23.

NOTE ON THE EQUATION IN NUMBERS OF THE FIRST DEGREE BETWEEN ANY NUMBER OF VARIABLES WITH POSITIVE COEFFICIENTS.

[*Philosophical Magazine*, XVI. (1858), 369—371.]

I PROPOSE to show that all the systems of values $(x, y, z \dots w)$ which satisfy a given equation in integers,

$$ax + by + cz + \dots + lw = n,$$

$((a, b, c \dots l)$ being all positive, and the number of systems therefore definite), may be made to depend on algebraical equations whose coefficients are known functions of $a, b, c \dots l$ and n. The fact is somewhat surprising, the proof easy, being an immediate consequence of the theorem I have given* in the *Quarterly Journal of Mathematics*, and also in Tortolini's *Annali* for January 1857, of the problem of the partition of numbers.

For my present purpose, this theorem may be with advantage presented under a somewhat modified form as follows:—Let $\Theta(Ft)$ be used to denote the coefficient of $\frac{1}{t}$ in the expansion of Ft in ascending powers of t. Let N stand for the number of solutions of the equation

$$ax + by + cz + \dots + lw = n;$$

let m be the least common multiple of $a, b, c, \dots l$,

ρ be any *primitive* root of $\rho^m = 1$,

and ρe^{-pt} be called Λp; then

$$N = \Sigma\Theta \left\{\frac{\Lambda(-n)}{(1-\Lambda a)(1-\Lambda b)\dots(1-\Lambda l)}\right\}.$$

If now we call N' what N becomes when, in lieu of the equation

$$ax + by + cz + \dots + lw = n, \tag{1}$$

we write

$$ax' + ax'' + by + cz + \dots + lw = n, \tag{2}$$

[* p. 90 above.]

it is clear that

$$N' = \Sigma\Theta \left\{ \frac{\Lambda(-n)}{(1-\Lambda a)^2 (1-\Lambda b)\dots(1-\Lambda l)} \right\}.$$

But it is also clear that all the solutions of equation (2) may be derived from those of equation (1), by writing for each value of x

$$x' + x'' = x; \tag{3}$$

and as the number of solutions of equation (3) is evidently $x+1$, we have $N' = \Sigma x + N$, or

$$\Sigma x = \Sigma\Theta \left\{ \frac{\Lambda a \,.\, \Lambda(-n)}{(1-\Lambda a)^2 (1-\Lambda b)\dots(1-\Lambda l)} \right\}.$$

In like manner, if we write

$$ax' + ax'' + ax''' + by + cz + \dots + lw = n,$$

the solutions of this equation spring from those of equation (1) by making $x' + x'' + x''' = x$, the number of solutions of which equality is $\frac{1}{2}(x+1)(x+2)$: wherefore

$$\Sigma\, \frac{x^2 + 3x + 2}{2} = \Sigma\Theta \left\{ \frac{\Lambda(-n)}{(1-\Lambda a)^3 (1-\Lambda b)\dots(1-\Lambda l)} \right\};$$

from which we may readily deduce, by aid of what has been already shown,

$$\Sigma x^2 = \Sigma\Theta\, \frac{(\Lambda a)(1+\Lambda a)\,\Lambda(-n)}{(1-\Lambda a)^3 (1-\Lambda b)\dots(1-\Delta l)},$$

and so in general,

$$\Sigma x^i = \Sigma\Theta\, \frac{\Lambda(a)(1+\Lambda a)\dots\{(i-1)+\Lambda a\}}{(1-\Lambda a)^{i+1} (1-\Lambda b)\dots(1-\Lambda l)}.$$

Again, if we write

$$ax + by_1 + by_2 + \dots + by_\epsilon + cz + \dots + lw = n, \tag{4}$$

we shall find by parity of reasoning (seeing that in this last equation the solutions may be derived from those of equation (1) by keeping $x, z, \dots w$ all unaltered, whilst we give to $y_1, y_2 \dots y_\epsilon$ all the values compatible with $y_1 + y_2 + \dots + y_\epsilon = y$), the value of Σx^i in equation (4) will be the same as that of

$$\Sigma x^i \,.\, \frac{(y+1)(y+2)\dots(y+\epsilon)}{1\,.\,2\dots\epsilon}$$

in equation (1). Wherefore we shall evidently obtain

$$\Sigma x^i \,.\, y^\epsilon = \Sigma\Theta\, \frac{\Lambda a(1+\Lambda a)\dots\{(i-1)+\Lambda a\} \times \Lambda b(1+\Lambda b)\dots\{(\epsilon-1)+\Lambda b\}}{(1-\Lambda a)^{i+1}(1-\Lambda b)^{\epsilon+1}(1-\Lambda c)\dots(1-\Lambda l)};$$

the extension of the theorem to $\Sigma x^i \,.\, y^\epsilon \,.\, z^\omega \dots$ is too obvious to need further allusion.

Thus, then, to find x_1, $x_2 \ldots x_N$, we may begin by forming an equation of the Nth degree, whose coefficients are known, because the sums of the powers of the roots are given. Supposing these roots to consist of N_1 values x_1, N_2 values $x_2, \ldots N_\mu$ values x_μ, the solution of μ simple equations will enable us to find the sum of the N_1 values of y corresponding to x_1, the sum of the N_2 values of y corresponding to $x_2 \ldots$, and the sum of the N_μ values of y corresponding to x_μ. To effect this, we have only to write down the values of Σxy, $\Sigma x^2 y, \ldots \Sigma x^\mu y$. In like manner we may find the sum of the N_1 values of y^2 corresponding to x_1, the N_2 values of y^2 corresponding to x_2, &c., and so in general for y^ω. Thus, then, we may obtain the requisite number of sets of equations for determining independently by means of equations of the degrees $N_1, N_2, \ldots N_\mu$ respectively the values of y corresponding to each of the distinct values of x; and in like manner for all the other variables. The principal interest of this note consists, however, in the appreciation of the fact that we can represent algebraically, as has been shown above, the value of $\Sigma x^\alpha . y^\beta . z^\gamma \ldots$, where the sign of summation extends over all the simultaneous solutions of

$$ax + by + cz + \&\text{c.} = n.$$

This is a considerable advance upon the conception (itself before my discovery entirely unrecognized*) of the explicit representability of the mere number of the solving systems $x, y, z \ldots$ by general algebraical formulæ. By this new theorem we pass, as it were, from the shadow to the substance.

* As witness the comparatively unfructuous labours of Paoli, Herschel, Kirkman, and even of Cayley. But as honest labour is seldom entirely wasted, so in the present case it was my valued friend Mr Kirkman's Manchester memoir on partitions which first drew and fixed my attention on the subject.