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KINEMATICS AND STATICS OF SMALL SUPERPOSED DEFORMATIONS

8/1968

WARSZAWA

N a p r a w a c h r ę k o p i s u

D o u ż y t k u w e w n ę t r z n e g o

Zakład Mechaniki Ośrodków Ciągłych IPPT PAN

Nakład 200 egz. Ark. wyd. 1.1. Ark. druk. 1.5
Oddano do drukarni w maju 1968 r.
Wydrukowano w czerwcu 1968 r. Nr zam. 408/0/68

Warszawska Drukarnia Naukowa , Warszawa ,
ul.Śniadeckich 8

Kinematics and statics of small superposed deformations

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1. Introduction

The problems concerning an infinitesimal displacement field superimposed on finite deformations of continuous media are of great importance for investigation of small vibrations and phenomena of stability loss. The theory of small deformations superposed on large fundamental deformations of non-linear elastic materials was initiated by Green, Rivlin and Shield in 1952, [1] /cf. also [2]/. This theory, formulated in convected coordinates description, constituted the basis for further supplements and applications. In 1961 Pipkin and Rivlin [3], applying the description of deformations in a fixed Cartesian reference frame, developed the theory of small additional deformations in materials with fading memory. Similar problems of small additional motions superposed on a fundamental motion of viscoelastic materials were discussed by the present author [4,5].

The main aim of the paper is to present a review of the most important kinematic and static relations obtained so far in the field and to compare the results taking into consideration two kinds of description of motion, i.e. in convected /moving/ coordinates as well as in fixed spatial coordinates. For this purpose some formally new relations have been added and some transformations from one system of coordinates into another have been presented.

We also wish to emphasize that, from general point of view, two methods of description presented are entirely equivalent, however, in some particular problems either spatial coordinates or convected coordinates are more suitable in leading up to simpler final results.

2. Equations of motion

Consider a body in its undeformed state $\mathring{B}$ at some instant of time $\tau = 0$. Let the coordinates of a generic particle be X^A in an arbitrary curvilinear system of coordinates (X^A) with covariant metric tensor $\mathring{G}_{AB}$ and contravariant metric tensor $\mathring{G}^{AB}$ respectively. This is what is called the material system of coordinates. Denoting by (x^i) an arbitrary fixed spatial system of coordinates with metric tensors g_{ij} and g^{ij} respectively, we can specify the deformation of a body giving x^i as a function of X^A and time τ, so that

$$/2.1/ \qquad x^i = x^i(X^A, \tau) \ , \quad X^A = X^A(x^i, \tau) \ ,$$

providing that regularity conditions ensuring the existence of the inverse second relation are satisfied. The above equations of motion can be also written in the equivalent vector form

$$/2.2/ \qquad \underset{\sim}{r} = \underset{\sim}{r}(X^A, \tau) \ , \quad \underset{\sim}{r}_{,A} = \underset{\sim}{r}_{,A}(X^B, \tau) = \underset{\sim}{G}_A(X^B, \tau) \ ,$$

where $\underset{\sim}{r}$ is the radius vector of a material particle corresponding to the deformed state B at some current instant τ, and A following the comma denotes partial differentiation with respect to X^A.

If we consider that the system (X^A) moves and deforms together with a body, we have the system of convected coordinates, the metric tensors of which /at instant τ / are connected with corresponding basis vectors with the following well known relations:

$$/2.3/ \quad G_{AB}(\tau) = \underset{\sim}{G}_A \cdot \underset{\sim}{G}_B \ , \quad G^{AB}(\tau) = \underset{\sim}{G}^A \cdot \underset{\sim}{G}^B \ , \quad G = \det G_{AB} \ .$$

The deformation gradients

$$/2.4/ \qquad \frac{\partial x^i}{\partial X^A} = x^i_{,A} \quad , \quad \frac{\partial X^A}{\partial x^i} = X^A_{,i} \quad ,$$

being the measures of a local deformation in the neighbourhood of a particle X^A , become the corresponding coefficients of tensor transformation from (X^A) to (x^i) if we use the notion of convected coordinates.

We obtain the following expressions for Christoffel symbols of the second kind in a system of convected coordinates:

$$/2.5/ \quad \left\{ {A \atop B\,C} \right\} \stackrel{\mathrm{df}}{=} \frac{\partial X^A}{\partial \xi^\mu} \frac{\partial}{\partial X^B} \left(\frac{\partial \xi^\mu}{\partial X^C} \right) =$$

$$= \frac{1}{2} G^{AM} \left(G_{MB,C} + G_{MC,B} - G_{BC,M} \right) ,$$

$$/2.6/ \quad -\left\{ {A \atop B\,C} \right\} = \frac{\partial \xi^\mu}{\partial X^B} \frac{\partial}{\partial X^C} \left(\frac{\partial X^A}{\partial \xi^\mu} \right) ,$$

and the following expressions in an arbitrary system of spatial coordinates:

$$/2.7/ \quad \left\{ {i \atop j\,k} \right\} = \frac{\partial x^i}{\partial \xi^\mu} \frac{\partial}{\partial x^j} \left(\frac{\partial \xi^\mu}{\partial x^k} \right) = \frac{1}{2} g^{im} \left(g_{mj,k} + g_{mk,j} - g_{jk,m} \right) ,$$

$$/2.8/ \quad -\left\{ {i \atop j\,k} \right\} = \frac{\partial \xi^\mu}{\partial x^j} \frac{\partial}{\partial x^k} \left(\frac{\partial x^i}{\partial \xi^\mu} \right) .$$

By (ξ^μ) we have denoted an auxiliary system of fixed rectangular Cartesian coordinates. This is evident that the transformation coefficients $\partial \xi^\alpha / \partial x^j$ and $\partial x^j / \partial \xi^\alpha$ do not depend on motion of a particle X^A ; thus for simplicity and without loss of generality we shall assume hereafter that the spatial system (x^i) coincides with Cartesian system (ξ^α) .

Taking into account the relations /2.1/, all scalar, vector and tensor quantities describing any process of deformation can be regarded either as functions of the variables x^i, τ , or as functions of the variables X^A, τ .

3. Small additional motion superposed on a fundamental motion

Let us suppose that at instant $\tau = t$ a material particle following some disturbances takes a new position $\overset{*}{\underset{\sim}{r}}(X^A, t)$ in the neighbourhood of $\underset{\sim}{r}(X^A, t)$ determined by the small displacement vector $\eta \underset{\sim}{w}(X^A, t)$, where η is a small dimensionless parameter such that its squares and higher degree powers can be neglected in comparison with those of first degree. Thus, for the state $\overset{*}{B}$ we have

$$/3.1/ \qquad \overset{*}{\underset{\sim}{r}}(X^A, t) = \underset{\sim}{r}(X^A, t) + \eta \underset{\sim}{w}(X^A, t),$$

or

$$/3.2/ \qquad \overset{*}{x}{}^i(X^A, t) = x^i(X^A, t) + \eta w^i(X^A, t),$$

where by w^i we have denoted contravariant components of the vector $\underset{\sim}{w}$ in the basis g_i of the fixed spatial system of coordinates (x^i) . According to /2.2/, /2.3/, we can write

$$/3.3/ \quad \overset{*}{\underset{\sim}{r}}_{,A}(X^B, t) = \underset{\sim}{r}_{,A}(X^B, t) + \eta \underset{\sim}{w}_{,A}(X^B, t) = \underset{\sim}{G}_A + \eta \underset{\sim}{G}'_A \;,$$

$$/3.4/ \quad \underset{\sim}{G}'_A = \underset{\sim}{G}_M \nabla_A w^M = \underset{\sim}{G}^M \nabla_A w_M \;, \quad \underset{\sim}{G}'^A = -\underset{\sim}{G}^N \nabla_N w^A = -\underset{\sim}{G}_N \nabla^N w^A,$$

where primes denote the increments of basis vectors, ∇ is the symbol of covariant differentiation, w_A and w^A denote components of the vector $\underset{\sim}{w}$ in the basis $\underset{\sim}{G}^A$ and $\underset{\sim}{G}_A$ respectively. The vector $\underset{\sim}{w}$ can be also represented in the new basis $\overset{*}{\underset{\sim}{G}}{}^{A}$ or $\overset{*}{\underset{\sim}{G}}{}_{A}$, however, the differences between corresponding components are immaterial being of order $O(\eta^2)$.

The gradients /2.4/ in the disturbed state $\overset{*}{B}$ are as follows:

$$/3.5/ \quad \overset{*}{x}{}^{i}{}_{,A} = x^{i}{}_{,A} + \eta w^{i}{}_{,A} \ , \quad \overset{*}{X}{}^{A}{}_{,i} = X^{A}{}_{,i} + \eta X'^{A}{}_{,i} \ ,$$

and bearing in mind that

$$/3.6/ \quad X^{A}{}_{,r} x^{r}{}_{,B} = \overset{*}{X}{}^{A}{}_{,r} \overset{*}{x}{}^{r}{}_{,B} = \delta^{A}_{B} \ , \quad X^{M}{}_{,j} x^{i}{}_{,M} = \overset{*}{X}{}^{M}{}_{,j} \overset{*}{x}{}^{i}{}_{,M} = \delta^{i}_{j} \ ,$$

we obtain

$$/3.7/ \quad X'^{A}{}_{,i} = - w^{r}{}_{,M} X^{A}{}_{,r} X^{M}{}_{,i} = - w^{r}{}_{,i} X^{A}{}_{,r} \ .$$

The above results can be applied to transform tensor quantities in the state $\overset{*}{B}$ from the system of convected coordinates (X^A) to the system of spatial coordinates (x^i) and vice versa. The linearized increments of all tensors, vectors, or scalars can be expressed in components of the vector $\underset{\sim}{w}$ either in convected or in spatial reference frame.

According to /2.3/, /2.5/, /2.6/, we can write the increments of metric tensors of the system (X^A) in the state $\overset{*}{B}$ in the following form /cf. [1,2] /:

$$/3.8/ \quad \overset{*}{G}_{AB} = \overset{*}{\underset{\sim}{G}}{}_{A} \cdot \overset{*}{\underset{\sim}{G}}{}_{B} = G_{AB} + \eta G'_{AB} \ , \quad \overset{*}{G}{}^{AB} = \overset{*}{\underset{\sim}{G}}{}^{A} \cdot \overset{*}{\underset{\sim}{G}}{}^{B} = G^{AB} + \eta G'^{AB} \ ,$$

$$/3.9/ \quad G'_{AB} = \nabla_A w_B + \nabla_B w_A = 2\nabla_{(A} w_{B)} \ , \quad G'^{AB} = -(\nabla^A w^B + \nabla^B w^A) = -2\nabla^{(A} w^{B)} \ ,$$

$$/3.10/ \quad G' = (\det G_{AB})' = 2G \nabla_M w^M .$$

Similarly, for the increments of Christoffel symbols of the second kind, we obtain

$$/3.11/ \quad \overset{*}{\left\{ {A \atop B\,C} \right\}} = \left\{ {A \atop B\,C} \right\} + \eta \left\{ {A \atop B\,C} \right\}' = \overset{*}{X}{}^{A}_{,\mu} \, \overset{*}{\xi}{}^{\mu}_{,BC} = - \overset{*}{\xi}{}^{\mu}_{,B} \, \overset{*}{X}{}^{A}_{,\mu C} \, ,$$

$$/3.12/ \quad \left\{ {A \atop B\,C} \right\}' = \frac{1}{2}\Big[G^{AM}\left(G'_{MB,C} + G'_{MC,B} - G'_{BC,M} \right) +$$

$$+ \, G'^{AM}\left(G_{MB,C} + G_{MC,B} - G_{BC,M} \right)\Big] = \nabla_B \nabla_C w^A \, ,$$

where Greek indices refer to an auxiliary system of fixed rectangular Cartesian coordinates.

The increments of basis vectors and metric tensors of any system of curvilinear spatial coordinates corresponding to the disturbed position of a material particle are as follows:

$$/3.13/ \quad \overset{*}{\underset{\sim}{g}}_i \equiv \underset{\sim}{g}_i(\overset{*}{x}{}^k, t) = \underset{\sim}{g}_i(x^k, t) + \eta \, \underset{\sim}{g}_{i,r}(x^k, t) \, w^r(x^k, t) =$$

$$= \underset{\sim}{g}_i + \eta \, \underset{\sim}{g}'_i \, ,$$

$$/3.14/ \quad \underset{\sim}{g}'_i = \underset{\sim}{g}_{i,r} w^r = \underset{\sim}{g}_s \left\{ {s \atop i\,r} \right\} w^r , \quad \underset{\sim}{g}'^i = \underset{\sim}{g}{}^i_{,r} w^r = - \underset{\sim}{g}{}^s \left\{ {i \atop s\,r} \right\} w^r ,$$

$$/3.15/ \quad g'_{ij} = g_{ij,r} w^r = \left(\left\{ {s \atop i\,r} \right\} g_{sj} + \left\{ {s \atop r\,j} \right\} g_{is} \right) w^r ,$$

$$/3.16/ \quad g'^{ij} = g^{ij}_{,r} w^r = - \left(\left\{ {i \atop s\,r} \right\} g^{sj} + \left\{ {j \atop r\,s} \right\} g^{is} \right) w^r .$$

Taking into account the definition /2.7/, we also obtain

$$/3.17/ \quad \left\{ {i \atop j\,k} \right\}' = \left\{ {i \atop j\,k} \right\}_{,r} w^r = \frac{1}{2}\Big[g^{im}_{,r} \left(g_{mj,k} + g_{mk,j} - g_{jk,m} \right) w^r +$$

$$+g^{im}(g_{mj,k}+g_{mk,j}-g_{jk,m})_{,\tau}w^{\tau}\Big] .$$

It can be shown that the relations /3.9/ and /3.15/, /3.16/ result directly from tensor transformations. To this end let us observe that the following auxiliary relations hold:

/3.18/ $$X^{A}_{,\tau}x^{\tau}_{,BC}=X^{A}_{,\mu}\xi^{\mu}_{,\tau}\left(x^{\tau}_{,\nu}\xi^{\nu}_{,B}\right)_{,C}=$$

$$=X^{A}_{,\mu}\xi^{\mu}_{,\tau}\left(x^{\tau}_{,\nu}\xi^{\nu}_{,BC}+\xi^{\nu}_{,B}x^{\tau}_{,\nu C}\right)=\left\{{A \atop BC}\right\}+X^{A}_{,\tau}\xi^{\mu}_{,B}x^{\tau}_{,\mu C},$$

/3.19/ $$a^{M}b_{N}x^{\tau}_{,MC}X^{N}_{,\tau}=a^{M}b_{N}\left\{{N \atop MC}\right\}+a^{B}b_{\tau}\xi^{\mu}_{,C}x^{\tau}_{,\mu B}=$$

$$=a^{M}b_{N}\left\{{N \atop MC}\right\}+a^{m}b_{\tau}\xi^{\mu}_{,m}x^{\tau}_{,\mu n}x^{n}_{,C}=$$

$$=a^{M}b_{N}\left\{{N \atop MC}\right\}-a^{m}b_{\tau}\left\{{\tau \atop mn}\right\}x^{n}_{,C} ,$$

where a^{M} and b_{N} denote the components of two arbitrary vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Transforming, for example $\overset{*}{G}_{AB}$ to the system of spatial coordinates (x^{i}) , and taking into account /3.15/, /3.16/, /3.18/, /3.19/, we obtain

/3.20/ $$\overset{*}{g}_{ij}=g_{ij}+\eta g'_{ij}=\overset{*}{G}_{AB}\overset{*}{X}{}^{A}_{,i}\overset{*}{X}{}^{B}_{,j}=$$

$$=\left(G_{AB}+\eta G'_{AB}\right)\left(X^{A}_{,i}+\eta X'^{A}_{,i}\right)\left(X^{B}_{,j}+\eta X'^{B}_{,j}\right),$$

/3.21/ $$g'_{ij}=G'_{AB}X^{A}_{,i}X^{B}_{,j}+G_{AB}X^{A}_{,i}X'^{B}_{,j}+G_{AB}X'^{A}_{,i}X^{B}_{,j}=$$

$$= (\nabla_A w_B + \nabla_B w_A) X^A_{,i} X^B_{,j} - G_{AB} X^A_{,i} (w^N x^r_{,N})_{,M} X^M_{,j} X^B_{,r} - G_{AB} X^B_{,j} (w^N x^r_{,N})_{,M} X^M_{,i} X^A_{,r} =$$

$$= (\nabla_A w_B + \nabla_B w_A) X^A_{,i} X^B_{,j} - G_{AB} X^A_{,i} X^M_{,j} \left(w^B_{,M} + \left\{ {B \atop NM} \right\} w^N \right) -$$

$$- G_{AB} X^B_{,j} X^M_{,i} \left(w^A_{,M} + \left\{ {A \atop NM} \right\} w^N \right) - G_{AB} X^A_{,i} X^M_{,j} X^B_{,r} \xi^\mu_{,N} x^r_{,\mu M} w^N$$

$$- G_{AB} X^B_{,j} X^M_{,i} X^A_{,r} \xi^\mu_{,N} x^r_{,\mu M} w^N =$$

$$= - g_{ir} w^p X^N_{,p} \xi^\mu_{,N} x^r_{,\mu M} X^M_{,j} - g_{jr} w^p X^N_{,p} \xi^\mu_{,N} x^r_{,\mu M} X^M_{i} =$$

$$= - g_{ir} w^p \xi^\mu_{,p} x^r_{,\mu j} - g_{jr} w^p \xi^\mu_{,p} x^r_{,\mu i} =$$

$$= \left(\left\{ {r \atop pj} \right\} g_{ir} + \left\{ {r \atop pi} \right\} g_{jr} \right) w^p \quad .$$

Similarly, transforming $\overset{*}{g}_{ij}$ in the state $\overset{*}{B}$ to the system (X^A) , it is possible to carry out the relations equivalent to /3.9/$_1$.

4. Material time derivetive

The material time derivative of an absolute tensor connected with a material particle is defined as the time rate of that tensor watched by "an observer" moving together with a material particle /cf. e.g. [6,7] /.

Let us consider two positions of a material particle corresponding to the state B at instant t , and to the state $\hat{B}$ immediately following B at instant $t+\Delta t$. In other words these positions of a particle differ only with an infinitesimal process of deformation. The material time derivative of an absolute tensor $\underset{\sim}{\phi}$ can be defined as follows:

$$/4.1/ \qquad \dot{\underset{\sim}{\phi}} \overset{df}{=} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\hat{\underset{\sim}{\phi}}(X^A, t+\Delta t) - \underset{\sim}{\phi}(X^A, t) \right]$$

The representations of the tensor $\dot{\underset{\sim}{\phi}}(x^i, t)$ in the fixed system of spatial coordinates (x^i) can be determined according to the definition /4.1/ /cf. [6,7] /

$$/4.2/ \qquad \dot{\phi}^{i\cdot\cdot}_{\cdot\cdot j} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\hat{\phi}^{i\cdot\cdot}_{\cdot\cdot j}(X^A, t+\Delta t) - \phi^{i\cdot\cdot}_{\cdot\cdot j}(X^A, t)\right] =$$

$$= \frac{D}{Dt}\phi^{i\cdot\cdot}_{\cdot\cdot j} = \frac{\partial}{\partial t}\phi^{i\cdot\cdot}_{\cdot\cdot j} + v^r \nabla_r \phi^{i\cdot\cdot}_{\cdot\cdot j} \ ,$$

where $\partial/\partial t$ denotes partial time differentiation holding x^i = const, and v^r are contravariant spatial component of the velocity of a particle.

To obtain representations of the tensor $\dot{\phi}(X^A, t)$ in the system of convected coordinates (X^A), we can also use the definition /4.1/, however, we must remember that components of various tensors referred to different moving frames cannot be directly compared; first they should be transformed to any fixed common frame, e.g. to the Cartesian system (ξ^α) . As the next step we can pass to the limit and again return to the system (X^A) .

Let us observe that the quantities

$$/4.3/ \qquad \hat{x}^i_{,A}(X^B, t+\Delta t) = x^i_{,A}(X^B, t) + \Delta t\, v^i_{,A}(X^B, t) \ ,$$

$$/4.4/ \qquad \hat{X}^A_{,i}(X^B, t+\Delta t) = X^A_{,i}(X^A, t) + \Delta t\, X^A_{,ik} v^k(X^B, t) \ ,$$

tend to $x^i_{,A}$ and $X^A_{,i}$ respectively, when $\Delta t \to 0$, and that

$$/4.5/ \qquad \frac{D}{Dt}(x^i_{,A}) = \left(\frac{Dx^i}{Dt}\right)_{,A} = v^i_{,A} \ , \quad \frac{D}{Dt}(\xi^\alpha_{,A}) = \left(\frac{D\xi^\alpha}{Dt}\right)_{,A} = v^\alpha_{,A} \ ,$$

$$/4.6/ \qquad \frac{D}{Dt}(X^A_{,i}) = -X^A_{,r} X^M_{,i} \frac{D}{Dt}(x^r_{,M}) = -X^A_{,r} X^M_{,i} v$$

where D/Dt denotes partial time differentiation holding X^A = const.

Taking into account /4.1/ and our previous remarks, we can write

$$\dot{\phi}^{A\cdot\cdot}_{\cdot\cdot B} = X^{A}_{,\alpha}\cdots\xi^{\beta}_{,B}\dot{\phi}^{\alpha\cdot\cdot}_{\cdot\cdot\beta} =$$

$$= X^{A}_{,\alpha}\cdots\xi^{\beta}_{,B}\lim_{\Delta t\to 0}\left\{\frac{1}{\Delta t}\left[\xi^{\alpha}_{,C}\cdots\hat{X}^{D}_{,\beta}\hat{\phi}^{C\cdot\cdot}_{\cdot\cdot D} - \xi^{\alpha}_{,C}\cdots X^{D}_{,\beta}\phi^{C\cdot\cdot}_{\cdot\cdot D}\right]\right\}$$

$$= X^{A}_{,\alpha}\cdots\xi^{\beta}_{,B}\frac{D}{Dt}\left(\xi^{\alpha}_{,C}\cdots X^{D}_{,\beta}\phi^{C\cdot\cdot}_{\cdot\cdot D}\right) =$$

/4.7/
$$= X^{A}_{,\alpha}\cdots\xi^{\beta}_{,B}\left[\xi^{\alpha}_{,C}\cdots X^{D}_{,\beta}\frac{D}{Dt}\phi^{C\cdot\cdot}_{\cdot\cdot D} +\right.$$

$$\left.+ v^{\alpha}_{,C}\cdots X^{D}_{,\beta}\phi^{C\cdot\cdot}_{\cdot\cdot D} + \cdots - \xi^{\alpha}_{,C}\cdots v^{\mu}_{,M}X^{D}_{,\mu}X^{M}_{,\beta}\phi^{C\cdot\cdot}_{\cdot\cdot D}\right]$$

$$= \frac{D}{Dt}\phi^{A\cdot\cdot}_{\cdot\cdot B} + \left(v^{A}_{,M} + \left\{{A \atop MC}\right\}v^{C}\right)\phi^{M\cdot\cdot}_{\cdot\cdot B} + \cdots - \left(v^{M}_{,B} + \left\{{M \atop DB}\right\}v^{D}\right)\phi^{A\cdot\cdot}_{\cdot\cdot M}.$$

Thus, we obtain finally /cf. e.g. [7] /

/4.8/
$$\dot{\phi}^{A\cdot\cdot}_{\cdot\cdot B} = \frac{D}{Dt}\phi^{A\cdot\cdot}_{\cdot\cdot B} + \sum{}' \phi^{M\cdot\cdot}_{\cdot\cdot B}\nabla_{M}v^{A} - \sum \phi^{A\cdot\cdot}_{\cdot\cdot M}\nabla_{B}v^{M},$$

where $\sum'$ and $\sum$ denote the sums in all upper contravariant and all lower covariant indices respectively.

We wish to emphasize that the symbols $(\dot{\ })$ and D/Dt have in general different meanings, however, for vectors or scalars considered as functions of a particle and time they are formally equivalent. For example, we have

/4.9/
$$\underset{\sim}{v}(X^{A},t) = \frac{D}{Dt}\underset{\sim}{r}(X^{A},t) = \dot{\underset{\sim}{r}}(X^{A},t),$$

$$v^{i}(X^{A},t) = \frac{D}{Dt}x^{i}(X^{A},t) = \dot{x}^{i}(X^{A},t).$$

5. Velocity and acceleration

Taking into account /3.1/, /3.2/ and /4.9/, we can express the velocity of a particle in the disturbed state $\overset{*}{B}$ in the following vector form:

$$\overset{*}{\underset{\sim}{v}}(X^A,t) = \underset{\sim}{v}(X^A,t) + \eta\dot{\underset{\sim}{w}}(X^A,t) = \underset{\sim}{v} + \eta\underset{\sim}{v}' \;. \tag{5.1}$$

Multiplying both sides of the above relation by the vectors $\overset{*}{\underset{\sim}{g}}_i$ and $\overset{*}{\underset{\sim}{g}}{}^i$ determined in the disturbed position of a material particle, and taking into account /3.13/, /3.14/, we obtain

$$v'^i = \dot{w}^i - v^s\left\{{i \atop s\,r}\right\}w^r = \frac{\partial}{\partial t}w^i + v^s\nabla_s w^i - v^s\left\{{i \atop s\,r}\right\}w^r, \tag{5.2}$$

$$v'_i = \dot{w}_i + v_s\left\{{s \atop i\,r}\right\}w^r = \frac{\partial}{\partial t}w_i + v^s\nabla_s w_i + v_s\left\{{s \atop i\,r}\right\}w^r \tag{5.3}$$

The components of the velocity vector referred to convected coordinates in the state $\overset{*}{B}$ can be obtained multiplying both sides of /5.1/ by the vectors $\overset{*}{\underset{\sim}{G}}_A$ or $\overset{*}{\underset{\sim}{G}}{}^A$. This leads to

$$\overset{*}{v}_A(X^B,t) = v_A(X^B,t) + \eta v'_A(X^B,t) = v_A + \eta(\dot{w}_A + v_N\nabla_A w^N), \tag{5.4}$$

$$\overset{*}{v}{}^A(X^B,t) = v^A(X^B,t) + \eta v'^A(X^B,t) = v^A + \eta(\dot{w}^A - v^N\nabla_N w^A), \tag{5.5}$$

where we have used /3.4/ and the following additional relations:

$$\frac{D}{Dt}(\underset{\sim}{G}_A\cdot\underset{\sim}{G}^B) = 0\,, \quad \frac{D}{Dt}\underset{\sim}{G}_A = \underset{\sim}{G}_M\nabla_A v^M, \quad \frac{D}{Dt}\underset{\sim}{G}^A = -G^M\nabla_M v^A. \tag{5.6}$$

According to /4.8/ the increments of velocity components can be written in the following form:

$$v'_A = \frac{D}{Dt} w_A - w_M \nabla_A v^M + v_N \nabla_A w^N \,, \tag{5.7}$$

$$v'^A = \frac{D}{Dt} w^A + w^M \nabla_M v^A - v^N \nabla_N w^A \,, \tag{5.8}$$

where $D/Dt = (\partial/\partial t)\ X^A = \text{const}$. It can be seen easily that two last terms on the right hand side of /5.7/, /5.8/ are exclusively of convected character and they vanish instantly when the fundamental motion does not exist.

In a similar way, we obtain for the acceleration vector

$$\overset{*}{\dot{\underset{\sim}{v}}}(X^A,t) = \dot{\underset{\sim}{v}}(X^A,t) + \eta \ddot{\underset{\sim}{w}}(X^A,t) \,, \tag{5.9}$$

$$\overset{*}{\dot{v}}{}^i = \dot{v}^i + \eta v'^i = \dot{v}^i + \eta\left(\ddot{w}^i - \dot{v}^s \left\{ {i \atop s\ r} \right\} w^r\right), \tag{5.10}$$

$$\overset{*}{\dot{v}}_i = \dot{v}_i + \eta v'_i = \dot{v}_i + \eta\left(\ddot{w}_i + \dot{v}_s \left\{ {s \atop i\ r} \right\} w^r\right), \tag{5.11}$$

and

$$\overset{*}{\dot{v}}_A = \dot{v}_A + \eta \dot{v}'_A = \dot{v}_A + \eta\left(\ddot{w}_A + \dot{v}_M \nabla_A w^M\right), \tag{5.12}$$

$$\overset{*}{\dot{v}}{}^A = \dot{v}^A + \eta \dot{v}'^A = \dot{v}^A + \eta\left(\ddot{w}^A - \dot{v}^N \nabla_N w^A\right). \tag{5.13}$$

The corresponding increments in convected coordinates can be written in an explicit form as follows:

$$\dot{v}'_A = \frac{D^2}{Dt^2} w_A - 2\nabla_A v^M \frac{D}{Dt} w_M - \frac{D}{Dt}(\nabla_A v^M) w_M + w_M \nabla_N v^M \nabla_A v^N + \nabla_A w^N \left(\frac{D}{Dt} v_N - v_M \nabla_N v^M\right), \tag{5.14}$$

$$/5.15/ \quad \overset{\cdot}{v}'^A = \frac{D^2}{Dt^2}w^A + 2\nabla_M v^A \frac{D}{Dt}w^M + \frac{D}{Dt}(\nabla_M v^A)w^M + w^M\nabla_M v^N \nabla_N v^A -$$
$$- \nabla_N w^A \left(\frac{D}{Dt}v^N + v^M \nabla_M v^N \right) .$$

Successive accelerations of higher orders can be dealt with in a similar manner. We have for example

$$/5.16/ \quad \overset{(n)}{v}{}'_A = \overset{(n+1)}{w}_A + \overset{(n)}{v}_N \nabla_A w^N \quad , \quad \overset{(n)}{v}{}'^A = \overset{(n+1)}{w}{}^A - \overset{(n)}{v}{}^N \nabla_N w^A ,$$

where (n) denotes the material differentiation performed n times, according to the definition of the n-th acceleration.

It can be also shown that the increments /5.7/, /5.8/ or /5.14/, /5.15/ can be derived from /5.3/ or /5.11/ by direct tensor transformations. Let us consider for example

$$/5.17/ \quad \overset{*}{\dot{v}}{}^A = \overset{*}{\dot{v}}{}^i \overset{*}{X}{}^A_{,i} = \dot{v}^A + \eta \left(X^A_{,i}\dot{v}'^i + X'^A_{,i}\dot{v}^i \right) ,$$

and next taking into account /5.3/ and /3.7/, we obtain

$$\dot{v}'^A = X^A_{,i}\left(\dot{w}^i - v^s \left\{ {i \atop s r} \right\} w^r \right) - w^r_{,M} X^A_{,r} X^M_{,i} \dot{v}^i =$$
$$= \dot{w}^A - X^A_{,i} v^s \left\{ {i \atop s r} \right\} w^r - v^M \left(w^N x^r_{,N} \right)_{,M} X^A_{,r} =$$
$$/5.18/ \quad = \dot{w}^A - v^M w^A_{,M} - v^M w^N X^A_{,r} x^r_{,MN} - X^A_{,i} v^s \left\{ {i \atop s r} \right\} w^r =$$
$$= \dot{w}^A - v^M \left(w^A_{,M} + \left\{ {A \atop MN} \right\} w^N \right) - v^m w^n \xi^\mu_{,m} x^r_{,\mu n} X^A_{,r} -$$
$$- v^s w^r \left\{ {i \atop s r} \right\} X^A_{,i} = \dot{w}^A - v^M \nabla_M w^A .$$

Also

$$/5.19/ \quad \overset{*}{\dot{v}}_A = \overset{*}{x}{}^i_{,A} \overset{*}{\dot{v}}_i = \dot{v}_A + \eta \left(x^i_{,A} \dot{v}'_i + w^i_{,A} \dot{v}_i \right) ,$$

and

$$\dot{v}'_A = x^i_{,A}(\ddot{w}_i + \dot{v}_s\{^s_{ir}\}w^r) + (w^N x^i_{,N})_{,A}\dot{v}_M X^M_{,i} =$$

$$= \ddot{w}_A + \dot{v}_M w^M_{,A} + w^N \dot{v}_M x^i_{,NA} X^M_{,i} + x^i_{,A}\dot{v}_s\{^s_{ir}\}w^r =$$

/5.20/
$$= \ddot{w}_A + \dot{v}_M(w^M_{,A} + \{^M_{AN}\}w^N) + w^m \dot{v}_r \xi^\mu_{,m} x^r_{,\mu s} x^s_{,A} + \\ + w^r \dot{v}_s\{^s_{ir}\}x^i_{,A} =$$

$$= \ddot{w}_A + \dot{v}_M \nabla_A w^M - w^m \dot{v}_r\{^r_{ms}\}x^s_{,A} + w^r \dot{v}_s\{^s_{ri}\}x^i_{,A} = \\ = \ddot{w}_A + \dot{v}_M \nabla_A w^M$$

6. Strain tensors and Rivlin-Ericksen kinematic tensors

The Cauchy-Green strain tensors follow from the definitions /cf. [3,6] /

/6.1/
$$c_{ij} = G_{AB} X^A_{,i} X^B_{,j} \quad , \quad C_{AB} = g_{ij} x^i_{,A} x^j_{,B} \; .$$

When the system of material coordinates (X^A) moves and deforms together with a body /convected coordinates/ the second definition /6.1/ is also valid in that system. The tensor c_{ij} and C_{AB} evaluated at the same current instant of time t are numerically equivalent to the metric tensors g_{ij} and $G_{AB}(t)$ respectively. In particular, if in the initial undeformed state $\overset{\circ}{B}$ the system (X^A) coincides with (x^i) then

/6.2/
$$C_{AB}(0) = \overset{\circ}{G}_{AB}(0) = g_{ij} \qquad \text{at} \quad \tau = 0 \; .$$

Similarly, if the above systems of coordinates coincide in the deformed state B , we have

/6.3/
$$C_{AB}(t) = G_{AB}(t) = g_{ij} \qquad \text{at} \quad \tau = t \; .$$

The increments of the Cauchy-Green tensors are determined as differences between the disturbed state $\overset{*}{B}$ and the deformed state B . According to /6.1/ and /3.9/, /3.21/, we obtain

$$/6.4/ \qquad C'_{AB}(t) = G'_{AB}(t) = g_{ij}\left(x^i_{,A} w^j_{,B} + w^i_{,A} x^j_{,B}\right) + g'_{ij} x^i_{,A} x^j_{,B}$$

$$= \nabla_A w_B + \nabla_B w_A$$

In any rectangular Cartesian reference frame the above definition is equivalent to the definition of the classical strain tensor /cf. [3] /, thus

$$/6.5/ \qquad C'_{AB}(t) = G'_{AB} = 2\, e_{AB} \;, \quad e_{AB} = \frac{1}{2}\left(w_{A,B} + w_{B,A}\right)$$

It can be also shown, in a way similar to /3.15/, /3.16/, that in the present notation $C'_{ij}(t) \equiv 0$.

Frequently, other definitions of strain tensors are more advantageous in convected coordinates. Let us introduce the following strain tensor /cf. [2] /:

$$/6.6/ \qquad 2E_{AB}(t) \overset{df}{=} G_{AB}(t) - \overset{\circ}{G}_{AB}(0) = C_{AB}(t) - C_{AB}(0) \;,$$

whence by simple comparison with /6.4/ we have: $E'_{AB} = \frac{1}{2} G'_{AB} = \nabla_{(A} w_{B)}$.

The Rivlin-Ericksen kinematic tensors [8] , are of great importance in the theory of non-linear visco-elastic materials /cf. also [6,3] /. Their definition in the curvilinear fixed system of coordinates (x^i) can be presented as follows:

$$/6.7/ \qquad \overset{(0)}{A}_{ij.} = \nabla_i v_j + \nabla_j v_i = 2\, d_{ij} \;,$$

$$/6.8/ \quad A^{(\nu+1)}_{ij} = \dot{A}^{(\nu)}_{ij} + A^{(\nu)}_{ir}\nabla_j v^r + A^{(\nu)}_{rj}\nabla_i v^r \cdot \quad (\nu = 0,1,2,\cdots),$$

where d_{ij} denotes the strain rate tensor. Transforming the above expression to the system of convected coordinates, we obtain

$$/6.9/ \quad A^{(0)}_{AB} = A^{(0)}_{ij} x^i_{,A} x^j_{,B} = \nabla_A v_B + \nabla_B v_A = 2d_{AB},$$

$$/6.10/ \quad A^{(\nu+1)}_{AB} = A^{(\nu+1)}_{ij} x^i_{,A} x^j_{,B} = \dot{A}^{(\nu)}_{AB} + A^{(\nu)}_{AM}\nabla_B v^M + A^{(\nu)}_{MB}\nabla_A v^M .$$

On expanding $\dot{A}^{(\nu)}_{AB}$ according to the definition /4.8/, we have finally

$$/6.11/ \quad A^{(\nu+1)}_{AB} = \frac{D}{Dt} A^{(\nu)}_{AB} = \frac{D^{\nu+1}}{Dt^{\nu+1}} A^{(0)}_{AB} \quad (\nu = 0,1,2,\cdots).$$

In view of /6.7/ and /6.8/, the increments of the Rivlin-Ericksen tensors in any spatial system of coordinates take the form

$$/6.12/ \quad \overset{*}{A}{}^{(0)}_{ij} = A^{(0)}_{ij} + \eta A'^{(0)}_{ij} = \nabla_i v_j + \nabla_j v_i + \eta\left(\nabla_i v'_j + \nabla_j v'_i - 2v_s \left\{ {s \atop ij} \right\}'\right),$$

$$/6.13/ \quad A'^{(0)}_{ij} = 2\nabla_{(i} v'_{j)} - 2v_s \left\{ {s \atop ij} \right\}' = 2d'_{ij}$$

$$/6.14/ \quad A'^{(\nu+1)}_{ij} = \left(\dot{A}^{(\nu)}_{ij}\right)' + A'^{(\nu)}_{ir}\nabla_j v^r + A^{(\nu)}_{ir}\nabla_j v'^r + A'^{(\nu)}_{rj}\nabla_i v^r +$$

$$+ A^{(\nu)}_{rj}\nabla_i v'^r + A^{(\nu)}_{ir}\left\{ {r \atop js} \right\}' v^s + A^{(\nu)}_{rj}\left\{ {r \atop is} \right\}' v^s,$$

where on account of /4.2/

$$\left(\dot{A}^{(\nu)}_{ij}\right)' = \frac{\partial}{\partial t} A'^{(\nu)}_{ij} + v'^{r} \nabla_r A^{(\nu)}_{ij} + v^r \nabla_r A'^{(\nu)}_{ij} - \tag{6.15}$$
$$- v^r \left\{ {s \atop jr} \right\}' A^{(\nu)}_{is} - v^r \left\{ {s \atop ir} \right\} A'^{(\nu)}_{sj} \; .$$

To derive the corresponding increments referred to the system of convected coordinates (X^A) , let us take into consideration the following simple relations:

$$\underset{\sim}{V}_{,A} = \underset{\sim}{G}^M \nabla_A v_M = \underset{\sim}{G}_M \nabla_A v^M \;, \quad \underset{\sim}{W}_{,A} = \underset{\sim}{G}^M \nabla_A w_M = \underset{\sim}{G}_M \nabla_A w^M ,$$
$$\dot{\underset{\sim}{W}}_{,A} = \left(\frac{D}{Dt}\underset{\sim}{W}\right)_{,A} = \frac{D}{Dt}\underset{\sim}{W}_{,A} = \underset{\sim}{G}^M \frac{D}{Dt}(\nabla_A w_M) + \nabla_A w_M \frac{D}{Dt}\underset{\sim}{G}^M . \tag{6.16}$$

Hence, we have in the state $\overset{*}{B}$

$$\overset{*}{\nabla}_A \overset{*}{v}_B = \overset{*}{\underset{\sim}{V}}_{,A} \cdot \overset{*}{\underset{\sim}{G}}_B = \left(\underset{\sim}{V}_{,A} + \eta \overset{\circ}{\underset{\sim}{W}}_{,A}\right)\left(\underset{\sim}{G}_B + \eta \underset{\sim}{G}'_B\right) =$$
$$= \underset{\sim}{V}_{,A} \cdot \underset{\sim}{G}_B + \eta \left(\underset{\sim}{V}_{,A} \cdot \underset{\sim}{G}'_B + \underset{\sim}{G}_B \cdot \frac{D}{Dt}\underset{\sim}{W}_{,A}\right) = \tag{6.17}$$
$$= \nabla_A v_B + \eta \left[\frac{D}{Dt}(\nabla_A w_B) + \nabla_A v^M \nabla_B w_M - \nabla_B v^N \nabla_A w_N \right] .$$

According to /6.9/, /6.11/, the above transformation leads to the results

$$A'^{(0)}_{AB} = \frac{D}{Dt}\left(\nabla_A w_B + \nabla_B w_A\right) = 2\, d'_{AB} \; , \tag{6.18}$$

$$A'^{(\nu+1)}_{AB} = \frac{D}{Dt} A'^{(\nu)}_{AB} = \frac{D^{\nu+1}}{Dt^{\nu+1}} A'^{(0)}_{AB} \qquad (\nu = 0, 1, 2, \cdots) \; . \tag{6.19}$$

Bearing in mind the definition of the angular velocity vector or the spin tensor /cf. [6] /

$$/6.20/ \qquad \omega^A = \epsilon^{ABC}\omega_{BC} = \frac{1}{2}\epsilon^{ABC}\left(\nabla_B v_C - \nabla_C v_B\right),$$

where ϵ^{ABC} is the permutation symbol, we obtain the following result:

$$/6.21/ \qquad \omega'_{AB} = \frac{1}{2}\frac{D}{Dt}\left(\nabla_A w_B - \nabla_B w_A\right) + \nabla_A v^N \nabla_B w_N - \nabla_B v^N \nabla_A w_N .$$

In other words, increments of the spin tensor in the state $\overset{*}{B}$ depend as well on the displacement vector $\underset{\sim}{w}$ as on the velocity of fundamental motion in the state B. Moreover, it can be observed that the relation /6.18/ is equivalent to

$$/6.22/ \qquad A'^{(0)}_{AB} = 2d'_{AB} = \frac{D}{Dt}G'_{AB} = \frac{D}{Dt}C'_{AB} = 2\frac{D}{Dt}E_{AB} .$$

The relations /6.18/, /6.19/ can be obtained directly on the base of /6.12/ to /6.15/ throughout the corresponding tensor transformation in the disturbed state $\overset{*}{B}$. For example, taking into account the definitions /6.8/ and /4.8/, we arrive at

$$/6.23/ \qquad \begin{aligned} \overset{*}{A}{}^{(\nu+1)}_{AB} &= A^{(\nu+1)}_{AB} + \eta A'^{(\nu+1)}_{AB} = \overset{*}{A}{}^{(\nu+1)}_{ij}\overset{*}{x}{}^i_{,A}\overset{*}{x}{}^j_{,B} = \\ &= \left(\dot{\overset{*}{A}}{}^{(\nu)}_{ij} + \overset{*}{A}{}^{(\nu)}_{ir}\overset{*}{\nabla}_j\overset{*}{v}{}^r + \overset{*}{A}{}^{(\nu)}_{rj}\overset{*}{\nabla}_i\overset{*}{v}{}^r\right)\overset{*}{x}{}^i_{,A}\overset{*}{x}{}^j_{,B} = \\ &= \dot{\overset{*}{A}}{}^{(\nu)}_{AB} + \overset{*}{A}{}^{(\nu)}_{AM}\overset{*}{\nabla}_B\overset{*}{v}{}^M + \overset{*}{A}{}^{(\nu)}_{MB}\overset{*}{\nabla}_A\overset{*}{v}{}^M = \\ &= A^{(\nu+1)}_{AB} + \eta\frac{D}{Dt}A'^{(\nu)}_{AB} . \end{aligned}$$

In a similar way, taking into account /6.8/, /6.14/, we obtain

$$\left(\dot{\overset{*}{A}}{}^{(\nu)}_{ij} + \overset{*}{A}{}^{(\nu)}_{ir}\overset{*}{\nabla}_j\overset{*}{v}{}^r + \overset{*}{A}{}^{(\nu)}_{rj}\overset{*}{\nabla}_i\overset{*}{v}{}^r\right)\overset{*}{x}{}^i_{,A}\overset{*}{x}{}^j_{,B} =$$

/6.24/

$$= A^{(\nu+1)}_{ij}x^i_{,A}x^j_{,B} + \eta\left(A'^{(\nu+1)}_{ij}x^i_{,A}x^j_{,B} + A^{(\nu+1)}_{ij}w^i_{,A}x^j_{,B} + A^{(\nu+1)}_{ij}x^i_{,A}w^j_{,B}\right),$$

and

/6.25/ $$A'^{(\nu+1)}_{AB} = A'^{(\nu+1)}_{ij}x^i_{,A}x^j_{,B} + A^{(\nu+1)}_{ij}\left(w^i_{,A}x^j_{,B} + x^i_{,A}w^j_{,B}\right).$$

We wish to emphasize that any raising of indices in $A'^{(\nu)}_{ij}$ and $A'^{(\nu)}_{AB}$ should be done in the disturbed state $\overset{*}{B}$ using the metric tensors $\overset{*}{g}{}^{ij}$ and $\overset{*}{G}{}^{AB}$ respetively. This leads to the results

/6.26/ $$\overset{*}{A}{}^{(0)k}_{\ \ i} = \overset{*}{A}{}^{(0)}_{ir}\overset{*}{g}{}^{kr} = A^{(0)k}_{\ \ i} + \eta A'^{(0)k}_{\ \ i} = \nabla_i v^k + \nabla^k v_i + \eta\left(\nabla_i\dot{w}^k + \nabla^k\dot{w}_i\right),$$

/6.27/ $$\overset{*}{A}{}^{(\nu+1)k}_{\ \ i} = \overset{*}{A}{}^{(\nu+1)}_{ir}\overset{*}{g}{}^{kr} = A^{(\nu+1)k}_{\ \ i} + \eta A'^{(\nu+1)k}_{\ \ i} = A^{(\nu+1)k}_{\ \ i} + \eta\left[\left(\dot{A}^{(\nu)k}_{\ \ i}\right)' + A'^{(\nu)}_{ir}\nabla^k v^r + A^{(\nu)}_{ir}\nabla^k\dot{w}^r + A'^{(\nu)k}_{\ \ r}\nabla_i v^r + A^{(\nu)k}_{\ \ r}\nabla_i\dot{w}^r\right].$$

Similarly, bearing in mind that

/6.28/ $$\overset{*}{A}{}^{(0)C}_{\ \ A} = \overset{*}{A}{}^{(0)}_{AM}\overset{*}{G}{}^{CM} = A^{(0)C}_{\ \ A} + \eta A'^{(0)C}_{\ \ A} = A^{(0)C}_{\ \ A} + \eta\left(A^{(0)}_{AM}G'^{CM} + A'^{(0)}_{AM}G^{CM}\right),$$

and taking into account

$$/6.29/ \quad \frac{D}{Dt}\left(G^{AM}G_{CM}\right)=0\,, \quad \frac{D}{Dt}G_{AB}=2\nabla_{(A}v_{B)}\,, \quad \frac{D}{Dt}G^{AB}=-2\nabla^{(A}v^{B)}\,,$$

we obtain

$$/6.30/ \quad A'^{(0)C}_{\;\;A}=\frac{D}{Dt}\left(\nabla_A w^C+\nabla^C w_A\right)+2\nabla_{(A}w_{M)}A^{(0)CM}-2\nabla^{(C}w^{M)}A^{(0)}_{AM}\,,$$

$$/6.31/ \quad A'^{(\nu)C}_{\;\;A}=A'^{(\nu)}_{AM}G^{CM}+A^{(\nu)}_{AM}G'^{CM}=G^{CM}\frac{D^{\nu}}{Dt^{\nu}}A'^{(0)}_{AM}-2\nabla^{(C}w^{M)}A^{(\nu)}_{AM}\,.$$

For the increments of mixed components of the skew-symmetric spin tensor the following relations are valid [cf. /6.20/, /6.21/]:

$$/6.32/ \quad \omega'^{\cdot C}_{A}=\frac{D}{Dt}\left(\nabla_A w^C-\nabla^C w_A\right)+d_{AM}\nabla^C w^M-\omega_{AM}\nabla^M w^C+$$
$$+d^{CN}\nabla_N w_A-\omega^{CN}\nabla_A w_N\,,$$

$$/6.33/ \quad \omega'^{A}_{\cdot C}=-\omega'^{\cdot A}_{C}$$

In other words, the increments of mixed components of the spin tensor depend not only on the angular velocity of fundamental motion but also on the corresponding strain rate tensor.

7. Equations of equilibrium and boundary conditions

Equations of equilibrium and surface relations between the stress vector and the stress tensor can be written in the following equivalent forms /cf. [2] /:

$$\nabla_M \sigma^M_A + \varrho f_A = \varrho \dot{v}_A \qquad \text{or} \qquad \nabla_M \sigma^{AM} + \varrho f^A = \varrho \dot{v}^A , \tag{7.1}$$

$$t_A = \sigma^M_A n_M \qquad \text{or} \qquad t^A = \sigma^{AM} n_M , \tag{7.2}$$

where σ^A_B and σ^{AB} are mixed and contravariant components of the symmetric Cauchy stress tensor, t_A and t^A are covariant and contravariant components of the surface stress vector $\underset{\sim}{t}$, f^A and f_A denote components of a body force per unit mass, ϱ is the density of a body, and n_A - components of the unit normal vector. The above equations have been written in convected coordinates, however, they have the same form in an arbitrary fixed system of coordinates (x^i).

Let us suppose that in any disturbed state $\overset{*}{B}$ all the quantities mentioned above undergo some increments which we denote by primes. Thus, for the state $\overset{*}{B}$ expressed in the coordinates (x^i) , we have: $\sigma^i_j + \eta\sigma'^i_j$, $t_i + \eta t'_i$, $f_i + \eta f'_i$, $\varrho + \eta\varrho'$, $n_i + \eta n'_i$, etc. Similarly, in the system (X^A), we have: $\sigma^A_B + \eta\sigma'^A_B$, $t^A + \eta t'^A$, etc. Because of different properties of the systems (x^i) and (X^A) , the corresponding increments of tensors and vectors in one representation differ from those in the other, however, the relations between them can be established without any essential difficulty.

Substituting formally new values of the quantities considered into the equations of equilibrium and the surface relations in the state $\overset{*}{B}$, we obtain

$$\begin{aligned} &\nabla_k \sigma'^k_i - \left\{ {m \atop k\,i} \right\}' \sigma^k_m + \left\{ {k \atop k\,r} \right\}' \sigma^r_i + \varrho f'_i + \varrho' f_i = \varrho' \dot{v}_i + \varrho \dot{v}'_i , \\ &\nabla_k \sigma'^{ik} + \left\{ {i \atop k\,m} \right\}' \sigma^{km} + \left\{ {k \atop k\,m} \right\}' \sigma^{im} + \varrho f'^i + \varrho' f^i = \varrho' \dot{v}^i + \varrho \dot{v}'^i , \end{aligned} \tag{7.3}$$

$$/7.4/\quad t'_i = \sigma'^k_i n_k + \sigma^k_i n'_k\ , \quad t'^i = \sigma'^{ik} n_k + \sigma^{ik} n'_k\ .$$

Bearing in mind the relations /3.12/, /5.12/, /5.13/, and remembering that in the state $\overset{*}{B}$

$$/7.5/\quad \overset{*}{\nabla}_M \overset{*}{\sigma}^M_A = \overset{*}{\sigma}^M_{A,M} + \left\{ {\overset{*}{M} \atop MN} \right\} \overset{*}{\sigma}^N_A - \left\{ {\overset{*}{N} \atop MA} \right\} \overset{*}{\sigma}^M_N\ ,$$

we also arrive at the following equations /cf. [1,2] /:

$$/7.6/\quad \nabla_M \sigma'^M_A + \sigma^N_A \nabla_M \nabla_N w^M - \sigma^N_M \nabla_A \nabla_N w^M + \varrho' f_A + \varrho f'_A =$$
$$= \varrho' \dot{v}_A + \varrho(\ddot{w}_A + \dot{v}_N \nabla_A w^N)\ ,$$

$$/7.7/\quad \nabla_M \sigma'^{AM} + \sigma^{AN} \nabla_M \nabla_N w^M + \sigma^{NM} \nabla_N \nabla_M w^A + \varrho' f^A + \varrho f'^A =$$
$$= \varrho' \dot{v}^A + \varrho(\ddot{w}^A - \dot{v}^N \nabla_N w^A)\ ,$$

$$/7.8/\quad t'_A = \sigma'^M_A n_M + \sigma^M_A n'_M\ , \quad t'^A = \sigma'^{AM} n_M + \sigma^{AM} n'_M\ .$$

The relations between the increments σ'^A_B and σ'^i_j result from the following transformation:

$$\overset{*}{\sigma}^i_j = \overset{*}{\sigma}^A_B \overset{*}{x}^i_{,A} \overset{*}{X}^B_{,j} = (\sigma^A_B + \eta\sigma'^A_B)(x^i_{,A} + \eta w^i_{,A})(X^B_{,j} + \eta X'^B_{,j}) =$$

$$/7.9/\quad = \sigma^i_j + \eta\left(\sigma'^A_B x^i_{,A} X^B_{,j} + \sigma^A_B w^i_{,A} X^B_{,j} + \sigma^A_B x^i_{,A} X'^B_{,j}\right),$$

so that

$$/7.10/\quad \sigma'^i_j = \sigma'^A_B x^i_{,A} X^B_{,j} + \sigma^M_B w^i_{,M} X^B_{,j} - \sigma^M_N w^N_{,B} x^i_{,M} X^B_{,j}\ .$$

Similarly, we obtain

/7.11/ $$\sigma'^{A}_{B} = \sigma'^{i}_{j} x^{j}_{,B} X^{A}_{,i} + \sigma^{i}_{j} w^{j}_{,B} X^{A}_{,i} - \sigma^{k}_{j} w^{i}_{,k} x^{j}_{,B} X^{A}_{,i} .$$

It can be also shown that the equations /7.6/, /7.7/ can be derived from /7.3/ as a result of simple tensor transformation.

Let us assume for further simplicity of transformations that the fixed system of spatial coordinates (x^i) coincides with the system of rectangular Cartesian coordinates (ξ^α). In more general cases, with no loss of generality, we can always transform the equations considered from one fixed system of coordinates into another fixed system.

Under the above assumption, the equations of equilibrium in the state $\overset{*}{B}$, written in the form

/7.12/ $$\overset{*}{\sigma}{}^{k}_{i,k} + \overset{*}{\rho}(\overset{*}{f}_i - \dot{\overset{*}{v}}_i) = 0 ,$$

can be transformed as follows:

$$\overset{*}{\sigma}{}^{k}_{i,k} + \overset{*}{\rho}(\overset{*}{f}_i - \dot{\overset{*}{v}}_i) = \overset{*}{X}{}^{M}_{,k}(\overset{*}{\sigma}{}^{N}_{P} \overset{*}{x}{}^{k}_{,N} \overset{*}{X}{}^{P}_{,i})_{,M} + \overset{*}{\rho}(\overset{*}{f}_P - \dot{\overset{*}{v}}_P)\overset{*}{X}{}^{P}_{,i} =$$

$$= \overset{*}{X}{}^{M}_{,k}(\overset{*}{\sigma}{}^{N}_{P,M} \overset{*}{x}{}^{k}_{,N} \overset{*}{X}{}^{P}_{,i} + \overset{*}{\sigma}{}^{N}_{P} \overset{*}{X}{}^{P}_{,i} \overset{*}{x}{}^{k}_{,NM}) + \overset{*}{\sigma}{}^{N}_{P} \overset{*}{x}{}^{k}_{,N} \overset{*}{X}{}^{P}_{,ik} +$$

$$+ \overset{*}{\rho}(\overset{*}{f}_P - \dot{\overset{*}{v}}_P)\overset{*}{X}{}^{P}_{,i} =$$

/7.13/ $$= \overset{*}{X}{}^{A}_{,i}\left[\overset{*}{\sigma}{}^{N}_{A,N} + \left\{ {\overset{*}{M} \atop MN} \right\}\overset{*}{\sigma}{}^{N}_{A} - \left\{ {\overset{*}{M} \atop NA} \right\}\overset{*}{\sigma}{}^{N}_{M} + \overset{*}{\rho}(\overset{*}{f}_A - \dot{\overset{*}{v}}_A)\right] =$$

$$= \overset{*}{X}{}^{A}_{,i}\left[\nabla_N \sigma^{N}_{A} + \rho(f_A - \dot{v}_A) + \eta\left(\nabla_N \sigma'^{N}_{A} + \left\{ {M \atop MN} \right\}' \sigma^{N}_{A} - \right.\right.$$

$$\left.\left. - \left\{ {M \atop NA} \right\}' \sigma^{N}_{M} + \rho' f_A + \rho f'_A - \rho' \dot{v}_A - \rho \dot{v}'_A\right)\right] = 0,$$

where we have used /2.5/, /2.6/, /3.11/ rejecting all terms of higher order than $O(\eta)$. Taking into account /3.12/, /5.12/, /7.1/$_1$ it can be observed that the condition for /7.13/ to be satisfied

$$/7.14/ \qquad \nabla_N \sigma'^N_A + \left\{ {M \atop MN} \right\}' \sigma^N_A - \left\{ {M \atop NA} \right\}' \sigma^N_M + \rho'(f_A - \dot{v}_A) + \rho(f'_A - \dot{v}'_A) = 0,$$

is equivalent to already written equation /7.6/. In a similar way the equation /7.7/ for contravariant representation of the stress tensor can be derived starting from the equation of equilibrium in the form /7.1/$_2$.

Let us note at the end of our considerations that for incompressible materials many of the relations discussed so far can be simplified considerably. This is achieved with the following incompressibility conditions:

$$/7.15/ \qquad G = \det G_{AB} = 1 \ , \ \rho = \mathrm{const} \ ; \ G' = 2G \nabla_M w^M \ , \ \rho' = 0 \ .$$

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