

THE SPATIAL DISTRIBUTION OF LOW FLOWS IN POLAND NOT EXCEEDED AT AN ASSUMED PROBABILITY

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Abstract: Since most hydrological quantities are regarded as random variables, it is important that their distributions of probability be identified. This paper therefore presents the results of a fitting of most frequently-applied probability distributions to a series of minimum yearly flows. Specifically, the series of minimal annual discharges derived from 119 stream gauges located throughout Poland and relating to the period 1971-1990. The main task was to adjust one of the probability distribution functions applied most frequently (Fig. 1). Maps generated provide for the identification of several regularities.

Key words: hydrology, Poland, flow, rivers, frequency distributions, low-flow

INTRODUCTION

Annual minimal flows play an important role in hydrology, since knowledge of them, their temporal variability, periodicity and spatial diversity is crucial if the use of water resources on the local and national scales is to be rational. Annual minimal flows can be used to generate a series of parameters and characteristics of economic and management value (navigation parameters of rivers, inviolable flows, available flows, etc.). Many authors use low discharges to estimate the boundary values used in setting low flows, and it is equally important that there be knowledge and an appropriate interpretation of low discharges in the estimation of inviolable flows, i.e. the minimum flow rate needing to be maintained in a particular discharge section line of a river in line with biological and social considerations.

The conventional characteristic flows applied in Poland (e.g. Average Low Flow, Smallest Low Flow), are usually values not

associated with any known probability of appearance. For this reason, flows at an assumed probability are used to calculate constructional profiles (for example as water intakes on rivers are planned).

A Low-flow Frequency Curve (LFFC) shows the proportion of years in which a flow is exceeded (or equivalently the average interval in years ('return period' or 'recurrence interval') after which a river once again falls below a given discharge) (Smakhtin 2001). The LFFC is constructed on the basis of a series of annual flow minima (daily or monthly minimum discharges or flow volumes), which are extracted from available original continuous flow series (one value from every year recorded).

METHOD

As most hydrological characteristics are regarded as random variables, it is im-

portant that distributions of probability for them be identified. The procedure involved usually includes two stages: estimation of the parameters to the accepted probability distribution and verification of the hypothesis that a real probability distribution of a given value is involved.

As the ‘true’ probability distributions of low flows are unknown, the practical problem is to identify a reasonable ‘functional’ distri-

bution and to quantify the parameters thereof. The procedure here entails fitting several theoretical distribution functions to observed low-flow data and deciding which distribution fits the data best, on the basis of statistically- and graphically-based tests. Among the distribution functions, most of those referred to frequently in the literature in connection with low-flows are different forms of the Weibull, Gumbel, Pearson Type III and log-normal distributions.

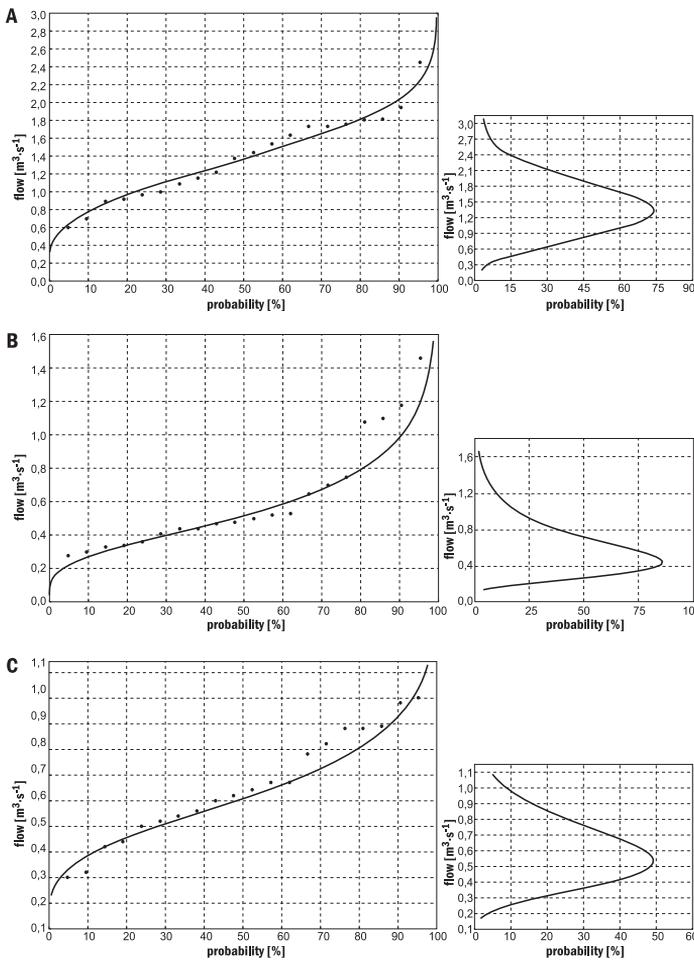


Figure 1. Examples of fitted types of distribution of analysed annual low flow series from twenty-year period (1971-1990) (flow duration and density curves). A – Fisher-Tippet III distribution (Biała Łądecka-Żelazno), B – log-normal distribution (Gowienica-Widzieńsko), C – gamma distribution (Łososina-Jakubkowiec).

Many studies have been devoted to the probability distributions most suitable for fitting the sequences of annual minimum flows in different regions and for minima of different averaging intervals, as well as evaluating methods by which to estimate distribution parameters (Matalas 1963; Jozeph 1970; Prakash 1981; Beran and Rodier 1985; Loganathan *et al.* 1985; McMahon and Mein 1986; Singh 1987; Waylen and Woo 1987; Khan and Mawdsley 1988; Sefe 1988; Leppajarvi 1989; Polarski 1989; FRIEND 1989; Nathan and McMahon 1990b; Russell 1992; Loaiciga *et al.* 1992; Durrans 1996; Lawal and Watt 1996a; Lawal and Watt 1996b; Bulu and Onoz 1997, Jakubowski 2005).

The author's task was not to investigate the real distribution of examined series but only to adjust one of the functions applied most frequently to ensure fulfilment of the Kolmogorov λ test condition of conformability. The scheme applied to achieve this entailed:

- an attempt to adjust a Fisher-Tippett III distribution, frequently recommended in reference to minimum flows, by means of the maximum likelihood method;
- adjustment of the function of the gamma distribution to the series for which the Kolmogorov λ test indicated lack of compatibility with the aforementioned Fisher-Tippett III distribution;
- further verification followed by adjustment of the log-normal distribution to the remaining series by means of the moments method, as well as verification of compatibility with the empirical distribution (test above).

The distribution functions for all investigated basins were recognized in this way. Matalas (1963) recommended the types of distributions listed above for the analysis of low flows in the United States.

The use of the Kolmogorov λ test in deciding on the sufficient level of adjustment of a theoretical distribution to an empirical one can be controversial. It is sometimes considered to be too gentle (Brzeziński

1995; Ozga-Zielińska, Brzeziński 1997). However, on the other hand, its computational simplicity makes it popular and recommended officially for application (*Zasady obliczania ...* 1973).

The basic information relating to the fit types of the probability distribution is as presented below. Further details on properties of these functions and questions concerning estimation of parameters for them are to be found in many papers on the subject (Kaczmarek 1970; Byczkowski 1972; Kite 1988; Węglarczyk 1998). Examples of three fit distributions of probability are shown in Fig. 1.

The Fisher-Tippett type III distribution (Weibull) – Fig. 1 A

Examined by Fisher and Tippett in 1928 (Byczkowski 1996), this asymptotic distribution is well conditioned when sample size proceeds to infinity. Theoretically, then, it should only be applied when sample size is sufficiently large. On the other hand, there is no general criterion allowing it to be evaluated if a given size is indeed sufficiently large (Kaczmarek 1960). The probability density function is:

$$f(x; \alpha, \beta, \varepsilon) = \frac{\beta}{\alpha - \varepsilon} \cdot \left(\frac{x - \varepsilon}{\alpha - \varepsilon} \right)^{\beta - 1} \cdot e^{-\left(\frac{x - \varepsilon}{\alpha - \varepsilon} \right)^\beta}, \quad (1.)$$

$$\alpha > 0, (\beta x) > 0, x \geq \varepsilon$$

where α is a scale parameter equal to the order of the lowest derivative of the probability function that is not zero at $x = \varepsilon$, β the characteristic drought (a location or central value parameter) and ε the lower limit to x .

Equations are a solution employing the method of maximum likelihood:

$$(\hat{\beta} - 1) \cdot \sum_{i=1}^n (x_i - \hat{\varepsilon})^{-1} - \frac{n \hat{\beta} \sum_{i=1}^n (x_i - \hat{\varepsilon})^{\hat{\beta} - 1}}{\sum_{i=1}^n (x_i - \hat{\varepsilon})^{\hat{\beta}}} = 0 \quad (2.)$$

$$n + \hat{\beta} \cdot \sum_{i=1}^n \ln(x_i - \hat{\varepsilon}) - \frac{n \hat{\beta} \cdot \sum_{i=1}^n (x_i - \hat{\varepsilon})^{\hat{\beta}} \cdot \ln(x_i - \hat{\varepsilon})}{\sum_{i=1}^n (x_i - \hat{\varepsilon})^{\hat{\beta}}} = 0 \tag{3.}$$

No further simplification is possible and equations 2 and 3 must be solved as simultaneous equations. The procedure follows the method of Condie and Nix (1975). Knowing α and ε , the β parameter can be solved using the formula:

$$\hat{\alpha} = \hat{\varepsilon} + \left[\frac{\sum_{i=1}^n (x_i - \hat{\varepsilon})^{\hat{\beta}}}{n} \right]^{\frac{1}{\hat{\beta}}} \tag{4.}$$

Gamma distribution (Pearson type III) – Fig. 1 C

This is one of the distributions applied most frequently in hydrology. It was worked out at the beginning of the 20th century by British statistician Karl Pearson, as one of the biggest systems of the function. The gamma distribution has gained widespread use in the smoothing of hydrological data series. The density probability distribution of the Pearson type III distribution is:

$$f(x; \alpha, \beta, \varepsilon) = \frac{1}{\alpha \cdot \Gamma(\beta)} \cdot \left(\frac{x - \varepsilon}{\alpha} \right)^{\beta-1} \cdot e^{-\left(\frac{x-\varepsilon}{\alpha}\right)}, \tag{5.}$$

$$x \geq \varepsilon, \alpha > 0, \beta > 0$$

where α , β and ε are parameters to be defined, and $\Gamma(\beta)$ the gamma function.

The Pearson type III function is asymmetrical in relation to the straight line crossing the mode point (i.e. is positively skewed), limited at the point $\varepsilon=0$, and asymptotic, approaching the vertical axis in the upper part. There is also one maximum in the mode point.

The likelihood function is established as the logarithm of the product of the density probability distributions. Differentiation

with respect to α , β and ε thus equating to zero give the equations:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{1}{\alpha^2} \cdot \sum_{i=1}^n (x_i - \varepsilon) - \frac{n\beta}{\alpha} = 0 \tag{6.}$$

$$\frac{\partial \ln L}{\partial \beta} = -\frac{n \cdot \Gamma'(\beta)}{\Gamma(\beta)} + \sum_{i=1}^n \ln(x_i - \varepsilon) - n \cdot \ln \alpha = 0 \tag{7.}$$

$$\frac{\partial \ln L}{\partial \varepsilon} = \frac{n}{\alpha} - (\beta - 1) \cdot \sum_{i=1}^n \left(\frac{1}{x_i - \varepsilon} \right) = 0 \tag{8.}$$

Further transformations are solved numerically. Maximum likelihood methods may not always be applicable, and in general it is not possible for any automatic procedure to guarantee determination of the minimum variance solution each time. Each dataset requires careful investigation of the shape of the function.

The three-parameter lognormal distribution (Galton) – Fig. 1 B

The three-parameter lognormal represents the normal distribution of the logarithms of the reduced variable $(x-\varepsilon)$, where ε is a lower boundary. The density probability distribution is given by:

$$f(x; \mu, \delta, \varepsilon) = \frac{1}{(x - \varepsilon)\delta\sqrt{2\pi}} \cdot e^{-\left[\frac{[\ln(x-\varepsilon)-\mu]^2}{2\delta^2}\right]}, \tag{9.}$$

where μ and σ^2 are form and scale parameters, shown later to the mean and variance of the logarithms of $(x-\varepsilon)$.

The lognormal distribution curve is asymmetrical (displays positive skewness), limited at the point $\varepsilon=0$ (the range of variability is unlimited), and with a function of density that has one maximum. Using the formula presented below (Kite 1988), estimators of the distribution were calculated by means of the method of moments.

$$\delta = \sqrt{\ln(1+(z)^2)} \tag{10.}$$

$$\mu = \ln\left(\frac{\delta}{z}\right) - \frac{\ln(z+1)}{2} \quad (11.)$$

$$\varepsilon = \mu - \frac{\delta}{z} \quad (12.)$$

$$\text{where: } z = \frac{1 - \left(\frac{-c_s + (c_s^2 + 4)}{2}\right)^{2/3}}{\left(\frac{-c_s + (c_s^2 + 4)}{2}\right)^{1/3}} \quad (13.)$$

and c_s is the skewness coefficient.

MATERIALS

Studies were based on the series of annual minimal discharges (NQ) from 119 stream gauges closing off autochthonous basins. As the catchments referred to cover 19% of Poland, from the point of view of the size criterion of the sample, it is possible to affirm with 95% confidence that this is enough for derived analyses to be regarded as credible (Boczarow 1976). Flow series (Q) [m^3s^{-1}] were transformed into their equivalent series (q) [$\text{dm}^3\text{s}^{-1}\text{km}^{-2}$] of specific runoffs more suitable for comparative analyses and for the illustration of the spatial variability to a phenomenon. Areas of basins are between 104 and 1534 km^2 (small and medium-sized in Polish conditions).

Data used for the analysis come from the period 1971-1990. It was a phase of peculiar "hydrological anxiety" with frequent high-water events of wide amplitude (1979, 1982), as well as prolonged droughts extending over large areas of Poland (1983, 1989). At that time, the anthropogenic influence on hydrological processes increased. Equally important was the fact that relatively few analyses treated low flows during this period. The twenty-year period of 1971-1990 was in many respects atypical, however, and from the research point of view this repre-

sents a very significant feature, leading to its being described as a "multi-year period digest". Examining a shorter time-interval, we obtain information on the scale of variability to a phenomenon being addressed over a longer period.

There is now a common view among hydrologists that gauging records need to meet conditions of homogeneity so that they can be statistically tested and analysed (Ozga-Zielińska, Brzeziński 1994; Węglarczyk 1998). With a view to potential non-uniformity in series of NQ flows being identified, several statistical tests are recommended. To detect elements whose values stray more markedly from remaining values, the *Grubbs-Beck test* was applied. The following step of a uniformity analysis procedure examined all the records series for the independent occurrence of elements. The *Wald-Wolfowitz test* was applied to that end. The final stage of the analysis entailed checks for series stationarity of selected basin samples using a *rank Kruskal-Wallis sums test*.

The fact that all observed cuttings were detected with a *Grubbs-Beck test* is of a great importance, as this statistic only in fact allows for detection of elements with values differing markedly from others. Next a *Wald-Wolfowitz series test* and a *Kruskal-Wallis test* qualified the researched series as uniform at the $\alpha = 1\%$ level of significance. At the same time, in accordance with Neyman's and Scott's statement (1971), some probability distributions including the *log-normal*, *Pearson* and *Weibull* were entirely resistant to outstanding elements in a records series. This means that the occurrence of elements outstanding in the sequence described by means of those distributions of random variables should be treated as totally natural, and the elements in question should not be rejected. As is demonstrated in the following part of this paper, all examined series of records present distributions from this particular group.

ANALYSIS

The results of fitting probability distributions to the studied series of annual minima for discharges are presented in Fig. 2. Only about 15% of the empirical distributions of studied series (17 basins) were recognised as Fisher-Tippett distributions. Likewise, only about 15% of the studied cases fitted the gamma distribution well. Overall, in more than 70% of the examined series the log-normal distribution curve approximated the empirical distribution sufficiently. This type of distribution may be assumed to predominate in Poland.

The spatial distribution of the types of probability distribution adjusted to the series of minimum yearly flows does not indicate any considerable order (Fig. 3). However, several regularities may be observed. It seems, for ex-

ample, that the rivers with minimum flow series have a similar distribution to the Fisher-Tippett function, being grouped in areas of high retentiveness. This is especially true of the rivers in young glacial areas, as well as upland rivers relatively abundant in water. Lake District basins are examples here, like those of the Glomia,

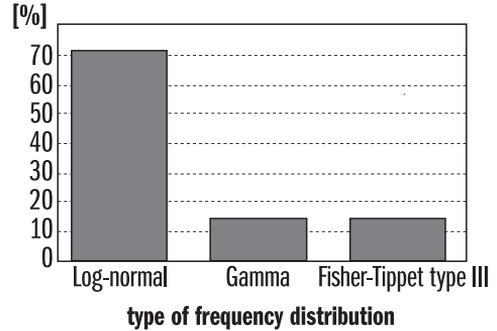


Figure 2. Frequency of distribution types of analysed set of annual low flow series (NQ).

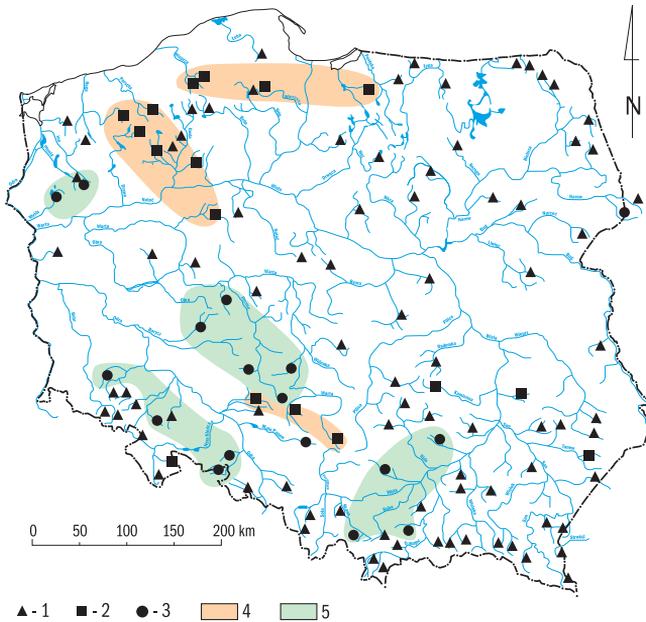


Figure 3. Types of frequency distributions of annual low flow series. 1 – log-normal distribution, 2 – Fisher-Tippett III distribution, 3 – gamma distribution, 4 – areas in which basins with yearly minima coming under Fisher-Tippett III distribution are grouped, 5 – areas in which basins with yearly minima coming under gamma distribution are grouped.

Rega and Parseta, as well as some rivers flowing off the Lublin Upland, Roztocze Hills and Kraków-Czestochowa Upland (e.g. the Tanew, Warta and Liswarta). Most of these series are characterized by high negative coefficients of skewness.

Many regions of Poland are found to have pairs of adjacent rivers whose minima for low flows assume a gamma distribution (like the Myśla and Mała Ina; the Orla and Czarna Struga; and the Osobłoga and Biała). Such a situation occurs often applies in Lower Silesia and on the Wielkopolska Lowland. Beyond these areas, series of minimum annual flows to which the gamma distribution fits best occur singly and relatively seldom.

The great majority of the studied basins are distinguished by a lognormal distribution, so we can assume that this type of distribution dominates in Poland where average-sized basins are concerned. Although the regularities observed are not marked, the author is of the opinion that they offer a strong hint as regards further research and analysis in this area.

EXTREMES IN LOW-FLOW SERIES

The spatial and temporal nature of extremes are very important in analyses of low flow. Conclusions are to be drawn with care, on account of the temporal inhomogeneity of the extremes. Thus, to maintain comparability of results and eliminate this problem, quantiles were accepted estimators of the minimum and maximum specific low runoff, calculated on the basis of probability distributions presented above. Specific low runoff of a probability not exceeding 10% was used as the estimator of minimum low flow, while the estimator of the maximum was probable runoff, measured in respect of specific low runoff of probability not exceeding 90%.

The estimators of maximum specific low runoff - $Nq_{(90\%)}$, change within the sample of basins, over a comparatively wide range (from 0.6 to 8.7 $\text{dm}^3\text{s}^{-1}\text{km}^{-2}$) - Fig. 4. The minimum was obtained for the Ochnia basin, while the maximum appeared in basins of the Pomeranian Lake District (the Studnica - 8.73 $\text{dm}^3\text{s}^{-1}\text{km}^{-2}$, the Wieprza - 8.61 $\text{dm}^3\text{s}^{-1}\text{km}^{-2}$). $Nq_{(90\%)}$ ranged from 1.8 to 3.4 $\text{dm}^3\text{s}^{-1}\text{km}^{-2}$ in half of the studied cases, and the median obtained for the whole sample of basins is equal to 2.57 $\text{dm}^3\text{s}^{-1}\text{km}^{-2}$. The distribution for these runoffs shows clear, if not very strong, positive skewness (coefficient of skewness = 1.37).

Some mountain areas are also represented in the group of river basins in which $Nq_{(90\%)}$ values differ significantly from the average, alongside the aforementioned Lake District basins (the Biała Łądecka and Łomnica in the Sudety Mountains and the Biały Dunajec in the Tatras) - Fig. 4.

High absolute minima are connected with the occurrence in these basins of waste-mantles of great thickness and considerable water-absorptiveness. According to Jokiel (1994), the resources of the zone of active exchange are considerable; much greater than on average in Poland.

In remaining mountain areas, foothills, uplands and the greater part of the lake districts, runoff $Nq_{(90\%)}$ estimators are much lower, and do not exceed 4 $\text{dm}^3\text{s}^{-1}\text{km}^{-2}$. Somewhat greater maxima are only noted for the basins in the Roztocze and Podhale regions, as well as in the Beskid Śląski and Beskid Żywiecki ranges. Low $Nq_{(90\%)}$'s also occur in the eastern part of the Małopolska Upland and among the Great Mazurian Lakes.

On the basis of the spatial distribution for the estimator of minimum specific low runoff ($Nq_{(10\%)}$) there is found to be a quite distinctive, compact area of low runoff accounting for a considerable part of the Polish Lowland (Wielkopolska and Mazowsze). The greatest runoffs appear consistently in

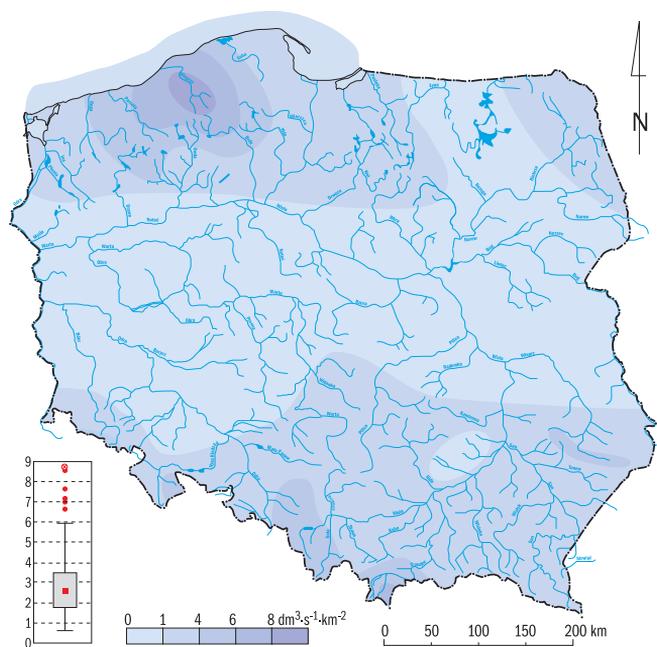


Figure 4. Spatial arrangement and diversity of maximal 90% specific low flow estimator – $Nq(90\%)$ [$\text{dm}^3 \cdot \text{s}^{-1} \cdot \text{km}^{-2}$]

the lake districts of Pomerania and Suwałki, as well as in the Tatra Mountains. Account should be taken of the way this relates to the specific hydrogeological conditions in this area. The greatest minimum runoffs occur in regions richest in near-surface underground waters. For example, there are basins in areas with large aquifers, such as those in the deposits of end moraine (of the Pomeranian and Suwałki Lake Districts), or in creviced karst rocks (of the Uplands: of Lublin and the Małopolska region).

The Carpathians (especially the Western Beskids) also feature elevated $Nq_{(10\%)}$ values. In this case, the estimator is diversified in relation to the affluence of aquifers and the speed with which they become dewatered. The differentiation to the environmental conditioning of low flows was the reason for the regional division of this area by Tlałka (1982). The results of her work

might also be related to low runoffs because of the identical origin of the two phenomena. The main characteristic of the Carpathian area is the more major role played by falls and evaporation processes in the formation of the runoff, as compared with other parts of Poland. A lack of prolonged dry periods is also an important feature of mountainous terrain. This property, and the fact that rivers are supplied by crevice waters (springs with a quite stable yield) during low-flow periods ensures that in mountain rivers a certain small amount of groundwater is always present.

The average estimator of minimum low runoff for the sample of basins in the period 1971-1990 was $0.99 \text{ dm}^3 \cdot \text{s}^{-1} \cdot \text{km}^{-2}$. Half of all estimated runoffs are in the range 0.56 to $1.46 \text{ dm}^3 \cdot \text{s}^{-1} \cdot \text{km}^{-2}$. Similarly, as in the earlier cases, a group of values significantly different from average is also apparent here. The

absolute maximum $Nq_{(10\%)}$ was observed in the Studnica basin ($6.36 \text{ dm}^3\text{s}^{-1}\text{km}^{-2}$). Large runoffs also occur in different basins of the Pomeranian Lake District, the Wieprza ($5.88 \text{ dm}^3\text{s}^{-1}\text{km}^{-2}$) and the Brda ($4.95 \text{ dm}^3\text{s}^{-1}\text{km}^{-2}$). Wielkopolska Lake District basins are characterized by the absolute lowest $Nq_{(10\%)}$ runoffs $Nq_{(10\%)}$. The distribution of the basins in the studied sample is positively asymmetric (skewness coefficient = 2.17) - Fig. 5.

LOW-FLOWS NOT EXCEEDED AT THE PROBABILITY OF 1%

As available observed flow records are normally insufficient for reliable frequency quantification of extreme low-flow events, different types of theoretical distribution functions are used to extrapolate beyond the limits of “observed” probabilities and to improve the accu-

racy of low-flow estimation. The previously analyzed estimators of extremes of low-flow were between the limits of empirical distribution (e.g. $Nq_{(10\%)}$ - 1 per 10 years in the case of 20-year series). The low probability, for short sequences, can be obtained through the extrapolation of the theoretical function. This means that, for probabilities of 1, 2 and 5% (once per 100, 50 or 20 years), we are not able to obtain quantiles on the basis of the empirical distribution. These were counted by using the distribution functions of the probability, which had been fitted previously.

Frequencies of specific low runoffs of the assumed probability not exceeding 1% were counted. More than 70% of all counted values (85 basins) are below $1 \text{ dm}^3\text{s}^{-1}\text{km}^{-2}$, and only 7% are runoffs of more than $2 \text{ dm}^3\text{s}^{-1}\text{km}^{-2}$. Among the counted runoffs of probability 1%, there are also values below zero. Theoretically, there is a 1% probability (8 basins) that these rivers can periodically be dry. However,

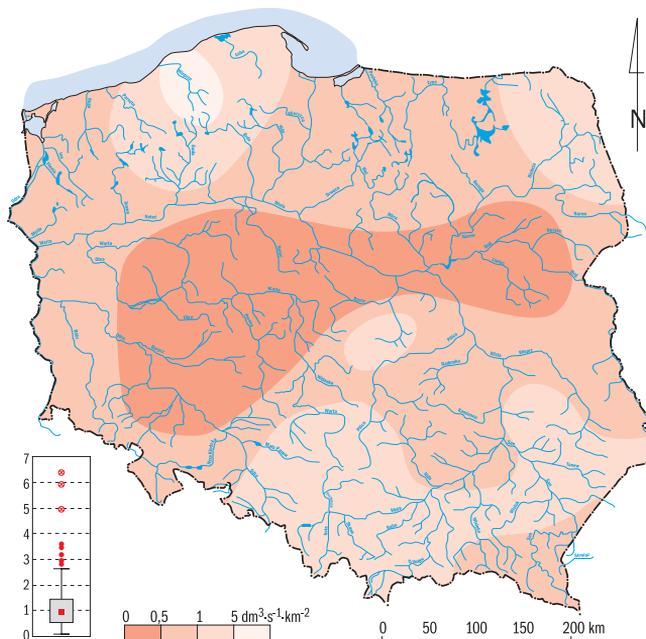


Figure 5. Spatial arrangement and diversity of minimal 10% specific low flow estimator – $Nq(10\%)$ [$\text{dm}^3\text{s}^{-1}\text{km}^{-2}$].

it should be remembered that this finding can only be the result of the mathematical smoothing points. Moreover, the lowest flows usually have the largest errors, because of rating curve inaccuracy. Nevertheless, it is visible that values below or close to zero are grouped in the majority of cases in the Wielkopolska region and in Mazowsze. These are areas poorest in water (e.g. the Rivers Ochnia, Liwiec, Gašawka and Noteć).

The highest specific low runoffs, counted for low-flow of probability 1% $Nq_{(1\%)}$ are locally greater than $4 \text{ dm}^3\text{s}^{-1}\text{km}^{-2}$, and they generally occur in basins abundant in water situated in the middle of the Pomeranian Lake District. These are areas of end moraine of particularly great thickness, as well as waterlogging (along the Studnica, Wieprza and Brda, for example) - Fig. 6.

A similar map of minimum specific low runoffs at a 1% exceedance probability can

also be found in the *Atlas Hydrologiczny Polski* [Hydrological Atlas of Poland] (1986). Despite the fact that this map relates to another time period (1951-1975), it is practically identical to the map presented here. The smallest runoffs (below $0.05 \text{ dm}^3\text{s}^{-1}\text{km}^{-2}$) occur in Wielkopolska and in the eastern part of the Mazowsze Lowland as well. The largest (as in this paper) – are in the Pomeranian Lake District, in the Tatra and Sudety Mountains, and in the centre of the Silesian Upland. This shows that the distribution of low runoffs does not depend on climatic conditions, but reflects overlapping hydrogeological and climatic conditions, and outflow due to anthropogenic activity.

The maps presented here can also be used as a background to more detailed inter-regional analysis. Observed regularities can also be helpful in various designs and tasks relating to opinions on water resources for the national economy.

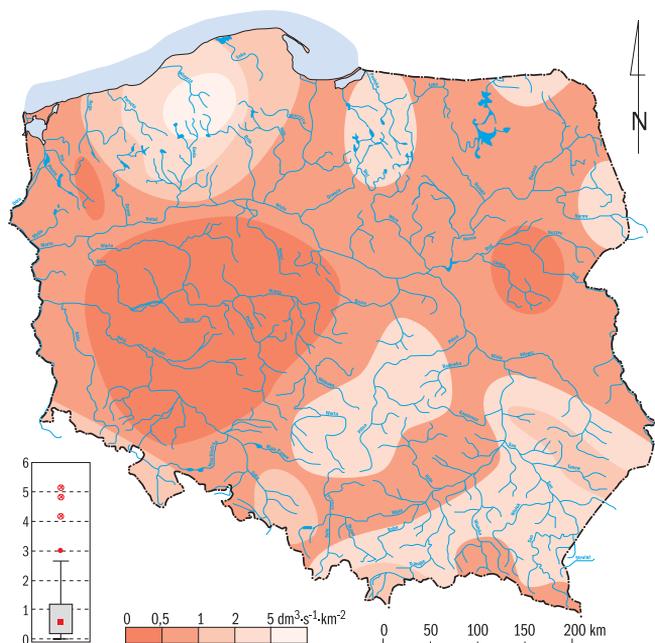


Figure 6. Spatial arrangement and diversity of specific low flows not exceeded at 1% probability – $Nq(1\%)$ [$\text{dm}^3\text{s}^{-1}\text{km}^{-2}$].

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