reason, and can by no effort be made sensible to the understanding. Observe that, in this point, the curve dips its beak, so to say, into the cusp of the parent first involute, and yet touches the original circle. Professor Cayley informs me he has met with the same kind of point in an investigation into the form of the parallels to an ellipse, and proposes to call it a triangular point, as consisting of the union of a node and two cusps. At this point, in the case before us, we have

$$p = \frac{a\phi^2}{2} - a, \quad \frac{ds}{d\phi} = p + p'' = \frac{a\phi^2}{2};$$

so that, it will be observed, $\frac{d^2s}{d\phi^2}$, as well as $\frac{ds}{d\phi}$, vanishes when ϕ is made zero.

This gives me occasion to make a remark which I do not remember having seen in the text-books, namely, that for any curve, while *in general* $\frac{ds}{d\phi} = 0$ indicates the existence of a cusp, this law is subject to the exception that if a succession of such derivatives

$$\frac{ds}{d\phi}, \quad \frac{d^2s}{d\phi^2}, \quad \frac{d^3s}{d\phi^3}, \dots$$

all vanish simultaneously, there will not be a cusp in fact unless the last of the series is of an *odd* order.

Fig. 5 exhibits the critical cases (1) of the double tangents in opposition, (2) on the point of quitting the central branch, (3) in coincidence. Mr Spottiswoode informs me that this figure has not been drawn with the same attention to mechanical exactitude as the other figures.

In fig. 5 are seen examples of the uncusped species. The Norwich spiral (of which a word or two more presently) belongs to this species, but is not drawn; its apse lies midway between the centre of the circle and the cusp of the first cyclode. In fig. 2 is seen an example of a symmetrical tricuspidal cyclode of the third order; in fig. 4, of a unicuspidal cyclode of the same order, where a loop replaces the missing cusps.

To return to the Norwich spiral; its radius of curvature ρ has been shown in the preceding rule to be always equal to its radius vector r, reckoned from the centre of the circle. Now it is easy to see that whilst $\int d\phi\rho$ represents the arc of any curve, $\int d\phi r$ will represent the corresponding arc of its first pedal; so that the spiral in question possesses the remarkable property (capable, one would think, of some practical kinematic application) that these two arcs always remain equal to each other. More generally, if $p^2+p'^2$*, where p is a rational integral function of ϕ, and p' its first derivative in respect to ϕ, is a perfect square, the arc of the curve and of its pedal will always remain algebraically related. Here, then, we are led to consider the possibility of satisfying this diophantine condition for cyclodes beyond the second order. At a first glance the problem might seem to be impossible. For if the condition is satisfied by $p=F\phi$, a rational integral quantic in ϕ of the order n, it obviously will be satisfied also by $F(\phi+\lambda)$, λ being an arbitrary constant; and consequently we have only $(n-1)$ and not n disposable constants (or ratios) wherewith to satisfy the n conditions involved in a function of ϕ of order $2n$ being a perfect square.

This objection, however, is only apparent, and may at once be seen so to be, at all events as regards cyclodes of an even order—say, of order $2m$. For we may suppose

$$p=F(\phi+\lambda)\{f(\phi+\lambda)\}^2,$$

a quantic of the order m in $(\phi+\lambda)^2$, then

$$F^2+F'^2=f^2+4(\phi+\lambda)^2f'^2$$

is a quantic of the order $2m$ in $(\phi+\lambda)^2$, and the m disposable constants in f are sufficient to make this a perfect square. Thus, then, the n conditions are not absolutely incompatible. Still the disproof of the incompatibility might *seem* to involve the necessity of F being a function of $(\phi+\lambda)^2$, that is of the cyclode being of the symmetrical kind. Moreover, if the problem be attacked by a direct exoscopic method for cyclodes of the second, fourth, and sixth orders, it will be found that the only cyclodes which possess the required property

* It will be remembered that $r^2=p^2+p'^2$. I may remark incidentally that this equation enables us to extend the well-known one, $p^2=r^2-a^2$, applicable to the first cyclode: the general theorem which includes this as a particular case is obviously

$$p^2=r^2-r'^2+r''^2+\ldots\pm a^2,$$

p being the perpendicular on the tangent of a cyclode of any order, and r, r', r'', ... the distances of the corresponding points in the cyclode and its successive evolutes from the centre of the originating circle.

are of the symmetrical kind, namely, for the second order, $p=\frac{a}{2}(\phi^2-1)$, for the fourth, $p=\frac{a}{2}(\phi^2-4)^2$, and for the sixth,

$$p=\frac{a}{2}(\phi^2-9)^3*, \quad \text{or} \quad p=\frac{a}{2}(\phi^2-9)(\phi^2-36)^2.$$

The inference, then, might appear to be almost irresistible as to the necessity of the symmetrical form holding good. But it is *not* so; *it is* true that only cyclodes of *even* orders are reducible, that is capable of giving r as a *rational* integral function of ϕ; but after the sixth order, that is beginning with the eighth, non-symmetrical reducible cyclodes come into existence, and, as the order rises, become infinitely more numerous than those of the symmetrical kind.

Calling $2m$ the order, every distinct mode of making the partitions of numbers expressed by the two simultaneous equations

$$\begin{Bmatrix} x_1+x_2+\ldots+x_i=m \\ y_1+y_2+\ldots+y_i=m \end{Bmatrix},$$

where i takes all possible values, gives rise to a system of equations yielding in general many solutions; and it is only when $x_1=y_1$, $x_2=y_2$, ..., $x_i=y_i$ that the solutions are of the symmetrical kind. Moreover, even in that case, in *general*, and subject only to rare cases of exception, the reducing system of equations gives two distinct groups of solutions, one corresponding to

* It is very easy to see that there is always one *reducible* symmetrical cyclode of the order $2m$ defined by the equation

$$p=\frac{a}{(2m)!}(\phi^2-m^2)^m,$$

corresponding to which

$$r=\frac{a}{(2m)!}(\phi^2-m^2)^{m-1}(\phi^2+m^2).$$

Thus, when $m=2$,

$$p=\frac{a}{24}(\phi^2-4)^2,$$

$$r=\frac{a}{24}(\phi^4-16);$$

when $\phi=0$, we have

$$p=\frac{2a}{3}, \quad p'=0, \quad p''=-\frac{2a}{3};$$

whence we may derive the following construction :—Draw an *uncusped* secondary cyclode with a tail equal to one-third of the radius; unwind from this a ternary cyclode beginning from the apse, which will become a cusp in the cyclode so engendered; and from this last cyclode, beginning at its cusp, again unwind a new cyclode, which will possess a *triangular* point at the apse of its atavian secondary cyclode. This will be a quartic reducible cyclode, and, as regards *form* (irrespective of position and magnitude), the only one that exists. By the way, it may be noticed that a system of coordinates consisting of the vectorial angle and angle of contingence furnishes what may be termed a *form* equation, that is, one in which actual magnitude is ignored. Thus, for example, $\tan\theta=k\tan\phi$ is the *form* equation to a conic.

symmetrical and the other to non-symmetrical cyclodes*. This wonderful theory, this outlying and unexplored region of geometry, in which the two great continents of algebra and arithmetic trend towards and come into contact at more than one point with one another, forms the subject of a communication to be brought by the author of this Note before the Mathematical Society of London, simultaneously and under the same roof with Mr Norman Lockyer's announcement to the Royal Society of his equally, but not more surprising and certain to be prolific discovery of the sun's unsuspected chromosphere, the analogue of the ocean of forms of which the isolated power-forms $[(\phi^2 - n^2)^n]$ correspond to the piled-up rose-coloured prominences.

* It is to be understood that every x and y must be an *actual* integer, *zero* being for this purpose to be regarded, not as a number, but as a negation of number. Furthermore, if the x and y numbers are not only respectively equal each to each, but have all the same value (as for example, *unity*), the corresponding system of equations become *incompatible*; or, to speak more philosophically, the order of the system becomes *zero*, which here *per contra* ought to be regarded as a number rather than as a negation of number; for the order of the system of equations is always lowered, not only by every x becoming equal to every y, but also by any number of x's or of y's becoming equal to each other; so that the order of the system sinking to zero, in consequence of all the x's and all the y's becoming equal, is only an extreme instance of this general law. If we go to the wider case of algebraical spirals, where $p = \frac{f(\phi)}{F(\phi)}$, the difference between the degrees of f and F being still an even integer $2m$, where m is positive or negative, and require $p^2 + \left(\frac{dp}{d\phi}\right)^2$ to be made a perfect square, precisely the same method of solution is applicable as when F is of the degree zero. If we call the degrees of f and ϕ κ and k respectively, so that $\kappa - q = 2m$, we have to make

$$x_1 + x_2 + \ldots + x_\epsilon - \xi_1 - \xi_2 - \ldots - \xi_\eta = m,$$
$$y_1 + y_2 + \ldots + y_\lambda - \eta_1 - \eta_2 - \ldots - \eta_\mu = m,$$

$\epsilon + \eta = \lambda + \mu = i$, where i takes all possible values,

$$x_1 + x_2 + \ldots + x_\epsilon + y_1 + y_2 + \ldots + y_\lambda = \kappa,$$
$$\xi_1 + \xi_2 + \ldots + \xi_\eta + \eta_1 + \eta_2 + \ldots + \eta_\mu = q.$$

Every such system of partitions gives rise to a system of equations containing solutions of the diophantine problem in question, that is, the problem of making r a rational function of ϕ. When the degree of p in ϕ, that is, $\kappa - q$ (and consequently m) is zero, the order of all the equation-systems undergoes a marked depression.