

ON THE INTEGRAL OF THE GENERAL EQUATION IN DIFFERENCES.

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THE most general form which can be given to a linear equation in differences may easily be seen to be reducible to the following,

$$a_x u_x + b_x u_{x-1} + c_x u_{x-2} + \&c. \text{ ad lib. } = 0,$$

with the initial conditions

$$u_0 = 1, \quad u_{-e} = 0.$$

Consequently to find u_n , or let us rather say to find

$$(-)^n a_1 a_2 \dots a_n u_n,$$

is really the problem of finding the value of a determinant belonging to a matrix of n^2 terms, whereof all the places below the diagonal line, with the exception of those in the oblique line immediately under the diagonal, are occupied by zeros, but of which all the other places are or may be occupied by finite quantities. For instance, supposing n to be 4, such a determinant would be

$$\begin{vmatrix} b_4, & c_4, & d_4, & e_4 \\ a_3, & b_3, & c_3, & d_3 \\ 0, & a_2, & b_2, & c_2 \\ 0, & 0, & a_1, & b_1 \end{vmatrix}.$$

Let us for a moment consider more particularly this determinant. If, using double indices to denote each coefficient, we were to write the above according to the usual method of notation as below,

$$\begin{vmatrix} 4.4, & 4.3, & 4.2, & 4.1 \\ 3.4, & 3.3, & 3.2, & 3.1 \\ 0, & 2.3, & 2.2, & 2.1 \\ 0, & 0, & 1.2, & 1.1 \end{vmatrix}$$

the law of formation of the general term would be very far from becoming evident on a cursory inspection; but a slight change, suggested by the very

system of equations in which the determinant originates, makes the law at once obvious: Nothing is more natural than that we should use $r.s$ or $s.r$, where $r > s$, to denote the coefficient of u_s in the equation of which r is the highest subindex of u ; with this modification, the above determinant changes into the following:—

$$\begin{vmatrix} 4.3, & 4.2, & 4.1, & 4.0 \\ 3.3, & 3.2, & 3.1, & 3.0 \\ . & 2.2, & 2.1, & 2.0 \\ . & . & 1.1, & 1.0 \end{vmatrix}$$

(the terms with equal indices appearing not now in the diagonal, but in the oblique line below it). With this notation it becomes apparent (and the reason of the rule may be deduced by the most simple reasoning from following the course of the successive substitutions in the system of equations giving rise to the determinant) that to find the general term we must write all the descending series of integers which can be formed, beginning with 4 and ending with zero, namely,

- 43210
- 4310
- 4210
- 4320
- 430
- 420
- 410
- 40

and read them off respectively into products as below:—

$$\begin{aligned} & 4.3 \times 3.2 \times 2.1 \times 1.0 \\ & (4.3 \times 3.1 \times 1.0) \times (-2.2) \\ & (4.2 \times 2.1 \times 1.0) \times (-3.3) \\ & (4.3 \times 3.2 \times 2.0) \times (-1.1) \\ & (4.3 \times 3.0) \times (2.2 \times 1.1) \\ & (4.2 \times 2.0) \times (3.3 \times 1.1) \\ & (4.1 \times 1.0) \times (2.2 \times 3.3) \\ & (4.0) \times (1.1 \times 2.2 \times 3.3). \end{aligned}$$

The sum of the above terms is the value of the determinant in question. And so in general, if we define u_n by means of the equation

$$(n.n) u_n + (n.n-1) u_{n-1} + (n.n-2) u_{n-2} + \dots = 0;$$

with the initial conditions as above stated, the value of u_n to a factor *près* will be represented by

$$\Sigma (n, n_1, n_2, \dots, n_\omega, 0),$$

where $n > n_1 > n_2 \dots > n_\omega$ [$\omega = 0, 1, 2, \dots (n-1)$] and $(n, n_1, n_2, \dots, n_\omega, 0)$ is to be interpreted as meaning

$$M \times n \cdot n_1 \times n_1 \cdot n_2 \times \dots \times n_\omega \cdot 0,$$

where to find M we write the complementary integers

$$m_1, m_2, m_3, \dots, m_{n-\omega+1},$$

which together with $n_1, n_2, \dots, n_\omega$ make up the complete tally of all the integers from 1 to $(n-1)$, and then write

$$M = (-)^{n-\omega+1} (m_1 \cdot m_1) \cdot (m_2 \cdot m_2) \dots (m_{n-\omega+1} \cdot m_{n-\omega+1}).$$

In order to form by an exhaustive process all the descending series above described, we may if we please consider the differences of the terms of any such series, and write

$$\delta = n - n_1, \delta_1 = n_1 - n_2 \dots \delta_\omega = n_\omega,$$

we have then

$$\delta + \delta_1 + \delta_2 + \dots + \delta_\omega = n.$$

So that the question is reducible to that of finding all the partitions of n , and of permuting in every possible manner the terms in each such system of partitions; for it is obvious that in general the value of $(n, n_1, n_2, \dots, n_\omega, 0)$ depends not only on the magnitudes, but on the order of sequence of $\delta, \delta_1, \delta_2, \dots, \delta_\omega$.

If we suppose that the order of the differences is limited, as, for example, that the equation is of the i th order, then any such coefficient as $r \cdot s$ is to be considered as zero when $r \sim s > i$, and consequently the partitions of n are to be limited to parts none greater than i . Moreover, if in such case the coefficients become constant, so that $r \cdot s = \phi(r-s)$, it is apparent that the order of the arrangement of $\delta_1, \delta_2, \dots, \delta_\omega$ becomes indifferent, and consequently the value of u_n , defined by the equation

$$u_n = (1) u_{n-1} + (2) u_{n-2} + \dots + (i) u_{n-i},$$

becomes the coefficient of t^n in $\frac{1}{1 - (1)t - (2)t^2 - \dots - (i)t^i}$, as is well known.

The above rule may easily be extended to a linear equation in differences with any number of variables. Thus suppose, for greater simplicity, that we write

$$u_{x,y} = \sum \begin{pmatrix} x, x' \\ y, y' \end{pmatrix} u_{x',y'} \quad \left[\begin{array}{l} x' = x-1, x-2, \dots, 0 \\ y' = y-1, y-2, \dots, 0 \end{array} \right],$$

with the initial conditions $u_{0,0} = 1$, $u_{e,f} = 0$ wherever one or both of e, f are negative units; then to find the value of $u_{m,n}$ we must form all the possible descending series $\left[\begin{array}{l} m, m_1, m_2, \dots, m_\omega, 0 \\ n, n_1, n_2, \dots, n_\omega, 0 \end{array} \right]$, subject only to the law that there

is a descent either from m_i to m_{i+1} , or from n_i to n_{i+1} , or at one and the same time from m_i to m_{i+1} and from n_i to n_{i+1} . The value of $u_{m,n}$ then becomes

$$\Sigma \binom{m, m_1, m_2, \dots, m_\omega, 0}{n, n_1, n_2, \dots, n_\omega, 0},$$

with the understanding that the term within the parenthesis is to be read as meaning

$$\binom{m, m_1}{n, n_1} \times \binom{m_1, m_2}{n_1, n_2} \times \binom{m_2, m_3}{n_2, n_3} \dots \times \binom{m_\omega, 0}{n_\omega, 0}.$$

And in like manner and under a similar form we obtain the value of $u_{n_1, n_2, \dots, n_\epsilon}$ defined by the general equation

$$u_{n_1, n_2, \dots, n_\epsilon} = \Sigma \binom{n_1, v_1}{n_2, v_2} \binom{n_2, v_2}{\vdots, \vdots} \binom{\vdots, \vdots}{n_\epsilon, v_\epsilon} u_{v_1, v_2, \dots, v_\epsilon}.$$

In defining the relations which connect one u with another, we may suppose that (r, s) means the coefficient of u_s in the equation

$$u_r = \Sigma (r, s) u_s \quad [r > s, u_0 = 1, u_{-e} = 0];$$

but we may also suppose that (r, s) means the coefficient of v_r in the equation

$$v_s = \Sigma (r, s) v_r \quad [r > s, v_n = 1, v_{n+\epsilon} = 0];$$

the value of u_0 , on the latter supposition, it is obvious, becomes equal to that of u_n on the former—a fact that is well known, and deducible from the circumstance that u_n and v_0 will be represented by the same determinant turned round into a new position. But by means of our general representation for the case of any number ϵ of variables, we see that there is an analogous theorem which connects together 2^ϵ different results, and which is not so immediate a consequence of the theory of determinants.

To make my meaning more clear, if we suppose the four following systems of equations, in each of which $m > \mu, n > \nu$,

$$u_{m,n} = \Sigma \binom{m, \mu}{n, \nu} u_{\mu,\nu} [u_{0,0} = 1, u_{-e,f} = 0, u_{e,-f} = 0, u_{-e,-f} = 0]^*,$$

$$v_{\mu,n} = \Sigma \binom{m, \mu}{n, \nu} v_{m,\nu} [v_{m,0} = 1, v_{m+e,0} = 0, v_{m-e,-f} = 0, v_{m+e,-f} = 0],$$

$$w_{m,\nu} = \Sigma \binom{m, \mu}{n, \nu} w_{\mu,n} [w_{0,n} = 1, w_{0,n+f} = 0, w_{-e,n-f} = 0, w_{-e,n+f} = 0],$$

$$\omega_{\mu,\nu} = \Sigma \binom{m, \mu}{n, \nu} \omega_{m,n} [\omega_{m,n} = 1, \omega_{m+e,n-f} = 0, \omega_{m-e,n+f} = 0, \omega_{m+e,n+f} = 0],$$

we shall have $u_{m,n} = v_{0,n} = w_{m,0} = \omega_{0,0}$.

* Or, more simply and rather more accurately, in place of the three equations within the bracket it is better to write $u_{p,q} = 0$ when p or q or each of them is negative, and so analogously for the cases following:—

$$v_{p,q} = 0 \text{ when } m-p \text{ or } q \text{ or each of them is negative,}$$

$$w_{p,q} = 0 \text{ when } m \text{ or } n-q \text{ or each of them is negative,}$$

$$\omega_{p,q} = 0 \text{ when } m-p \text{ or } n-q \text{ or each of them is negative.}$$

The theorem $u_n = v_0$ above given, when the equation of differences is of the second order, expresses the well-known theorem that the cumulant $[a, b, c, \dots h, k, l]$

(the denominator of the continued fraction $\frac{1}{a+}, \frac{1}{b+}, \frac{1}{c+}, \dots \frac{1}{k+}, \frac{1}{l}$)

is the same as the cumulant $[l, k, h, \dots c, b, a]$.

There is no known property either of cumulants of this kind or those of the higher orders, nor can there be any found, but what does and must flow as an immediate consequence from the representation of the linear-difference integral above given. For instance, the law of formation of the above cumulant by rejecting consecutive pairs of terms becomes intuitive; for to meet this case we must write descending series of integers $n, n_1, n_2, \dots n_\omega, 0$, such that each difference between consecutive terms n_i, n_{i+1} is always 1 or 2, and when the latter, $(n_i, n_{i+1}) = 1$.

So more generally if we write $u_n = a_n u_{n-1} + u_{n-r}$, we obtain an analogous law for throwing out in every possible way groups of r consecutive terms in order to express u_n in terms of $a_n, a_{n-1}, a_{n-2}, \dots a_0$. So, too, if we write $u_n = u_{n-1} + b_n u_{n-\mu}$, we obtain Binet's law of "discontiguous" products given in his long memoir on the subject published in the *Mémoires* of the Institute,—the law of descent upon this supposition being that the difference between n_i and n_{i+1} is 1 or r ; and if the former, $(n_i, n_{i+1}) = 1$.

We have seen above the convenience of shifting the system of subindices so as, for instance, to be able to treat the question of finding u_0 when we suppose $u_n = 1$ and $u_{n+e} = 0$, as well as that of finding u_n when we suppose $u_0 = 1, u_{-e} = 0$. More generally there is an advantage in writing $u_m = 1$ and $u_{m-e} = 0$ when it is a question of expressing u_n , which may then be conveniently denoted indifferently by $m:n$ or $n:m$,—the law being that regularly descending or ascending series are to be formed beginning with n and ending with m in every possible manner, each of which expresses a known product consisting of two parts—one made up of factors denoted by the conjunction of the consecutive terms in every such series, the other by the duplication of the integers between n and m not appearing in the series.

It is, moreover, convenient in some cases to express the limit which the descents are not to exceed (corresponding to the order of the equation).

Thus $\frac{n:m}{i}$ may be used to denote the limitation of the differences in $n:m$ not to exceed i . The well-known theorem in continued fractions ordinarily denoted by the equation $pq' - p'q = \pm 1$ may then be expressed in a somewhat more general form in the manner following.

(To be continued.)