22.

NOTE ON THE ALGEBRAICAL THEORY OF DERIVATIVE
POINTS OF CURVES OF THE THIRD DEGREE.

Two years and upwards have elapsed since I discovered the extraordinary theorem in the doctrine of cubic forms which I am about to state, but which has never yet been published by me, although communicated in confidence to a few friends, including Mr Cayley. It arose out of purely arithmetical speculations relating to such forms, to some of which I may make a brief allusion in the course of this note.

If we suppose the general homogeneous equation of the third degree in x, y, z reduced to the canonical form

$$x^3 + y^3 + z^3 + mxyz = 0,$$

any solution $x = a$, $y = b$, $z = c$ of this equation is of course one of a group of six obtained by the permutations of the three letters a, b, c, and having an obvious relation to one another through the medium of the points of inflexion. So, too, it is manifest if we take the equation to the curve in its most general form, from any given solution, a group of six, including the given one, may be formed, *the characteristics* of each of which will be linear functions of one another. For the purpose of the theorem about to be enunciated, such a group of solutions will be treated as a single solution; and then we can affirm the proposition following, in which a solvent system means a system of values of the variables x, y, z satisfying the equation $f(x, y, z) = 0$, and free from any common factor.

Let a, b, c *be any solvent system to a cubic homogeneous equation in* x, y, z; *then from* a, b, c *we may derive a new solvent system,* a′, b′, c′, *where* a′, b′, c′ *are each of them functions of the fourth degree of* a, b, c, *and another system* a″, b″, c″ *of the ninth degree in* a, b, c, *and another* a‴, b‴, c‴ *of the sixteenth*

degree, and so in general a new solvent system of the degree n^2 *in* a, b, c. *One such derivative system, and only one, of the degree* n^2 *can be formed, and none of any intermediate degree.*

Thus, for instance, the coordinates of the tangential (the name adopted from me by Mr Cayley to express the point of intersection of a tangent to a cubic curve at any point with the curve) being called a', b', c', these last letters are *biquadratic* functions of a, b, c^*.

So again, as I also suggested to Mr Cayley, the point in which the conic of closest contact with a cubic curve cuts the curve will necessarily have a derivative system of coordinates of a square-numbered degree in respect of the original ones, which by actual trial Mr Cayley has found to be the 25th. Mr Salmon, I believe, has obtained in certain geometrical investigations derivatives of the 49th degree.

I am in possession of the equations by means of which the successive systems of the fourth, ninth, &c. degrees, which I incline to call the first or primary, the second, third, &c. derivative systems, may be formed explicitly by successive derivation from one another; so that, for instance, as soon as I am informed that the system investigated by Mr Cayley is of the twenty-fifth degree or fifth order, I can find them without any reference to the geometry of the question, the quantities belonging to the nth derivative being in fact a known algebraical function of n! I was led to the discovery of this surprising and unique law by a statement of a friend, *not since verified, and which, for aught that has yet been shown, may or may not be true,* that the number 5 *could be* divided into two rational cubes: assuming this to be the fact, it necessitated (by virtue of my investigations) the coincidence to a factor *près* of two functions obtained by apparently independent algebraical processes, which coincidence by actual comparison of the functions I found to obtain.

With reference to the connexion of this theory of derivation with the arithmetic of equations of the third degree between three variables with integer coefficients, it is after this kind. Fermat has taught us that a certain class of such equations, viz. the equation $x^3 + y^3 + z^3 = 0$, is absolutely insoluble in integers (abstraction made of the trivial solutions of the type $x = 0$, $y + z = 0$). I have greatly multiplied the classes of such known insoluble equations, as may be seen by a communication from me to Tortolini's *Annali* in 1856 [p. 63 above]. But over and above such equations I have ascertained the existence of a large class of equations, soluble, or possibly so, it is true, but enjoying the property that all their solutions in integers, when they exist, are *monobasic*; that is to say, all their solutions are known functions of one

---

* This derivative solution (though not as corresponding to the *tangential*) was known also to Euler for a particular case, as will be seen by reference to his *Algebra*.

of them, which I term the *base*, and which is characterized by this property—that of all the solutions possible it is the one for which the *greatest* of the three variables is the *smallest* number possible. If this solution be laid down as a point in the curve corresponding to the given cubic, all the other solutions possible in integers will be represented by points in this curve, which are derivatives (in the sense previously employed in this note) to the given point, having coordinates respectively of the 4th, 9th, 16th, &c. degrees, in respect of the coordinates of the *basic* point*.

If my memory serves me truly, I have found (as a particular case) that all cubic equations in numbers of the form

$$x^3 + y^3 + z^3 = imxyz,$$

where i is 1 or 3 or 6 (I cannot at the moment remember which), are either *insoluble* or *monobasic*. The case of $im = 3$ must of course be exceptional, being satisfied by $x + y + z = 0$. This doctrine of derivation evidently conducts to a new branch of the grand doctrine of invariance. I hope to have tranquillity of mind ere long to give to the world my memoir, or a fragment of it, "On an Arithmetical Theory of Homogeneous and the Cubic Forms," the germ of which, now, alas! many weary years ago, first dawned upon my mind on the summit of the Righi, during a vacation ramble.

---

* This theorem is analogous to that relating to the integer solutions of $x^2 - Ay^2 = 1$, in so far as there is a *basic* solution to this equation in integers of which all the other solutions are derivatives, and not more than one such derivative exists of any given degree, but with the difference that there does exist one of every degree, and not merely (as in my theorem for cubic forms) of every square degree.