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On Two Methods of Determining the Ellipses and Ellipsoids of Positioning Accuracy of Robot Manipulators.

Abstract

When analysing the problem of the positioning accuracy of robot manipulators it is important to know how large may random deviations of the hand be from the desired position if the joint positioning errors possess a normal distribution. Two methods of determining the ellipses and ellipsoids of probability concentration are compared. First of them is based on the standard procedure of the probability calculus. The second approximate method consists in finding at first the polygon or polyhedron of the positioning accuracy, and then in finding the ellipse or ellipsoid of the principal axes and second order moments coinciding with those of the polygon or polyhedron respectively. Examples of application demonstrate that these two methods give very close results.

1. Introduction.

Small changes in the position of the hand of a manipulator are caused among others by inevitable small random deviations Δq_i from the desired (nominal) joint coordinate q_i^0. In the case of a revolute joint the deviation Δq_i corresponds to a small random rotation $\Delta\theta_i$ with respect to the desired joint positioning angle θ_i^0. In the case of a prismatic joint Δq_i corresponds to a small random linear deviation Δl_i from the desired joint positioning distance l_i^0. It is important to know how large may the random deviations from the desired position of the hand be if the joint positioning errors possess Gaussian distributions during the repeated cycles of manipulator's movement.

Following the paper by A.Kumar and K.J.Waldron [1] three sources of positioning errors may be distinguished:

1. Errors in positioning the joints accurately

2. Dynamic errors due to elastic deflections of individual members of the manipulator

3. Mechanical clearance in the system.

In the present paper only errors in positioning the hand accurately due to the random Gaussian errors in positioning the joints will be accounted for. A particular positioning error of the hand may be represented by a displacement vector whose components represent deviations from the desired coordinates of the hand. Since the joint positioning errors are random magnitudes in each of the repeated cycles, the end point of such a vector will have different random coordinates for each cycle of manipulator's movement. Analysing a large number of repeated cycles we have to deal with the problem of the probability concentration of the distribution of the end points of all random error displacement vectors.

For manipulators operating in two dimensions the probability concentration may be represented by a certain ellipse of equal probability. For a more general case when the manipulator operates in three dimensions the probability concentration may be represented by a certain ellipsoid of equal probability. The problem of determining such ellipsoids for robot manipulators has been shortly mentioned by A.Antshev et. all [2]. A more detailed study was presented by W.Szczepiński and Z.Wesołowski [3]. Below is presented comparison of two theoretical methods of determining the ellipses and ellipsoids of probability concentration of the positioning accuracy of robot manipulators.

The hand positioning errors are assumed to be caused by small random errors in joint positions. A joint position error is treated as a small random rotation or displacement from the desired position of the joint. The errors Δq_i in joint position are assumed to be distributed during the repeated cycles of manipulator's movement according to the Gaussian distribution. The density of probability that the joint positioning error is of the magnitude Δq_i is

$$\varphi(\Delta q_i) = \frac{1}{\sigma_i\sqrt{2\pi}}\, exp\left\{ -\frac{1}{2}\left(\frac{\Delta q_i}{\sigma_i}\right)^2 \right\}, \qquad (1)$$

where σ_i is the standard deviation, whose value depends on the accuracy of the joint in question labelled by the index i.

In the present work we shall use two approximate theoretical procedures for determining the ellipses and ellipsoids of probability concentration. The first "direct" procedure is based on the standard methods of the mathematical theory of statistics. The second "indirect" procedure proposed in the previous paper [3] is based upon the concept of the polygons or the polyhedrons of the positioning accuracy. A number of working examples have been solved with the use of these two procedures and then the results have been compared. It can be seen that both procedures give very close results. Thus both of them are of practical significance.

2. Manipulators with two-dimensional movements.

We shall now analyse the positioning accuracy of manipulators operating in two dimensions. Let us assume a Cartesian coordinate system X,Y. Any position of a chosen reference point of the manipulator's hand is defined by its two coordinates. Each of them is a certain function of the joint position parameters $q_i = q_i^0 + \Delta q_i$. Thus we can write:

$$X = X(q_1, q_2, \ldots, q_n)$$
$$Y = Y(q_1, q_2, \ldots, q_n) \tag{2}$$

To analyse the hand positioning errors we shall use a local Cartesian coordinate systems x,y, with the axes parallel to the corresponding axes of the basic system X,Y and the origin at the desired position of the reference point of the hand. The hand positioning error will be represented by a displacement vector **v** with the components

$$x = X - X^0, \qquad y = Y - Y^0, \tag{3}$$

where X^0, Y^0 define the desired position of the hand and X, Y stand for the actual coordinates of the hand position.

If the joint positioning errors Δq_i are given, the components (3) of the hand positioning error can be calculated from the linearized formulae:

$$x = \frac{\partial X}{\partial q_1}\Delta q_1 + \frac{\partial X}{\partial q_2}\Delta q_2 + \ldots + \frac{\partial X}{\partial q_n}\Delta q_n, \tag{4}$$

$$y = \frac{\partial Y}{\partial q_1}\Delta q_1 + \frac{\partial Y}{\partial q_2}\Delta q_2 + \ldots + \frac{\partial Y}{\partial q_n}\Delta q_n,$$

which may be also represented in the matrix form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial q_1} & \frac{\partial X}{\partial q_2} & \cdots & \frac{\partial X}{\partial q_n} \\ \frac{\partial Y}{\partial q_1} & \frac{\partial Y}{\partial q_2} & \cdots & \frac{\partial Y}{\partial q_n} \end{bmatrix} \cdot \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \\ \vdots \\ \Delta q_n \end{bmatrix} \tag{4a}$$

or simply

$$[\,\mathbf{v}\,] = [\,\mathbf{A}\,]\;[\,\Delta\mathbf{q}\,] \tag{4b}$$

2.1. Analytical procedure of determining ellipses of probability concentration

Let us assume that joint positioning errors Δq_i are distributed according to the Gaussian law (1) and that they are statistically independent. Then the distribution of the hand positioning errors calculated according to the linearized relation (4) is also Gaussian. The second order moments of the distribution of this error may be expressed as:

$$[\,\lambda_{ij}\,] \overset{\mathrm{def}}{=} [\,\mathbf{A}\,]\;[\,\sigma^2\,]\;[\,\mathbf{A}\,]^{\mathrm{T}} = \begin{bmatrix} \lambda_{xx} & \lambda_{xy} \\ \lambda_{yx} & \lambda_{yy} \end{bmatrix} \tag{5}$$

where

$$[\,\sigma^2\,] = \begin{bmatrix} \sigma_1^2 & 0 & \ldots & 0 \\ 0 & \sigma_2^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_n^2 \end{bmatrix} \tag{6}$$

is the matrix of variances of the positioning accuracy of all the joints [cf. formula (1)].

Elements of the matrix (5) are

$$\lambda_{xx} = \left(\frac{\partial X}{\partial q_1}\right)^2 \sigma_1^2 + \left(\frac{\partial X}{\partial q_2}\right)^2 \sigma_2^2 + \ldots + \left(\frac{\partial X}{\partial q_n}\right)^2 \sigma_n^2,$$

$$\lambda_{yy} = \left(\frac{\partial Y}{\partial q_1}\right)^2 \sigma_1^2 + \left(\frac{\partial Y}{\partial q_2}\right)^2 \sigma_2^2 + \ldots + \left(\frac{\partial Y}{\partial q_n}\right)^2 \sigma_n^2 , \qquad (7)$$

$$\lambda_{yx} = \lambda_{xy} = \frac{\partial X}{\partial q_1}\frac{\partial Y}{\partial q_1}\sigma_1^2 + \frac{\partial X}{\partial q_2}\frac{\partial Y}{\partial q_2}\sigma_2^2 + \ldots + \frac{\partial X}{\partial q_n}\frac{\partial Y}{\partial q_n}\sigma_n^2.$$

These elements are called respectively (cf. [4]):

$\lambda_{xx} = D_x = \sigma_x^2$ - dispersion of the component x of the error displacement vector **v**,

$\lambda_{yy} = D_y = \sigma_y^2$ - dispersion of the component y of the error displacement vector **v**,

$\lambda_{xy} = \lambda_{yx} = cov\{x,y\}$ - covariance of x and y or the correlation moment.

The Gaussian probability density function of two-dimensional distribution of coordinates x_1 and x_2 of the error displacement vector contains in its exponent the expression

$$\sum_{j=1}^{2}\sum_{k=1}^{2}\Lambda_{jk}\, x_j\, x_k = \Lambda_{xx}\, x^2 + 2\,\Lambda_{xy}\, x\, y + \Lambda_{yy}\, y^2, \qquad (8)$$

where the following notations have been used

$$x_1 = x, \qquad\qquad x_2 = y,$$

$$\Lambda_{11} = \Lambda_{xx}, \qquad \Lambda_{12} = \Lambda_{xy}, \qquad \Lambda_{22} = \Lambda_{yy}.$$

Coefficients Λ_{xx}, Λ_{xy} and Λ_{yy} constitute the elements of the inverse matrix $[\lambda_{ij}]^{-1}$

$$[\,\Lambda_{ij}\,] = [\,\lambda_{ij}\,]^{-1} = \begin{bmatrix}\Lambda_{xx} & \Lambda_{xy}\\ \Lambda_{yx} & \Lambda_{yy}\end{bmatrix} =$$

$$= \frac{1}{\lambda_{xx}\lambda_{yy} - \lambda_{xy}^2}\begin{bmatrix}\lambda_{yy} & -\lambda_{xy}\\ -\lambda_{yx} & \lambda_{xx}\end{bmatrix}.$$

This means that the ellipses of constant probability are

$$\frac{\lambda_{xx}\lambda_{yy}}{\lambda_{xx}\lambda_{yy} - \lambda_{xy}^2}\left(\frac{x^2}{\lambda_{xx}} - 2\frac{\lambda_{xy}}{\sqrt{\lambda_{xx}\lambda_{yy}}}\frac{x}{\sqrt{\lambda_{xx}}}\frac{y}{\sqrt{\lambda_{yy}}} + \frac{y^2}{\lambda_{yy}}\right) = const.$$

This equation may be written in a more convenient form (cf.[4])

$$\frac{1}{1 - \rho_{xy}^2}\left[\left(\frac{x}{\sigma_x}\right)^2 - 2\,\rho_{xy}\frac{x}{\sigma_x}\frac{y}{\sigma_y} + \left(\frac{y}{\sigma_y}\right)^2\right] = \lambda^2, \tag{9}$$

where

$$\rho_{xy} = \frac{\lambda_{xy}}{\sqrt{\lambda_{xx}\lambda_{yy}}} \tag{10}$$

is the correlation coefficient and $\lambda^2 = \chi_p^2(2)$. The value of λ^2 depends on the assumed probability

$$p = \Phi_{\chi^2(2)}(\lambda^2)$$

that the end point of the vector of a certain positioning error will lie inside the ellipse (9) of probability concentration.

2.2. Approximate procedure of determining the ellipses of probability concentration

The procedure described below has been used in the previous paper [3]. At first we shall assume that instead of the original Gaussian distribution (1) of joint positioning errors we have an auxiliary uniform distribution in which random values of errors are limited by two extreme values. We shall also assume that introduced auxiliary uniform distributions of Δq_i have the same standard deviations σ_i as the original Gaussian distributions. Thus the uniform distributions must be restricted to the intervals $\pm\sqrt{3}\,\sigma_i$ for $i = 1,2, \ldots ,n$.

It has been demonstrated in [5] that if joint positioning errors vary within two extreme values, then the end point of any error displacement vector of the hand will lie inside a certain polygon bounded by several pairs of parallel straight lines. The equations of these lines are

$$\frac{\partial Y}{\partial q_r} x - \frac{\partial X}{\partial q_r} y = \sum_{i=1}^{n} \begin{vmatrix} \frac{\partial X}{\partial q_i} & \frac{\partial X}{\partial q_r} \\ \frac{\partial Y}{\partial q_i} & \frac{\partial Y}{\partial q_r} \end{vmatrix} \Delta q_i. \tag{11}$$

Writing consecutively such an equation for all joint positioning parameters q_r, where r=1,2,...,n, we obtain the equations of n families of parallel straight lines. The end point of the vector of hand positioning error moves along one of such straight lines when the joint positioning error Δq_r is changing, while all remaining joint positioning errors are kept constant.

Extreme positions of lines (11) we shall obtain by taking appropriately the extreme values of joint positioning errors Δq_i^+ or Δq_i^- in order to obtain at first the largest possible value of the sum on the right-hand side of equation (11) and then its smallest possible value. These extreme positions constitute the edges of the polygon.

The extreme values of joint positioning errors $\Delta q_i^+ = +\kappa\sigma_i$ and $\Delta q_i^- = -\kappa\sigma_i$, which should be assumed when calculating the polygon, depend on the assumed probability p that the end point of displacement vector will lie inside the polygon. The polygon calculated for $\kappa=\sqrt{3}$ corresponds to the probability p = 1. For any $\kappa<\sqrt{3}$ the probability p may be calculated as the ratio of the areas of polygons obtained for $\Delta q_i = \pm\kappa\sigma_i$ and for $\Delta q_i = \pm\sqrt{3}\,\sigma_i$ respectively. Thus for the desired probability p the multiplier κ in two dimensional case should have the value $\kappa = \sqrt{3}\,\sqrt{p}$.

Having found the polygon we can calculate its second order (inertia) moments and then find the orientation of its principal axes 1,2 and second order principal moments J_1, J_2. Now the ellipse of probability concentration can be found, as the ellipse of the same principal axes and second order moments as the polygon (cf. Cramèr [6]). The principal radii a and b of the ellipse can be found by solving the system of equations

$$\frac{1}{4}\pi a b^3 = J_1, \qquad \frac{1}{4}\pi a^3 b = J_2, \tag{12}$$

where J_1 and J_2 are second order principal moments calculated for the polygon.

2.3 Examples of application

As the examples of application of these two procedures we shall determine the ellipses of probability concentration of the positioning accuracy for a simple manipulator with two revolute joints. The scheme of the mechanism of this manipulator is shown in Fig.1. Joint positions are determined by positioning angles θ_1 and θ_2.

The position of the hand point O is determined by the coordinates

$$X = l_1 \cos \theta_1 + l_2 \cos (\theta_1+\theta_2),$$
$$Y = l_1 \sin \theta_1 + l_2 \sin (\theta_1+\theta_2). \tag{13}$$

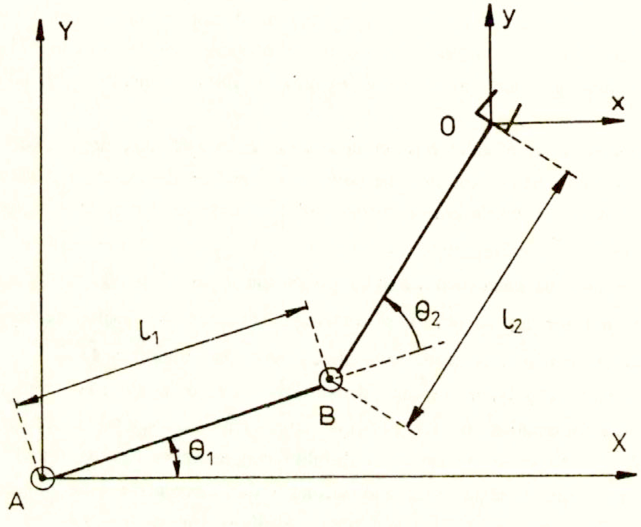

Fig.1.

These expressions represent certain functions of two independent variables θ_1 and θ_2. Linear dimensions l_1 and l_2 are assumed to be constant.

We shall calculate particular examples for

$$l_1 = l_2 = l = 1000 \; mm.$$

Standard deviations of normal distributions of joint positioning errors are taken to be

$$\sigma_{\theta_1} = \sigma_{\theta_2} = \sigma_{\theta} = 0.01 \; rad. \tag{14}$$

They are taken deliberately large in order to demonstrate that linear approach used in the theoretical procedure is justified. Ellipses of probability concentration will be calculated for three various positions of the manipulator.

Example 1.

Desired position of the hand is determined by the following joint positioning angles

$$\theta_1 = 0, \qquad \theta_2 = \frac{1}{2}\pi. \tag{15}$$

<u>Analytical procedure</u>

Using formulae (7) we obtain the values of the elements of the matrix of second order moments

$$\lambda_{xx} = \sigma_x^2 = \left[\left(\frac{\partial X}{\partial \theta_1}\right)^2 + \left(\frac{\partial X}{\partial \theta_2}\right)^2 \right] \sigma_\theta^2 = 2\, l^2 \sigma_\theta^2 = 200 \; mm^2,$$

$$\lambda_{yy} = \sigma_y^2 = \left[\left(\frac{\partial Y}{\partial \theta_1}\right)^2 + \left(\frac{\partial Y}{\partial \theta_2}\right)^2 \right] \sigma_\theta^2 = l^2 \sigma_\theta^2 = 100 \; mm^2, \tag{16}$$

$$\lambda_{xy} = cov\{X,Y\} = \left[\frac{\partial X}{\partial \theta_1}\frac{\partial Y}{\partial \theta_1} + \frac{\partial X}{\partial \theta_2}\frac{\partial Y}{\partial \theta_2} \right] \sigma_\theta^2 = -\, l^2 \sigma_\theta^2 = -\, 100 \; mm^2.$$

The correlation coefficient (10) is

$$\rho_{xy} = \frac{\lambda_{xy}}{\sqrt{\lambda_{xx}\lambda_{yy}}} = -\frac{1}{\sqrt{2}}.$$

Assuming that the probability that the end point of the vector of a certain hand positioning error lies inside the ellipse of probability concentration is p=0.683 we find from the tables that $\chi^2_p(2) = \lambda^2 = 2.26$. By substituting all the relevant magnitudes to equation (9) we find the following equation of the ellipse of probability concentration

$$x^2 + 2\,x\,y + 2\,y^2 = \lambda^2\,l^2\,\sigma_\theta^2 = 226. \tag{17}$$

The ellipse is shown in Fig.2. Its longer axis makes the angle $\gamma = 31^0 43'$ with the x-axis. The principal radii of the ellipse are

$$a = 24.32\ mm, \qquad b = 9.29\ mm. \tag{18}$$

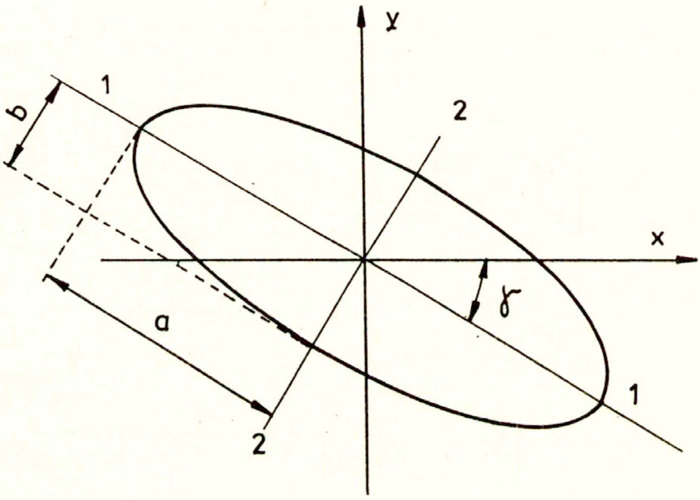

Fig.2.

As a parameter determining the elongation of the ellipse we shall take the relation

$$a/b = 2.62.$$

Approximate procedure

The polygon of the positioning accuracy is bounded by the extreme positions of straight lines determined by the equation [cf. Eqn (11)]

$$\frac{\partial Y}{\partial \theta_1} x - \frac{\partial X}{\partial \theta_1} y = \begin{vmatrix} \frac{\partial X}{\partial \theta_2} & \frac{\partial X}{\partial \theta_1} \\ \frac{\partial Y}{\partial \theta_2} & \frac{\partial Y}{\partial \theta_1} \end{vmatrix} \Delta\theta_2, \tag{19a}$$

for the lines of the first family, and

$$\frac{\partial Y}{\partial \theta_2} x - \frac{\partial X}{\partial \theta_2} y = \begin{vmatrix} \frac{\partial X}{\partial \theta_1} & \frac{\partial X}{\partial \theta_2} \\ \frac{\partial Y}{\partial \theta_1} & \frac{\partial Y}{\partial \theta_2} \end{vmatrix} \Delta\theta_1 \tag{19b}$$

for the lines of the second family.

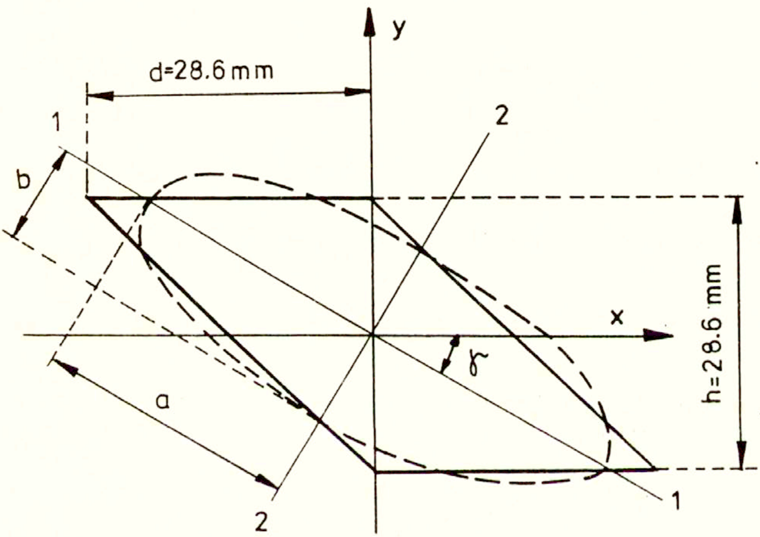

Fig.3

Determining the dimensions of the polygon we shall assume the following values of joint positioning errors [cf.(14)]

$$\Delta\theta_1 = \pm\kappa\sigma_\theta = \pm 0.0143 \; rad, \qquad \Delta\theta_2 = \pm\kappa\sigma_\theta = \pm 0.0143 \; rad.$$

Here for the probability p=0.683 for the ellipse of concentration the approximate value of the multiplier $\kappa = \sqrt{3}\,\sqrt{0.683} = 1.43$ has been assumed (cf. Section 2.2).

Making use of relations (13) and equations (19) we find that the polygon of the positioning accuracy is bounded by two pairs of parallel straight lines. These lines are determined by the equations

$$x + y = \pm\, l\, \sigma_\theta = \pm\; 14.3 \; mm,$$

$$y = \pm\, l\, \sigma_\theta = \pm\; 14.3 \; mm.$$

The polygon is shown in Fig.3.

The second order (inertia) moments of the polygon with respect to the reference axes x and y are

$$J_x = \frac{1}{12}\, d\, h^3 = 5.576 \cdot 10^4 \; mm^4,$$

$$J_y = \frac{1}{6}\, d^3 h = 11.152 \cdot 10^4 \; mm^4,$$

$$J_{xy} = \frac{1}{12}\, d^2 h^2 = 5.576 \cdot 10^4 \; mm^4.$$

The angle γ of inclination of the principal axis 1 with respect to the x-axis we obtain from the relation

$$tan\; 2\gamma = \frac{2J_{xy}}{J_y - J_x} = -2.$$

Thus we obtain the same value $\gamma = 31^o 43'$ as the angle of inclination of the principal axis of the ellipse shown in Fig.2.

The principal second order moments of the polygon are

$$J_1 = \frac{1}{2}\left(J_x + J_y\right) - \sqrt{\frac{1}{4}\left(J_x - J_y\right)^2 + J_{xy}^2} = 2.128\cdot 10^4\ mm^4,$$

$$J_2 = \frac{1}{2}\left(J_x + J_y\right) + \sqrt{\frac{1}{4}\left(J_x - J_y\right)^2 + J_{xy}^2} = 14.599\cdot 10^4\ mm^4.$$

Now by solving the system of equations (12) we obtain the principal radii of the ellipse of probability concentration

$$a = 26.41\ mm, \qquad b = 10.09\ mm.$$

The elongation parameter is

$$a/b = 2.62.$$

Note that the ellipses calculated with the use of above two procedures are geometrically similar. Their dimensions are only slightly different.

Example 2.

Desired position of the hand is determined by the joint positioning angles

$$\theta_1 = 0, \qquad \theta_2 = \frac{3}{4}\pi.$$

Analytical procedure

The elements of the matrix of second order moments are

$$\lambda_{xx} = l^2\sigma_\theta^2, \quad \lambda_{yy} = \left(2 - \sqrt{2}\right) l^2\sigma_\theta^2, \quad \lambda_{xy} = \left(1 - \frac{1}{\sqrt{2}}\right) l^2\sigma_\theta^2.$$

The correlation coefficient (10) takes the value

$$\rho_{xy} = \frac{1}{2}\sqrt{2 - \sqrt{2}} = 0.383.$$

Assuming as in the previous example p=0.683 and therefore $\lambda^2 = 2.26$ we obtain the equation of the ellipse of probability concentration

$$x^2 - x\,y + 1.707\,y^2 = 0.853\,\lambda^2\,l^2\,\sigma_\theta^2 = 192.78.$$

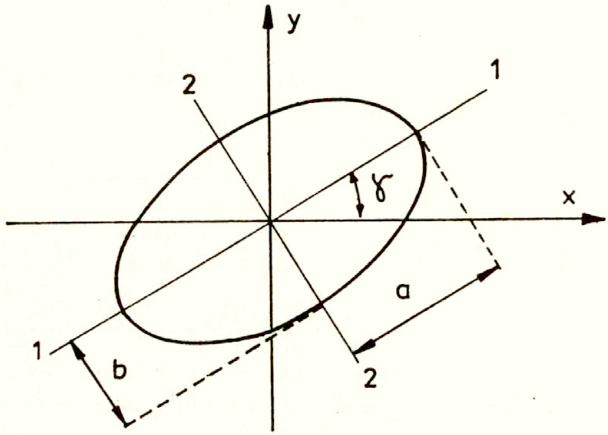

Fig.4

Its longer principal axis makes the angle $\gamma = 27^\circ 22'$ with the x-axis as shown in Fig.4. The principal radii of the ellipse are

$$a = 17.49\ mm, \qquad b = 10.72\ mm.$$

The relation a/b is

$$a/b = 1.630.$$

<u>Approximate</u> <u>procedure</u>

The polygon of the positioning accuracy is bounded by two pairs of parallel straight lines determined by the equations

$$(\sqrt{2}-1)x+y = \pm\, l\,\kappa\,\sigma_\theta = \pm\,14.3\ mm,$$

$$x - y = \pm\, l\,\kappa\,\sigma_\theta = \pm\,14.3\ mm.$$

The polygon is shown in Fig.5.

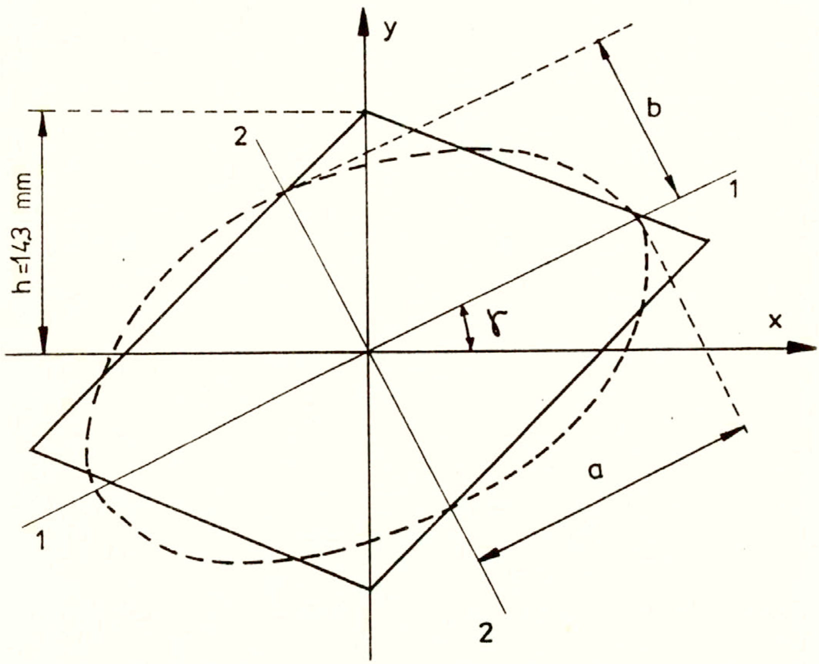

Fig.5.

The principal second order moments of the polygon are

$$J_1 = 17.112 \cdot 10^3 \ mm^4, \qquad J_2 = 45.410 \cdot 10^3 \ mm^4.$$

The angle of inclination of the principal axis 1 of the polygon with respect to the x-axis is $\gamma = 27^0 22'$. Thus it is of the same magnitude as the angle obtained with the use of the analytical procedure. The principal radii of the ellipse are

$$a = 17.51 \ mm, \qquad b = 10.75 \ mm.$$

The ratio of principal radii is $a/b = 1.628$. Thus it is almost the same as that obtained by means of the analytical procedure.

Example 3.

Desired position of the hand is determined by the joint positioning angles

$$\theta_1 = 0, \qquad \theta_2 = \frac{1}{4}\pi.$$

Analytical procedure

The elements of the matrix of second order moments are

$$\lambda_{xx} = l^2\sigma_\theta^2, \quad \lambda_{yy} = \left(2 + \sqrt{2}\right) l^2\sigma_\theta^2, \quad \lambda_{xy} = -\left(1 + \frac{1}{\sqrt{2}}\right) l^2\sigma_\theta^2.$$

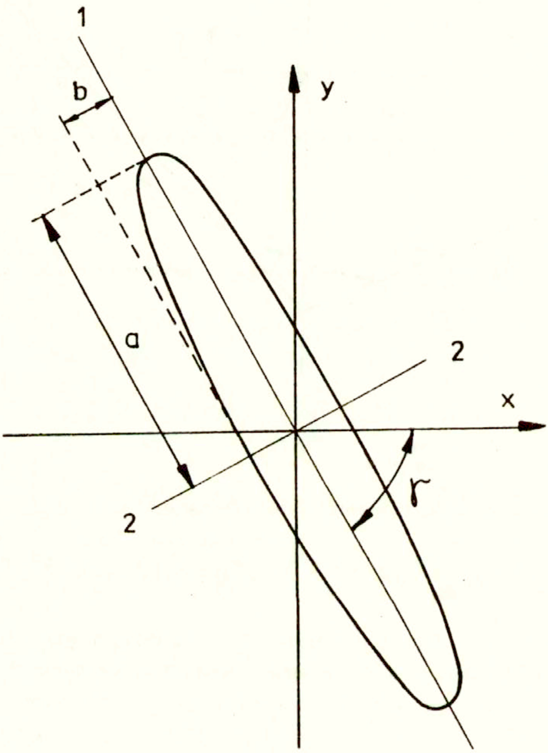

Fig.6.

The correlation coefficient (10) is

$$\rho_{xy} = -\frac{1}{2}\sqrt{2+\sqrt{2}} = -0.924.$$

Assuming as before p =0.683 we obtain the equation of the ellipse of probability concentration

$$x^2 + x\,y + 0.293\,y^2 = 0.146\,\lambda^2\,l^2\,\sigma_\theta^2 = 32.00.$$

Its longer principal axis makes the angle γ = 62°37' with the x-axis (Fig.6). The principal radii of the ellipse are

$$a = 30.56\ mm, \qquad b = 5.04\ mm.$$

The ratio of the principal radii is a/b = 6.062.

Approximate procedure

The polygon of the positioning accuracy (Fig.7) is bounded by two pairs of parallel straight lines determined by the equations

$$(1+\sqrt{2})x+y = \pm\, l\,\kappa\,\sigma_\theta = \pm\,14.3\ mm,$$

$$x + y = \pm\, l\,\kappa\,\sigma_\theta = \pm\,14.3\ mm.$$

The principal second order moments of the polygon are

$$J_1 = 0.458\cdot 10^4\ mm^4, \qquad J_2 = 9.189\cdot 10^4\ mm^4.$$

The angle which makes the principal axis 1 of the polygon with the x-axis is γ = 62°38'. This angle is almost of the same magnitude as that obtained above with the use of the analytical procedure. The principal radii of the ellipse resulting from equations (12) are

$$a = 33.83\ mm, \qquad b = 5.57\ mm.$$

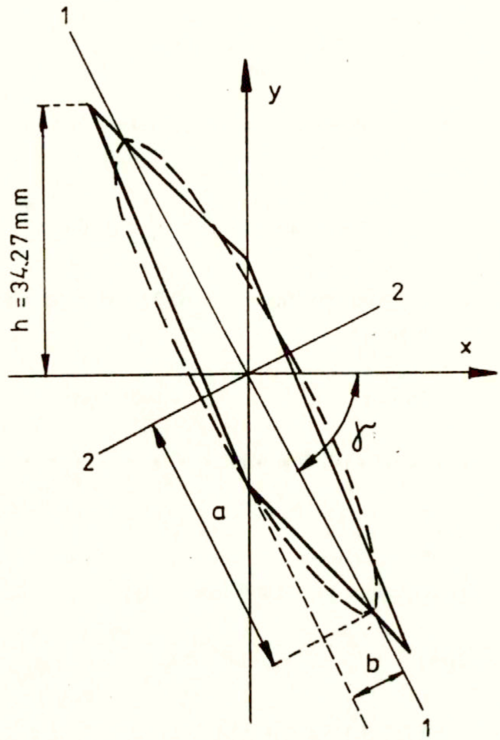

Fig.7

The ratio of principal radii is a/b = 6.078. It is almost the same as in the case when analytical procedure has been used.

3. Manipulators operating in three dimensions.

In this Section we shall analyse the problem of the positioning accuracy of manipulators of a general type operating in three dimensions. Let us assume a Cartesian coordinate system X, Y, Z. Any position of a reference point of the manipulator's hand is defined by its three coordinates. Each coordinate is a certain function of the joint position parameters $q_i = q_i^0 + \Delta q_i$

$$X = X(q_1, q_2, \dots, q_n),$$

$$Y = Y(q_1, q_2, \dots, q_n), \tag{20}$$

$$Z = Z(q_1, q_2, \dots, q_n)$$

To analyse the hand positioning errors we shall use a local Cartesian coordinate system x, y, z with the axes parallel to the corresponding axes of the basic system X, Y, Z, and the origin at the desired position of the reference point of the hand. The hand positioning error will be represented by a displacement vector **v** with the components

$$x = X - X^0, \qquad y = Y - Y^0, \qquad z = Z - Z^0, \tag{21}$$

where X^0, Y^0, Z^0 correspond to the desired (nominal) position of the hand and X, Y, Z stand for the actual coordinates of the hand position.

If the joint positioning errors Δq_i are given, the components (21) of the hand positioning error can be calculated from the linearized formulae

$$x = \frac{\partial X}{\partial q_1} \Delta q_1 + \frac{\partial X}{\partial q_2} \Delta q_2 + \dots + \frac{\partial X}{\partial q_n} \Delta q_n,$$

$$y = \frac{\partial Y}{\partial q_1} \Delta q_1 + \frac{\partial Y}{\partial q_2} \Delta q_2 + \dots + \frac{\partial Y}{\partial q_n} \Delta q_n, \tag{22}$$

$$z = \frac{\partial Z}{\partial q_1} \Delta q_1 + \frac{\partial Z}{\partial q_2} \Delta q_2 + \dots + \frac{\partial Z}{\partial q_n} \Delta q_n.$$

These relations may be also represented in the matrix form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial q_1} & \frac{\partial X}{\partial q_2} & \cdots & \frac{\partial X}{\partial q_n} \\ \frac{\partial Y}{\partial q_1} & \frac{\partial Y}{\partial q_2} & \cdots & \frac{\partial Y}{\partial q_n} \\ \frac{\partial Z}{\partial q_1} & \frac{\partial Z}{\partial q_2} & \cdots & \frac{\partial Z}{\partial q_n} \end{bmatrix} \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \\ \vdots \\ \Delta q_n \end{bmatrix} \tag{22a}$$

or simply

$$[\mathbf{v}] = [\mathbf{A}]\ [\Delta\mathbf{q}] \tag{22b}$$

3.1. Analytical procedure of determining the ellipsoids of probability concentration

For three-dimensional distribution of the hand positioning errors the matrix of the second order moments takes the form

$$\left[\lambda_{ij}\right] = \begin{bmatrix} \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{bmatrix}. \tag{23}$$

Elements of the matrix (23) are

$$\lambda_{xx} = \left(\frac{\partial X}{\partial q_1}\right)^2 \sigma_1^2 + \left(\frac{\partial X}{\partial q_2}\right)^2 \sigma_2^2 + \dots + \left(\frac{\partial X}{\partial q_n}\right)^2 \sigma_n^2,$$

$$\lambda_{yy} = \left(\frac{\partial Y}{\partial q_1}\right)^2 \sigma_1^2 + \left(\frac{\partial Y}{\partial q_2}\right)^2 \sigma_2^2 + \dots + \left(\frac{\partial Y}{\partial q_n}\right)^2 \sigma_n^2\ ,$$

$$\lambda_{zz} = \left(\frac{\partial Z}{\partial q_1}\right)^2 \sigma_1^2 + \left(\frac{\partial Z}{\partial q_2}\right)^2 \sigma_2^2 + \dots + \left(\frac{\partial Z}{\partial q_n}\right)^2 \sigma_n^2\ ,$$

$$\lambda_{xy} = \lambda_{yx} = \frac{\partial X}{\partial q_1}\frac{\partial Y}{\partial q_1}\sigma_1^2 + \frac{\partial X}{\partial q_2}\frac{\partial Y}{\partial q_2}\sigma_2^2 + \dots + \frac{\partial X}{\partial q_n}\frac{\partial Y}{\partial q_n}\sigma_n^2. \tag{24}$$

$$\lambda_{xz} = \lambda_{zx} = \frac{\partial X}{\partial q_1}\frac{\partial Z}{\partial q_1}\sigma_1^2 + \frac{\partial X}{\partial q_2}\frac{\partial Z}{\partial q_2}\sigma_2^2 + \dots + \frac{\partial X}{\partial q_n}\frac{\partial Z}{\partial q_n}\sigma_n^2.$$

$$\lambda_{zy} = \lambda_{yz} = \frac{\partial Y}{\partial q_1}\frac{\partial Z}{\partial q_1}\sigma_1^2 + \frac{\partial Y}{\partial q_2}\frac{\partial Z}{\partial q_2}\sigma_2^2 + \dots + \frac{\partial Y}{\partial q_n}\frac{\partial Z}{\partial q_n}\sigma_n^2.$$

The expression for the Gaussian probability density of the three-dimensional distribution of coordinates x_1, x_2, x_3 of the error displacement vector contains in its exponent the expression

$$\sum_{j=1}^{3} \sum_{k=1}^{3} \Lambda_{jk} \cdot x_j \cdot x_k = \Lambda_{11} x_1^2 + \Lambda_{22} x_2^2 + \Lambda_{33} x_3^2 + 2\Lambda_{12} x_1 x_2 + 2\Lambda_{23} x_2 x_3 + 2\Lambda_{13} x_1 x_3, \qquad (25)$$

where the following notations have been used

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z,$$

$$\Lambda_{11} = \Lambda_{xx}, \qquad \Lambda_{22} = \Lambda_{yy}, \qquad \Lambda_{33} = \Lambda_{zz},$$

$$\Lambda_{12} = \Lambda_{21} = \Lambda_{xy} = \Lambda_{yx}, \qquad \Lambda_{23} = \Lambda_{32} = \Lambda_{yz} = \Lambda_{zy},$$

$$\Lambda_{13} = \Lambda_{31} = \Lambda_{xz} = \Lambda_{zx}.$$

Coefficients Λ_{ij} constitute the elements of the inverse matrix $[\lambda_{ij}]^{-1}$

$$\left[\Lambda_{ij} \right] = \left[\lambda_{ij} \right]^{-1} = \begin{bmatrix} \Lambda_{xx} & \Lambda_{xy} & \Lambda_{xz} \\ \Lambda_{yx} & \Lambda_{yy} & \Lambda_{yz} \\ \Lambda_{zx} & \Lambda_{zy} & \Lambda_{zz} \end{bmatrix}. \qquad (26)$$

The expressions for the elements of this matrix can be found by solving with respect to them the systems of linear equations resulting from the definition that the product of these two matrices must be equal to the unit matrix. Thus we can write

$$\begin{bmatrix} \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{bmatrix} \cdot \begin{bmatrix} \Lambda_{xx} & \Lambda_{xy} & \Lambda_{xz} \\ \Lambda_{yx} & \Lambda_{yy} & \Lambda_{yz} \\ \Lambda_{zx} & \Lambda_{zy} & \Lambda_{zz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (27)$$

General expressions for the elements Λ_{ij} are rather long and complex and their usefulness in practical calculations is limited. In most practical cases some elements of matrix (23) are equal to zero and the solution of the system of equations resulting from (27) is much easier that in the general case. This will be demonstrated in the following example of application.

The equation of the ellipsoid of probability concentration takes the form [cf.(25)]

$$\Lambda_{xx}x^2 + \Lambda_{yy}y^2 + \Lambda_{zz}z^2 + 2\Lambda_{xy}x\,y + 2\Lambda_{yz}y\,z + 2\Lambda_{zx}z\,x = \lambda^2, \tag{28}$$

where $\lambda^2 = \chi^2_p(3)$ depends on the assumed probability that the end point of the vector of error displacement will lie inside the ellipsoid. Thus the ellipsoid (28) may be treated as the ellipsoid of probability concentration.

3.2. Approximate procedure of determining the ellipsoids of probability concentration

The approximate procedure described in this Section has been used in the previous paper [3]. Similarly as in two-dimensional problem (Section 2.2) we shall at first assume that it is limited by two extreme values $\pm\kappa\sigma_i$, where σ_i is the standard deviation of the actual Gaussian distribution of joint positioning errors. The multiplier κ should be appropriately chosen in order to obtain the assumed probability that the end point of the vector of hand positioning error will lie inside the ellipsoid of probability concentration.

It has been shown in [7] that when the joint positioning errors vary within certain limits (joint positioning tolerances), then the end points of all the vectors of the error displacement from the desired position of the hand will lie inside a certain polyhedron bounded by a family of pairs of parallel planes. The equations of these planes are

$$\begin{vmatrix} \frac{\partial Y}{\partial q_r} & \frac{\partial Y}{\partial q_s} \\ \frac{\partial Z}{\partial q_r} & \frac{\partial Z}{\partial q_s} \end{vmatrix} x + \begin{vmatrix} \frac{\partial Z}{\partial q_r} & \frac{\partial Z}{\partial q_s} \\ \frac{\partial X}{\partial q_r} & \frac{\partial X}{\partial q_s} \end{vmatrix} y + \begin{vmatrix} \frac{\partial X}{\partial q_r} & \frac{\partial X}{\partial q_s} \\ \frac{\partial Y}{\partial q_r} & \frac{\partial Y}{\partial q_s} \end{vmatrix} z = \sum_{i=1}^{n} \begin{vmatrix} \frac{\partial X}{\partial q_r} & \frac{\partial X}{\partial q_s} & \frac{\partial X}{\partial q_i} \\ \frac{\partial Y}{\partial q_r} & \frac{\partial Y}{\partial q_s} & \frac{\partial Y}{\partial q_i} \\ \frac{\partial Z}{\partial q_r} & \frac{\partial Z}{\partial q_s} & \frac{\partial Z}{\partial q_i} \end{vmatrix} \Delta q_i. \tag{29}$$

The end point of the vector of hand positioning error moves along one of such planes when two joint positioning errors Δq_r and Δq_s change, while all the remaining joint positioning errors are kept constant.

Two extreme positions of the planes belonging to one family we shall obtain by taking appropriately the extreme values of joint positioning errors $\Delta q_i^+ = +\kappa\sigma_i$ or $\Delta q_i^- = -\kappa\sigma_i$. Taking consecutively all the possible combinations of pairs of joint positioning errors Δq_r and Δq_s as changing parameters we obtain equations of various families of parallel planes and then their extreme positions forming the faces of the polyhedron of the positioning accuracy.

The polyhedron calculated for $\kappa=\sqrt{3}$ corresponds to the probability $p=1$. For any $\kappa<\sqrt{3}$ the probability p may be approximately calculated as the ratio of volumes of the polyhedron obtained by assuming $\Delta q_i=\pm\sqrt{3}\sigma_i$ and of the polyhedron calculated for $\Delta q_i = \pm\kappa\sigma_i$. Finally we obtain that for the desired probability p the multiplier is equal to $\kappa=\sqrt{3}\ \sqrt[3]{p}$. For example, for a particular value of probability $p=0.683$ we have $\kappa=1.525$.

Having found the polyhedron in the space of hand positioning errors we can calculate its second order moments (volume inertia moments) with respect to the reference axes x, y, z, and then find the orientation of its principal axes 1, 2, 3 and second order principal moments J_1, J_2, J_3.

Now the ellipsoid of probability concentration can be found, as the ellipsoid of the same principal axes and second order principal moments as the polyhedron. The principal radii a, b, c of the ellipsoid can be calculated by solving the system of equations

$$\frac{4}{15}\pi\, a\, b^3 c = J_1, \qquad \frac{4}{15}\pi\, a^3 b\, c = J_2, \qquad \frac{4}{15}\pi\, a\, b\, c^3 = J_3, \tag{30}$$

where J_1, J_2, J_3 are the second order principal moments calculated for the polyhedron.

3.3. Example of application

As a working example we shall determine the positioning accuracy ellipsoid for the hand of a simple *4-R* manipulator with four revolute kinematic pairs shown in Fig.8. Joint positions are determined by three positioning angles θ_1, θ_2, θ_3. Position of the joint with the axis 4-4 has no influence on the position of the hand.

The position of the hand is in the basic reference system X, Y, Z determined by the coordinates

$$X = [\, l_1 \cos\theta_2 + l_2 \cos(\theta_2+\theta_3)\,] \cos\theta_1,$$

$$Y = l_1 \sin\theta_2 + l_2 \sin(\theta_2+\theta_3), \tag{31}$$

$$Z = [\, l_1 \cos\theta_2 + l_2 \cos(\theta_2+\theta_3)\,] \sin\theta_1.$$

These expressions represent certain functions of three random independent variables θ_1, θ_2, θ_3. Linear dimensions l_1 and l_2 play the role of constant parameters. Thus for the problem in question general relations (20) are written in the particular form (31).

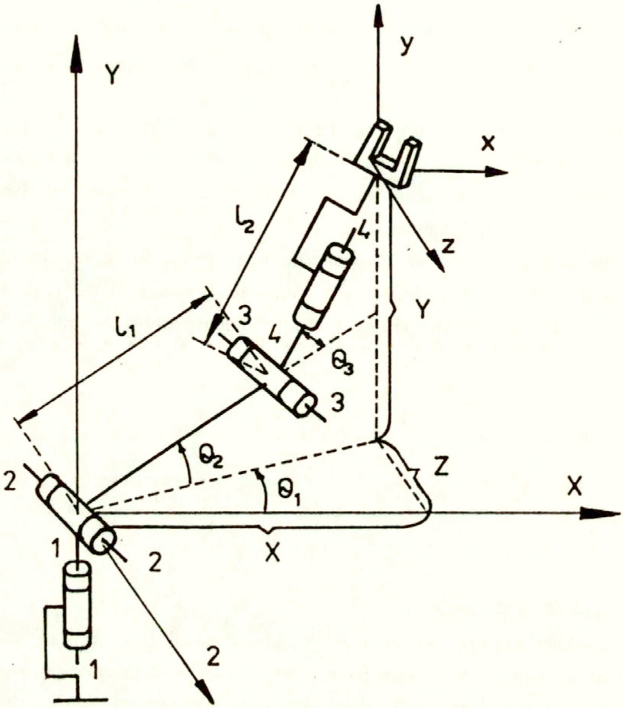

Fig.8

We shall calculate the numerical example for the following data

$$l_1 = l_2 = l = 1000\ mm. \tag{32}$$

Standard deviations for the Gaussian distribution of joint positioning errors are taken to be

$$\sigma_{\theta_1} = \sigma_{\theta_2} = \sigma_{\theta_3} = \sigma_{\theta} = 0.01 \; rad \tag{33}$$

They are taken deliberately large in order to demonstrate that both procedures discussed here may be used in a wide range of joint positioning errors similarly as in the case of the two-dimensional problem discussed in Section 2.

The ellipsoid of probability concentration will be calculated for a certain position of the manipulator determined by the following joint positioning angles

$$\theta_1 = 0, \qquad \theta_2 = 0, \qquad \theta_3 = \frac{1}{2}\pi. \tag{34}$$

The partial derivatives of functions (31) have in this case the following values

i	1	2	3
$\frac{\partial X}{\partial q_i}$	0	$-l$	$-l$
$\frac{\partial Y}{\partial q_i}$	0	l	0
$\frac{\partial Z}{\partial q_i}$	l	0	0

(35)

Analytical procedure

Using formulae (24) we obtain the values of the elements of the matrix of second order moments

$$\lambda_{xx} = \left(\frac{\partial X}{\partial \theta_1}\right)^2 \sigma^2_{\theta_1} + \left(\frac{\partial X}{\partial \theta_2}\right)^2 \sigma^2_{\theta_2} + \left(\frac{\partial X}{\partial \theta_3}\right)^2 \sigma^2_{\theta_3} = 2\, l^2 \sigma^2_{\theta}$$

$$\lambda_{yy} = \left(\frac{\partial Y}{\partial \theta_1}\right)^2 \sigma^2_{\theta_1} + \left(\frac{\partial Y}{\partial \theta_2}\right)^2 \sigma^2_{\theta_2} + \left(\frac{\partial Y}{\partial \theta_3}\right)^2 \sigma^2_{\theta_3} = l^2 \sigma^2_{\theta} \, ,$$

$$\lambda_{zz} = \left(\frac{\partial Z}{\partial \theta_1}\right)^2 \sigma^2_{\theta_1} + \left(\frac{\partial Z}{\partial \theta_2}\right)^2 \sigma^2_{\theta_2} + \left(\frac{\partial Z}{\partial \theta_3}\right)^2 \sigma^2_{\theta_3} = l^2 \sigma^2_{\theta},$$

$$\lambda_{xy} = \lambda_{yx} = \frac{\partial X}{\partial \theta_1}\frac{\partial Y}{\partial \theta_1} \sigma^2_{\theta_1} + \frac{\partial X}{\partial \theta_2}\frac{\partial Y}{\partial \theta_2} \sigma^2_{\theta_2} + \frac{\partial X}{\partial \theta_3}\frac{\partial Y}{\partial \theta_3} \sigma^2_{\theta_3} = -\, l^2 \sigma^2_{\theta},$$

$$\lambda_{xz} = \lambda_{zx} = \frac{\partial X}{\partial \theta_1}\frac{\partial Z}{\partial \theta_1}\sigma^2_{\theta_1} + \frac{\partial X}{\partial \theta_2}\frac{\partial Z}{\partial \theta_2}\sigma^2_{\theta_2} + \frac{\partial X}{\partial \theta_3}\frac{\partial Z}{\partial \theta_3}\sigma^2_{\theta_3} = 0,$$

$$\lambda_{zy} = \lambda_{yz} = \frac{\partial Y}{\partial \theta_1}\frac{\partial Z}{\partial \theta_1}\sigma^2_{\theta_1} + \frac{\partial Y}{\partial \theta_2}\frac{\partial Z}{\partial \theta_2}\sigma^2_{\theta_2} + \frac{\partial Y}{\partial \theta_3}\frac{\partial Z}{\partial \theta_3}\sigma^2_{\theta_3} = 0.$$

Thus the matrix (23) has now the particular form

$$[\lambda_{ij}]^{-1} = \begin{bmatrix} 2\, l^2 \sigma^2_\theta & -\, l^2 \sigma^2_\theta & 0 \\ -\, l^2 \sigma^2_\theta & l^2 \sigma^2_\theta & 0 \\ 0 & 0 & l^2 \sigma^2_\theta \end{bmatrix}.$$

The inverse matrix $[\Lambda_{ij}] = [\lambda_{ij}]^{-1}$ calculated according to the equation (27) is

$$[\Lambda_{ij}] = \begin{bmatrix} \Lambda_{xx} & \Lambda_{xy} & \Lambda_{xz} \\ \Lambda_{yx} & \Lambda_{yy} & \Lambda_{yz} \\ \Lambda_{zx} & \Lambda_{zy} & \Lambda_{zz} \end{bmatrix} = \begin{bmatrix} \frac{1}{l^2 \sigma^2_\theta} & \frac{1}{l^2 \sigma^2_\theta} & 0 \\ \frac{1}{l^2 \sigma^2_\theta} & \frac{2}{l^2 \sigma^2_\theta} & 0 \\ 0 & 0 & \frac{1}{l^2 \sigma^2_\theta} \end{bmatrix}.$$

Now assuming that the probability that the end point of the error displacement vector lies inside the ellipsoid is p = 0.683 we find from the tables that $\chi^2_p(3) = \lambda^2 = 0.35$. By substituting all the magnitudes to (28) we find the equation of the ellipsoid of probability concentration

$$x^2 + 2\,x\,y + 2\,y^2 + z^2 = l^2\,\sigma^2_\theta\,\lambda^2 = 350. \qquad (36)$$

The ellipsoid is shown in Fig.9. Its longest axis makes the angle $\gamma = 31^0 43'$ with the x-axis.

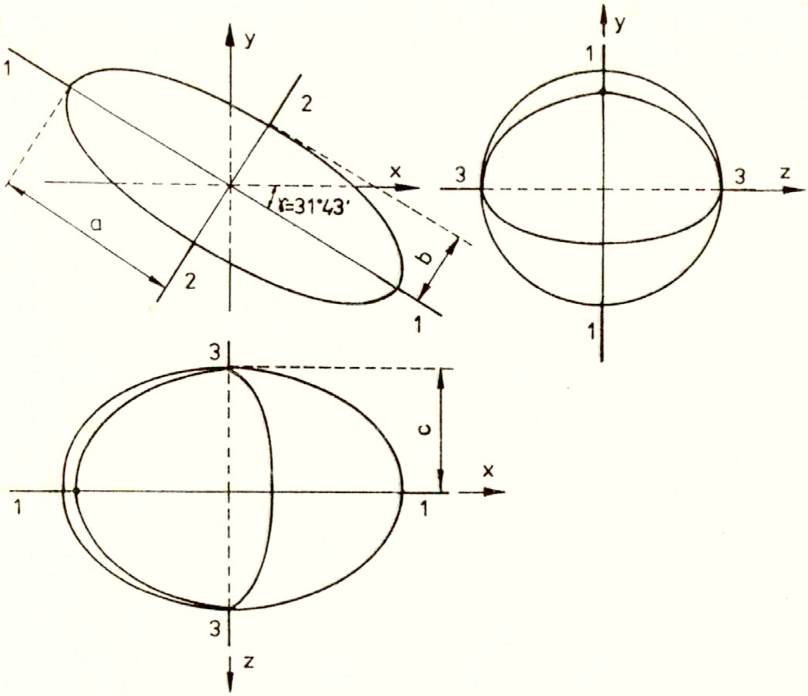

Fig.9

The principal radii of the ellipsoid are

$$a = 30.27\ mm, \qquad b = 11.56\ mm, \qquad c = 18.71\ mm. \tag{37}$$

The elongation of the ellipsoid may be estimated by the following parameters

$$\frac{a}{b} = 2.618, \qquad \frac{a}{c} = 1.618, \qquad \frac{b}{c} = 0.618.$$

Approximate procedure

The polyhedron of the positioning accuracy is bounded by the extreme positions of three pairs of planes determined by the general equation (29). Equations of the particular families of planes we shall determine by substituting to (29) appropriate values of the partial derivatives for all possible combinations of pairs $(r;s)$. For example when $\Delta\theta_1$ and $\Delta\theta_2$ are changing, the equation (29) has the following particular form

$$\begin{vmatrix} \frac{\partial Y}{\partial\theta_1} & \frac{\partial Y}{\partial\theta_2} \\ \frac{\partial Z}{\partial\theta_1} & \frac{\partial Z}{\partial\theta_2} \end{vmatrix} x + \begin{vmatrix} \frac{\partial Z}{\partial\theta_1} & \frac{\partial Z}{\partial\theta_2} \\ \frac{\partial X}{\partial\theta_1} & \frac{\partial X}{\partial\theta_2} \end{vmatrix} y + \begin{vmatrix} \frac{\partial X}{\partial\theta_1} & \frac{\partial X}{\partial\theta_2} \\ \frac{\partial Y}{\partial\theta_1} & \frac{\partial Y}{\partial\theta_2} \end{vmatrix} z = \begin{vmatrix} \frac{\partial X}{\partial\theta_1} & \frac{\partial X}{\partial\theta_2} & \frac{\partial X}{\partial\theta_3} \\ \frac{\partial Y}{\partial\theta_1} & \frac{\partial Y}{\partial\theta_2} & \frac{\partial Y}{\partial\theta_3} \\ \frac{\partial Z}{\partial\theta_1} & \frac{\partial Z}{\partial\theta_2} & \frac{\partial Z}{\partial\theta_3} \end{vmatrix} \Delta\theta_3.$$

Substituting the values of derivatives given in the table (35) we obtain the equation of the first family of planes

$$\begin{vmatrix} 0 & l \\ l & 0 \end{vmatrix} x + \begin{vmatrix} l & 0 \\ 0 & -l \end{vmatrix} y + \begin{vmatrix} 0 & -l \\ 0 & l \end{vmatrix} z = \begin{vmatrix} 0 & -l & -l \\ 0 & l & 0 \\ l & 0 & 0 \end{vmatrix} \Delta\theta_3.$$

Determining the dimensions of the polygon we shall assume [cf.(33)]

$$\Delta\theta_1 = \Delta\theta_2 = \Delta\theta_3 = \pm\,\kappa\,\sigma_\theta = \pm\,0.01525\ rad.$$

Here the probability $p = 0.683$ for the ellipsoid of probability concentration the approximate value $\kappa = \sqrt{3}\,\sqrt[3]{p} = 1.525$ has been assumed.

Repeating the procedure described above for all pairs of changing positioning errors $\Delta\theta_r$ and $\Delta\theta_s$ we arrive at the equations of extreme positions of planes forming the faces of the polyhedron. Equations of these planes are

$$x + y = \pm\,15.25\ mm,$$

$$y = \pm\,15.25\ mm, \tag{38}$$

$$z = \pm\,15.25\ mm,$$

The polyhedron is shown in Fig.10.

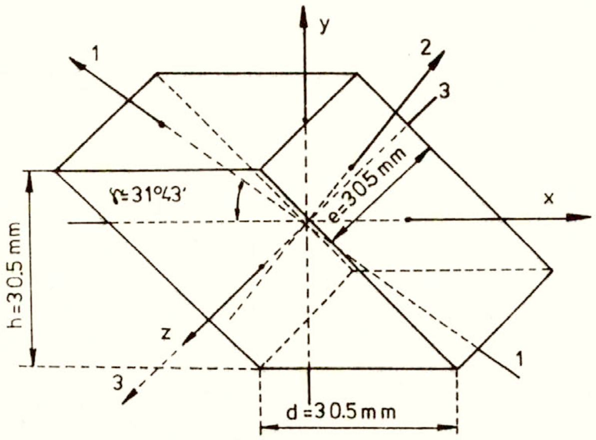

Fig.10.

The second order moments of the polyhedron with respect to the reference planes are

$$J_{xx} = \frac{1}{6}\, d^3 h\, e = 4.399 \cdot 10^6\ mm^5,$$

$$J_{yy} = \frac{1}{12}\, d\, h^3 e = 2.198 \cdot 10^6\ mm^5,$$

$$J_{zz} = \frac{1}{12}\, d\, h\, e^3 = 2.198 \cdot 10^6\ mm^5.$$

The mixed second order moment with respect to the planes x, z and y, z is

$$J_{x \cdot y} = -\frac{1}{12}\, d^2 h^2 e = -\ 2.198 \cdot 10^6\ mm^5,$$

The angle γ which makes the principal axis 1 of the polyhedron with the x-axis we obtain from the relation

$$tan\ 2\gamma = \frac{2\ J_{x\text{-}y}}{J_{xx} - J_{yy}} = -\ 2.$$

Thus we obtain the same value $\gamma = 31^0 43'$ as that of the angle of inclination of the principal axis of the ellipsoid shown in Fig.9.

The principal second order moments of the polyhedron are

$$J_1 = \frac{1}{2}\left(J_{xx} + J_{yy} \right) - \sqrt{\frac{1}{4}\left(J_{xx} - J_{yy} \right)^2 + J_{x\text{-}y}^2} = 0.842 \cdot 10^6\ mm^5,$$

$$J_2 = \frac{1}{2}\left(J_{xx} + J_{yy} \right) + \sqrt{\frac{1}{4}\left(J_{xx} - J_{yy} \right)^2 + J_{x\text{-}y}^2} = 5.756 \cdot 10^6\ mm^5,$$

$$J_3 = J_{zz} = 2.198 \cdot 10^6\ mm^5.$$

By solving the system of equations (30) we find the values of the principal radii of the ellipsoid of probability concentration

$$a = 31.11\ mm, \qquad b = 11.88\ mm, \qquad c = 19.22\ mm.$$

The elongation parameters of the ellipsoid are

$$\frac{a}{b} = 2.618, \qquad \frac{a}{c} = 1.618, \qquad \frac{b}{c} = 0.618.$$

Thus the ellipsoids calculated with the use of various procedures are geometrically similar, their dimensions being only slightly different.

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References

1. A.Kumar and K.J.Waldron, Numerical plotting of surfaces of positioning accuracy of manipulators, Mech.Mach.Theory, vol.16, No.4, 361-368, 1981.

2. A.Antshev, B.Bekiarov and L.Litov, Accuracy characteristics of mechanical systems with tree-like structure (in Russian), Advances in Mechanics, vol.11, No.1/2, 11-41, 1988.

3. W.Szczepiński and Z.Wesołowski, On determining the ellipses and ellipsoids of the positioning accuracy of robot manipulators, (in preparation).

4. G.A.Korn, Th.M.Korn, Mathematical Handbook for Scientists and Engineers, McGraw-Hill, New York 1968.

5. W.Szczepiński, On a certain method of determining tolerance field, (in Polish), Archiwum Budowy Maszyn, vol.2, No.3, 275-284, 1955.

6. G.Cramér, Mathematical Methods of Statistics, Princeton University Press 1946.

7. W.Szczepiński, Theory of polyhedrons of positioning accuracy of manipulators, Mech.Mach.Theory, vol.26, No.7, 697-709, 1991.