19.

AN ACCOUNT OF A DISCOVERY IN THE THEORY OF NUMBERS RELATIVE TO THE EQUATION $Ax^3 + By^3 + Cz^3 = Dxyz$.

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First General Theorem of Transformation.

If in the equation

$$Ax^3 + By^3 + Cz^3 = Dxyz, \qquad (1)$$

A and B are equal, or in the ratio of two cube numbers to one another, and if $27ABC - D^3$ (which I shall call the Determinant) is free from all single or square prime positive factors of the form $6n+1$, but without exclusion of *cubic* factors of such form, and if A and B are each odd, and C the double or quadruple of an odd number, or if A and B are each even and C odd, then, I say, the given equation may be made to depend upon another of the form

$$A'u^3 + B'v^3 + C'w^3 = D'uvw;$$

where

$$A'B'C' = ABC,$$

$$D' = D,$$

$$uvw = \text{some factor of } z.$$

The following are some of the consequences which I deduce from the above theorem. In stating them it will be convenient to use the term Pure Factorial to designate any number into the composition of which no single or square prime positive factor of the form $6n+1$ enters.

The equations

$$x^3 + y^3 + 2z^3 = Dxyz,$$

$$x^3 + y^3 + 4z^3 = Dxyz,$$

$$2x^3 + 2y^3 + z^3 = Dxyz,$$

are insoluble in integer numbers, provided that the Determinant in each case is a Pure Factorial.

The equation

$$x^3 + y^3 + Az^3 = 9Bxyz$$

is insoluble in integer numbers, provided that the Determinant, for which in this case we may substitute $A - 27B^3$, is a pure factorial whenever A is of the form $9n \pm 1$, and equal to $2p^{3i\pm1}$ or $4p^{3i\pm1}$, p being any prime number whatever.

I wish however to limit my assertion as to the insolubility of the equations above given. The theorem from which this conclusion is deduced does not preclude the possibility of two of the three quantities x, y, z being taken positive or negative *units*, either in the given equation itself or in one or the other of those into which it may admit of being transformed. Should such values of two of the variables afford a particular solution, then instead of affirming that the equations are insoluble, I should affirm that the *general solution* can be obtained by equations in finite differences*.

Second General Theorem of Transformation.

The equation

$$f^3x^3 + g^3y^3 + h^3z^3 = Kxyz \tag{2}$$

may always be made to depend upon an equation of the form

$$Au^3 + Bv^3 + Cw^3 = Duvw,$$

where

$$ABC = R^3 - S^3,$$

$$D = 3R;$$

and uvw = some factor of $fx + gy + hz$.

R representing $K + 6fgh$,

S „ $K - 3fgh$.

* Take for instance the equation $x^3+y^3+2z^3=9xyz$. The Determinant 27.25 is a Pure Factorial: consequently if the solution be possible, since in this case the transformed must be identical with the given equation, this latter must be capable of being satisfied by making x and y positive or negative units. Upon trial we find that $x=1$, $y=1$, $z=2$ will satisfy the equation. I believe, but have not fully gone through the work of verification, that these are the only possible values (prime to one another) which will satisfy the equation. Should they not be so, my method will infallibly enable me to discover and to give the law for the formation of all the others.

Here, then, under any circumstances, is an example, the first on record, of the complete resolution of a numerical equation of the third degree between three variables.

I have not leisure to show the consequences of this theorem of transformation in connexion with the one first given, but shall content myself with a single numerical example of its applications:

$$x^3 + y^3 + z^3 = -6xyz$$

may be made to depend on the equation

$$u^3 + v^3 + w^3 = 0,$$

and is therefore insoluble.

It is moreover apparent that the Determinant of equation (2) transformed is in general $-27R^3$, and is therefore always a Pure Factorial, and consequently the equation

$$f^3x^3 + g^3y^3 + h^3z^3 = Kxyz$$

will be itself insoluble, being convertible into an insoluble form, provided that $K + 6fgh$ is divisible by 9, and provided further that $(K + 6fgh)^3 - (K - 3fgh)^3$ belongs to the form m^3Q, where Q is of the form $9n \pm 1$, and also of one or the other of the two forms $2p^{3i\pm1}$, $4p^{3i\pm1}$, p being any prime number whatever.

Pressing avocations prevent me from entering into further developments or simplifications at this present time.

It remains for me to state my reasons for putting forward these discoveries in so imperfect a shape. They occurred to me in the course of a rapid tour on the continent, and the results were communicated by me to my illustrious friend M. Sturm in Paris, who kindly undertook to make them known on my part to the Institute.

Unfortunately, in the heat of invention I got confused about the law of oddness and evenness, to which the coefficients of the given equation are in the first theorem *generally* (in order for the successful application of my method as far as it is yet developed) required to be subject. I stated this law erroneously, and consequently drew erroneous conclusions from my Theorems of Transformation, which I am very anxious to seize the earliest opportunity of correcting. I venture to flatter myself that as opening out a new field in connexion with Fermat's renowned Last Theorem, and as breaking ground in the solution of equations of the third degree, these results will be generally allowed to constitute an important and substantial accession to our knowledge of the Theory of Numbers.