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SOLVABILITY OF THE LEAKAGE PROBLEM FOR THE HYDRODYNAMIC EULER EQUATIONS IN SOBOLEV SPACES

Contents

Part 1

1. Introduction

The present paper is devoted to mathematical analysis of the leakage problem as to one of the still outstanding fundamental problems of fluid mechanics.

Investigations of the existence and uniqueness of solutions for the initial boundary value problem for the Euler equations of an incompressible fluid started in the twentieth years of our century. N.M.Günter [Gü,1] as the first proved the uniqueness and existence of local solutions for the Cauchy problem in domain $\Omega = \mathbb{R}^3$ and later [Gü,2] for the initial boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^3$ moving with the fluid. In proofs of the above theorems he used the method of successive approximations. A few years later this problem was proved by L.Lichtenstein [Li,1]. In the two-dimensional case W.Wolibner [W,1] and most recently T.Kato [Ka,1] and V.I.Yudovich [Y,1] proved the existence of global in time solutions. In the three-dimensional case, for $\Omega = \mathbb{R}^3$, using the method of semigroups, T.Kato [Ka,2] proved the existence of local solutions. The solutions of Euler equations are obtained as the limit of vanishing viscosity solutions of the Navier-Stokes equations, as was used also by K.K.Golovkin [Go,1]. The uniqueness and existence of local solutions in a bounded domain $\Omega \subset \mathbb{R}^3$ with a vanishing normal component of velocity vector on the boundary was proved by D.Ebin and J.Marsden [E,1] using technics of the Riemann geometry on infinite dimensional manifolds and by J.P.Bourguignon and H.Brezis [Bo,1], who gave an alternate proof of the local existence, more analytical in character but relying still on geometrical technics. The same problem was also considered by O.A.Ladyzhenskaya [La,1] and

R.Temam [T,1] ,who used the Galerkin method.The two-dimensional flow in a nonsimply connected domain with nonvanishing normal component of the velocity vector on the boundary was considered by V.I.Yudovich [Y,2] .The domain had to be either nonsimply connected or with corners,because in the proof of the existence he had to assume that the normal component of the velocity on the boundary was nonvanishing.Yudovich considered only the first domains.The three-dimensional leakage problem (when the normal component of the velocity vector on the boundary is nonvanishing) was considered in [Ko,1] .This paper was edited by Dolidze from Kochin's notebook.However,the results presented in [Ko,1] are not correct since the vorticity ($\omega=$ $=\mathrm{rot}\,v$) is assumed to be arbitrary on the inflow part S_1 of the boundary .In that case the problem is overdetermined, because it appeared [Za,1] that boundary conditions must fulfill certain additional restrictions.Recently also Kazhikhov [Kaz,1,2,3,4] obtained the following results for the leakage problem.Let S_2 be a part of the boundary through which the fluid leaves the domain Ω and S_0 be another part of the boundary where the normal component of the velocity vector is equal to zero.In [Kaz,1] Kazhikhov proved the existence of solutions of the leakage problem in the two-dimensional case,where the velocity vector is given on S_1 and the pressure on S_2 . Moreover,between S_0 and S_i ,$i=1,2$,there are angles equal to $\frac{\pi}{2}$ .The existence was proved in the Hölder spaces.In papers [Kaz,2,3] the leakage problem for the three-dimensional case is formulated,where in addition to the normal component of velocity on the boundary the tangent components of vorticity vector on S_1 are given.Similarly as in [Kaz,1] ,the domain Ω has dihedral angles equal to $\frac{\pi}{2}$ between S_0 and S_i ,$i=1,2$.

Moreover, theorems of the existence of solutions of these problems (without proofs) are formulated in Hölder spaces. In the last paper [Kaz,4] it was shown that the problem with a given velocity vector on S_1 and a normal component of velocity on S_2 in a nonsimply connected domain $\Omega \subset \mathbb{R}^3$, where $S_1 \cap S_2 = \phi$, is well posed. The suitable method of successive approximations is constructed and the theorem about the existence of solutions of this problem in Hölder spaces is formulated. Simultaneously with [Kaz,1] the existence of solutions of the leakage problem in simply connected domain with a smooth boundary for a given tangent vorticity vector on S_1 was proved in Sobolev spaces and published by the present author [Za,1] . Moreover, the same problem in a domain with dihedral angles π/n was considered in [Za,3] . The existence of solutions of leakage problems in the two-dimensional domain with corners was proved in Sobolev spaces in [Za,6] . In [Za,2] the leakage problem in a domain with dihedral angles equal to π/n for a given velocity vector on S_1 and the pressure on S_2 in Sobolev spaces was considered.

The initial boundary value problem for the Euler equations in the case of a compressible perfect fluid was not so extensively treated as the previous incompressible case. First of all the uniqueness problem was considered by Graffi [Gr,1] and Serrin [Se,1] . T. Kato [Ka,3,4] proved the existence and uniqueness of solutions of the Cauchy problem for the Euler equations. These equations constitute a quasilinear hyperbolic symmetric system and the general result obtained for such systems was used. Recently the initial boundary value problem was considered in the case of vanishing normal component of the velocity vector on the boundary in the three-dimensional case [Ve,1,2,3] , [E,2] . In [Ve,1] results of a mixed problem for a hyperbolic equation of second order [Mi,1,2] , [Sa,1] are used.

The C^{∞} smoothness of solutions of the Euler equations with vanishing normal component of the velocity vector on the boundary for the incompressible and compressible fluid was shown in [T,2] , [Ve,4] ,respectively.

In Part 1 of this paper we consider the nonstationary leakage problem for the motion of incompressible and compressible fluids described by the Euler equations in bounded domains.By a leakage problem we understand a flow in a domain when a fluid enters the domain through one part of the boundary and leaves or does not leave it through another one (denoted in Section 2 by S_1 ,S_2 and S_0 ,respectively) .Moreover,the domains with a part of the boundary on which the normal component of the velocity vector vanishes are also considered.The boundary of simply connected domains with smooth boundary or with edges for the leakage problem must possess a part S_0 .We consider the two-dimensional and the three-dimensional domains. For illustration we show some typical types of the 3-dim. domains which will be considered

Fig.1

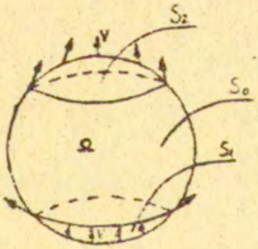


Fig.2

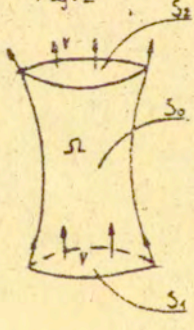
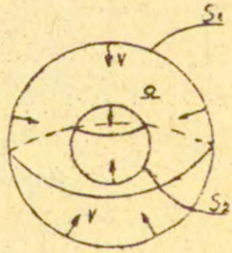


Fig.3



and similarly in the two-dimensional case.

To solve the leakage problem for the Euler equations we look at this problem with respect to the following properties of the fluid and thedomain:

- the compressibility of the fluid
- the connectedness of the domain
- the dimension of the domain
- the smoothness of the boundaries.

The domains with edges (in the three-dimensional case) or with corners (in the two-dimensional case) are an idealization of the physical leakage problem for a domain in a pipe or in a shallow river. Such problems are determined by given physical quantities on sections of the pipe or the river at the boundaries of the considered flow region.

To consider these problems in a most uniform way we must choose spaces in which the theorems of existence for most of these problems can be shown. We choose Sobolev spaces in which the following problems can be solved:

- The existence of solutions of the leakage problem for simply connected domains with smooth boundaries (Section 4). From the compatibility conditions for this case it follows that the normal component of the velocity vector must vanish in points of $S_0 \cap S_1$ of the boundary. This makes impossible to prove the existence of solutions in Hölder spaces (because of dividing by the normal component of velocity, see [Kaz, 1, 2, 3, 4], [Y, 2]). The nonvanishing normal component of velocity on the inlet part of the smooth boundary can be obtained only for nonsimply connected domains. Such cases with proofs in Hölder spaces are considered in [Kaz, 4], [Y, 2].

-The existence of solutions of the leakage problem in the two and three-dimensional domains with angles is treated in Section 5 .In this case the elliptic problems (B) ,(E) (Section 3) in domains with edges must be considered,therefore Kondratiev's results about the existence of solutions of elliptic problems in Sobolev spaces for domains with conical points [Kon,1,2],[M,1] must be used.Using these results in Sections D and E the existence of solutions of the boundary value problem in Sobolev spaces for the Laplace equation for the two and three-dimensional domains with edges is proved. The Sobolev spaces are most convenient because the results for elliptic problems in domains with non-smooth boundary are obtained in Sobolev spaces,and the existence of solutions of problems (B),(E) follows from the existence of the generalized solution in $H^1(\Omega)$ .It should be noticed that to consider these problems in the Hölder spaces we must use the existence of solutions of these problems in Sobolev spaces to obtain the Green function which is used to get the required estimates.However,the initial value problem in weighted Hölder spaces for the Dirichlet type problem was considered in papers [M,2,3],[So,4],but the Neumann problem,which occurs in leakage problem was not solved yet.

In the three-dimensional case the existence of solutions $v \in H^l(\Omega)$ of the problem (B),independently,solutions $p \in H^{l+1}(\Omega)$ of the problem (D) we show only if

(1.1) $\frac{\pi}{\alpha_0} > l+1$,and $\frac{\pi}{\alpha_0} > l$,respectively,

where $l \geqslant 1$ and α_0 is the maximal dihedral angle in the domain Ω .But the existence of solutions of the leakage problem (A,B) (Theorem 5.2) and of the leakage problem (D,E) (Theorem 5.4) is shown for $2(l-1) > n = 3$. Therefore (1.1) implies that $\alpha_0 < \frac{\pi}{4}$ for the problem (A,B) and $\alpha_0 < \frac{\pi}{3}$ for (D,E). In the two-dimensional domain the existence of solutions $v \in W_r^l(\Omega)$ of the problem (B), and independently solutions $p \in W_r^{l+1}(\Omega)$ of the problem (D) is proved if

$$(1.2) \qquad \frac{\pi}{\alpha_0} > l+1-\frac{2}{r} ,$$

where $l \geq 1$, $1 < r < \infty$ and α_0 is the maximal angle in the domain. But the existence of solutions of leakage problems (A,B) (Theorem 6.1) and (D,E) (Theorem 6.2) is obtained for $r(l-1) > n = 2$. Therefore (1.2) implies that $\alpha_0 < \frac{\pi}{2}$ for (A,B) and for (D,E).

For elliptic problems (B), (D) we can overcome the above restrictions (1.1), (1.2) on the maximal angle α_0 using the weighted Sobolev spaces $V^l_{r,\mu}(\Omega)$ [Kon,1,2], [M,1,2,3] with the following norm

$$\|u\|_{V^l_{r,\mu}(\Omega)} = \Big(\sum_{|\alpha| \leq l} \int_\Omega \varrho(x)^{r[\mu-(l-|\alpha|)]} |D^\alpha u|^r dx \Big)^{1/r}$$

where $\varrho(x)$ is the distance between the point $x \in \Omega$ and the next of edges, because in this case instead of (1.1) and (1.2) we have [Kon,1], [M,1]:

$$(1.3) \qquad \frac{\pi}{\alpha_0} > l+1-\mu \quad \text{or} \quad \frac{\pi}{\alpha_0} > l+2-\frac{2}{r}-\mu .$$

Hence, for μ sufficiently large we have no restriction on α_0. Moreover, the existence of solutions of the Neumann problem in $V^l_{r,\mu}(\Omega)$, r=2 [So,2] and the Dirichlet problem for an arbitrary $r>1$ [M,1,2,3] is shown.

However, it appears that estimates for solutions of evolution problems (A), (D) can not be obtained in weighted Sobolev spaces. We show this for the problem (A). Let us assume that $v \in C^2(\Omega^T)$, $\omega \in C^2(\Omega^T)$, then (3.8) implies

$$\int_\Omega \varrho^{r(\mu-1)}[\omega_t^s + v\cdot\nabla\omega^s - \omega\cdot\nabla v^s]\omega^s|\omega|^{r-2}dx + \int_\Omega \varrho^{r\mu}[\omega_t^s + v\cdot\nabla\omega^s +$$

(1.4)

$$-\omega\cdot\nabla v^s]_{,x^i}\omega^s_{x^i}|\omega_x|^{r-2}dx = \int_\Omega(\varrho^{r(\mu-1)}F^s\omega^s|\omega|^{r-2} + \varrho^{r\mu}F^s_{x^i}\omega^s_{x^i}|\omega_x|^{r-2})dx$$

We consider the following term of (1.4):

$$\int_\Omega \varrho^{r\mu}v\cdot\nabla\omega^s_{x^i}\omega^s_{x^i}|\omega_x|^{r-2}dx = \frac{1}{r}\int_\Omega \varrho^{r\mu}v\cdot\nabla|\omega_x|^r dx = \frac{1}{r}\int_{\partial\Omega}\varrho^{r\mu}v_n|\omega_x|^r ds - \mu\int_\Omega \varrho^{r\mu-1}v\cdot\nabla\varrho\,|\omega_x|^r dx,$$

where the last term must be estimated by $\max_\Omega \frac{|v\cdot\nabla\varrho|}{\varrho}\int_\Omega \varrho^{2\mu}|\omega_x|^r dx$ which is not bounded because $v\cdot\nabla\varrho$ do not vanish in any neighbourhood of edges.

- The existence of solutions of the leakage problem for a compressible fluid (Section 8). In this case we must prove the existence of solutions of a mixed problem for strictly hyperbolic equations of second order (Section F), which can be shown at the time being in Sobolev spaces $H^s(\Omega)$ only [Miy,1,2], [Sa,1], [Ve,1].

The essential advantage of using the Sobolev spaces is simplicity of obtaining estimates for evolution problems (A), (D), (G). We explain it shortly for the case of the problem (A) in which the method of deducing estimates is borrowed from papers [La,1], [T,1]. Let us assume that we want to obtain an estimate of ω in $W^1_r(\Omega)$. To do this we consider the following expression arising from [3.8]:

$$\int_{\Omega}(\omega_t + v\cdot\nabla\omega - \omega\cdot\nabla v)\cdot\omega|\omega|^{r-2}dx + \int_{\Omega}(\omega_t + v\cdot\nabla\omega - \omega\cdot\nabla v)_{x_i}\cdot\omega_{x_i}|\omega_x|^{r-2}dx = \tag{1.5}$$
$$= \int_{\Omega}(F\cdot\omega|\omega|^{r-2} + F_{x_i}\omega_{x_i}|\omega_x|^{r-2})\,dx\,,$$

where we assumed that v , $\omega \in C^2(\Omega\times[0,T])$ .This expression we rewrite in the form

$$\frac{1}{r}\frac{d}{dt}\int_{\Omega}(|\omega|^r + |\omega_x|^r)\,dx = -\int_{\Omega} v\cdot\nabla\omega_{x_i}\,\omega_{x_i}|\omega_x|^{r-2}dx + \tag{1.6}$$

+ another terms from (1.5).

For $\omega\in W^1_r(\Omega)$ the first term from the right-hand side of (1.6) can be estimated only by the following surface integral $-\int_{\partial\Omega} v\cdot\bar{n}|\omega_x|^r ds$,where $\bar{n}$ is the unit outward vector normal to the boundary.This term is positive for $v\cdot\bar{n}|_{\partial\Omega}<0$, in the part of the boundary where the fluid enters the domain.This term expresses the main difference between cases: $v_n|_{\partial\Omega}=0$ and $v_n|_{\partial\Omega}\neq 0$ .To estimate this term we must calculate the normal derivative of ω from the equation of the problem (A) and use the given function $\omega|_{S_1}$.

In this paper we consider simply connected domains only, but all obtained results are valid also for nonsimply connected domains.

Despite the use of Sobolev spaces the existence of solutions which satisfy the Euler equations in the classical sense is proved.

The assumption that the normal component of the velocity on a inlet part S_1 of the boundary is different from zero implies that we must prescribe additional boundary conditions. It appears that different boundary conditions may be assumed so we obtain different leakage problems.Namely,papers [Ko, 1], [Y,2] suggests that we can prescribe the vorticity vector on S_1 and normal component of the velocity vector on all boundary. In these papers the method of proving the existence of solutions is determined uniquely with respect to given boundary conditions, so instead of the Euler equations the authors consider the following system of two problems: the evolution problem on the vorticity vector and the elliptic problem on the velocity vector. We employ the same technique, see problems (A), (B), Section 3. Similarly, putting $v|_{S_1}$ and $p|_{S_2}$ andas reference functions the proof of the existence of solutions is completed in such a way that we consider the evolution problem on v (the problem (D), Section 3) and the elliptic problem on p (the problem (E), Section 3). In the case $v_n|_{\partial\Omega}=0$ the above two methods of proving the existence of solutions of the Euler equations are equivalent, because they lead to the existence of v from the same spaces and any additional boundary conditions are not required. In the above case the first method was used for instance in papers [Go,1], [Gü,1,2], [Ka,2], [La,1] and the second in [T,1]. The main idea of the above two methods of proving consists in replacing the Euler equations by a set of initial boundary value problems for some systems of differential equations, for which we know the admissable initial-boundary conditions which make the problems well posed. Knowing these boundary conditions we can deduce what boundary conditions are admissable for the leakage problem for the Euler equations.

In general the leakage problem for the Euler equations was not solved definitly because it was not shown that all possible groups of well posed problems replacing Euler equations are obtained. We know only few such problems, mentioned below. Namely for an incompressible and compressible fluid we can replace equations (3.1), (3.2) or (7.1), (7.2), respectively, by the following groups of well posed problems:

(a) An evolution problem for the vorticity vector ω, an elliptic problem for the velocity vector v with $\omega|_{S_1} \in TS_1$ and $v_n|_{\partial\Omega}$ as boundary conditions. This problem is denoted as the problem (A,B,C) (in Section 3).

(b) An evolution problem for the velocity vector, an elliptic problem for the pressure p, with $v|_{S_1}$ and $p|_{S_2}$ as boundary conditions. This problem is denoted as the problem (D,E) (in Section 3).

(c) An evolution problem for the vorticity vector, an elliptic problem for the velocity vector and an elliptic problem for the pressure. Inthis case we assume that $v|_{S_1}$ and $v_n|_{S_2}$ are given. This problem is denoted by (α, β, γ) (in Section 10).

(d) A hyperbolic equation for the density ϱ, an evolution problem for the vorticity vector ω and an elliptic problem for the velocity vector v. Therefore, to make these problems well posed the following boundary conditions must be given: $v_n|_{\partial\Omega}=0$. This problem is denoted by (F,G,H) (in Section 7).

The above problems are equivalent to the basic Euler equations with corresponding boundary conditions.

However, Sobolev spaces are not entirely convenient to formulate theorems about existence of solutions of problems (a), (b), (c), (d). In Section 10 it was shown that the following problems can not be solved in Sobolev spaces:

(a') The problem (A, B, C), where $\omega|_{S_1} \in TS_1$ such that (3.17) is not satisfied and $v_n|_{\partial\Omega}$ are given.

(b') The problem (α, β, γ), where $v|_{S_1}$ and $v_n|_{S_2}$ are given.

This follow from the smoothness property of traces on boundary of functions in $W_p^l(\Omega)$. If $u \in W_p^l(\Omega)$ then $u|_{\partial\Omega} \in W_p^{l-1/p}(\partial\Omega)$, but in apriori estimates of velocity for problems (a'), (b') it is necessary that $u \in W_p^l(\Omega)$ and $u|_{\partial\Omega} \in W_p^l(\partial\Omega)$.

At last Sections 3 and 10 imply that the following problems can be solved in Sobolev spaces:

(1) The problem (A, B), where $\omega|_{S_1} \in TS_1$, satisfying the equation (3.17), and $v_n|_{\partial\Omega}$ are given as boundary conditions.

(2) The problem (D, E).

(3) The problem (F, G, H) in the domain Ω such that $\partial\Omega = S_0$.

Now we shall revue results obtained for problems (1), (2) and (3) depending on the following properties of the boundary of Ω :

(I) nonsimply connected domains with smooth boundary, Fig. 3,

(II) simply connected domains with a smooth boundary, Fig. 1,

(III) simply connected domains with dihedral angles in the three-dimensional case and angles in the two-dimensional case equal to π/n, Fig. 2,

(IV) simply connected domains with dihedral angles ($\Omega \subset \mathbb{R}^3$) and with angles ($\Omega \subset \mathbb{R}^2$) less than $\pi/2$, Fig. 2.

In the case (1), (II) Theorem 4.1 shows that there exists a solution such that $v(t) \in W_r^2(\Omega)$, $\omega(t) \in W_r^1(\Omega)$, $r > n, n=2,3$. The proof of Theorem 4.1 implies that it is not possible to prove the existence of solutions with a smaller smoothness. The existence of solutions of the problem (1), (II) such that $v(t) \in W_r^{l+1}(\Omega)$, $\omega(t) \in W_r^l(\Omega)$, l is an arbitrary natural number, $r > \frac{n}{l}$, was not shown, but it can be proved in the same way as it was done in the proof of Theorem 4.1 for sufficiently smooth initial and boundary values.

The existence of solutions of the problem (1),(I) such, that $v(t)\in W_r^{l+1}(\Omega)$, $\omega(t)\in W_r^l(\Omega)$, and of the problem (2),(I) such that $v(t)\in W_r^{l+1}(\Omega)$, $p(t)\in W_r^{l+2}(\Omega)$, $r>\frac{n}{l}$, $l\geqslant 1$, $n=2,3$ can be proved in the same way as in Theorems 5.2,5.4,6.1,6.2 for domains with dihedral angles equal to π/n.

For the three-dimensional domains with edges the following possibilities were shown:
The existence of solutions of the problem (1),(III) such that $v(t)\in W_r^{l+1}(\Omega)$, $\omega(t)\in W_r^l(\Omega)$, $l\geqslant 1$, $r>\frac{3}{l}$ is proved in Theorem 5.2 and for l=1 in Theorem 5.1. The existence of solutions of the problem (2),(IV) such that $v(t)\in W_r^{l+1}(\Omega)$, $p(t)\in W_r^{l+2}(\Omega)$, $l\geqslant 1$, $r>\frac{3}{l}$ is proved in Theorem 5.4 and for l=1 in Theorem 5.3. The case l=1 is singled out because in this case we can prove, by our method, the existence of a widest class of solutions. The existence of solutions of the problem (1),(IV) such that $v(t)\in H^{l+1}(\Omega)$, $\omega(t)\in H^l(\Omega)$, $l\geqslant 2$, and of the problem (2),(IV) such that $v(t)\in H^{l+1}(\Omega)$, $p(t)\in H^{l+2}(\Omega)$, $l\geqslant 2$, is proved in Theorem 5.2 if $\frac{\pi}{\alpha_0}>l+2$ and in Theoerem 5.4 if $\frac{\pi}{\alpha_0}>l+1$, are satisfied, where α_0 is the maximal dihedral angle in the domain Ω. The solvability of the leakage problem in the two-dimensional case is expressed in Theorems 6.1,6.2 where the existence of solutions such that $v(t)\in W_r^{l+1}(\Omega)$, $\omega(t)\in W_r^l(\Omega)$ and $v(t)\in W_r^{l+1}(\Omega)$, $p(t)\in W_r^{l+2}(\Omega)$, $r>\frac{2}{l}$, $l\geqslant 1$ is shown for problems (1),(III); (1),(IV) and (2),(III); (2),(IV), respectively. For problems (1),(IV) and (2),(IV) we have the following restrictions on angles: $\frac{\pi}{\alpha_0}>2+l-\frac{2}{r}$, where α_0 is the maximal angle in the domain Ω.

In the compressible case the existence of solutions of the problem (3),(I) such that $\varrho(t)\in H^3(\Omega)$, $v(t)\in H^3(\Omega)$, $\omega(t)\in H^2(\Omega)$

(see Theorem 8.1) is shown only.

In Sections 4,5,6,8 we prove the existence of solutions of the considered problems using the weak star compactness of a bounded set in the space $L_\infty(0,T;W_p^l(\Omega))$ .Therefore,the uniqueness of solutions is not shown.Then we prove the uniqueness of solutions of problems $\{(1),(2)\}, \{(I),(II),(III),(IV)\}$ in Section 9.

Similarly as in [T,2], [Ve,4] it can be shown that solutions of problems $\{1,2,3\}\times\{I,II,III,IV\}$ belong to C^∞.

In the two-dimensional case we proved the existence of local solutions only,however the paper [Y,2] suggests a possibility of global solutions.This question for cases considered in this paper is open.

In Part 2 the following main results are obtained:

(α_0) The existence of solutions of problems (B),(E) in a bounded domain $\Omega\subset\mathbb{R}^3$ of the type(III) is proved in spaces $W_p^l(\Omega)$, $l\geq 1, p>1$ (Theorems C.1,C.2).These results are used in proofs of Theorems 5.1,5.2,5.3,5.4.These proofs follow from the possibility of an extension of a function given in a dihedral angle $\frac{\pi}{n}$ on all $\mathbb{R}^3$,by the method of reflection.

(β_0) The existence of solutions of problems (B),(E) in a bounded domain $\Omega\subset\mathbb{R}^3$ of the type(IV) is proved in spaces $H^l(\Omega)$,$l\geq 1$ (Theorems D.2,D.4).These results are used in proofs of Theorems 5.2,5.4.In this case we can not prove the existence of solutions in spaces $W_p^l(\Omega)$ .To do this it is necessary to construct Green's function in a dihedral angle equal to ϑ which could not be found.

(γ_0) The existence of solutions of problems (B),(E) in a bounded domain $\Omega\subset\mathbb{R}^2$ of the type(IV) is proved in spaces $W_p^l(\Omega)$ (Theorem E.2).This result is used to prove Theorems 6.1,6.2.

(δ_0) The existence of solutions of the mixed problem for the strictly hyperbolic equation (7.17) is proved in Sobolev spaces for domains of the kind (I) (Theorem F.1). In this case it must be assumed that the flow is subsonic in a neighbourhood of the boundary. This result is used in Theorem 8.1.

($\varkappa_0$) The existence of solutions of evolution problems (A), (D), (G) is proved in Sobolev spaces in Section A (Lemmas A.1, A.2). These proofs base on construction of solutions of problems (A), (D), (G).

(σ_0) In Section B the trace theorem for functions in Sobolev spaces in a domain with dihedral angles is proved. This result is used in proofs of all theorems from Sections C, D and E.

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2. Notations

We shall introduce the following notations.

We denote by $d_\vartheta=\{x\in\mathbb{R}^2: 0\le\varphi\le\vartheta\}$ an angle equal to $\vartheta\in(0,2\pi)$ on a plane $\mathbb{R}^2$. Let a boundary of d_ϑ consist of two half-lines γ_0 and γ_ϑ. We introduce a coordinate system such that γ_0 is determined by: $x_1\geq 0$, $x_2=0$ and γ_ϑ is leaning on the angle ϑ to γ_0. We mean by r, φ polar coordinates on a plane $\mathbb{R}^2$.

We denote by $\mathcal{D}_\vartheta\subset\mathbb{R}^3$ a dihedral angle $d_\vartheta\times\mathbb{R}^1$ with boundary planes $\Gamma_0=\gamma_0\times\mathbb{R}^1$, $\Gamma_\vartheta=\gamma_\vartheta\times\mathbb{R}^1$, and with an edge $\Gamma=\Gamma_0\cap\Gamma_\vartheta$. We shall choose the coordinates in $\mathcal{D}_\vartheta$: $x=(x',z)$, where $x'=(x_1,x_2)$ describes a point on d_ϑ and $z=$const is the angle d_ϑ.

In Sections 6 and E we assume that $\Omega\subset\mathbb{R}^2$ is a bounded simply connected domain and its boundary consists of four components S_ν, $\nu=1,\dots,4$ of a class C^l such that $\partial\Omega=\bigcup_{\nu=1}^{4}S_\nu$ and $S_2\cap S_4=\phi$, $S_1\cap S_3=\phi$. We denote angles between S_ν by α_k, $k=1,\dots,4$ and $\alpha_0=\max_k\alpha_k$.

In Sections 5,C,D we assume that $\Omega\subset\mathbb{R}^3$ is a bounded simply connected domain with the boundary $\partial\Omega=S_1\cup S_2\cup S_0$, where S_ν, $=0,1,2$ are surfaces of class C^l. There are dihedral angles between S_i and S_0, $i=1,2$. $L_i=S_i\cap S_0$, $i=1,2$, denote edges of Ω. We assume that $S_1\cap S_2=\phi$. Let T_xS_ν, $\nu=0,1,2$, be the space tangent to S_ν at the point x. Between tangent spaces T_xS_i and T_xS_0 at $x\in L_i$, $i=1,2$, there is a dihedral angle $\alpha_i(x)$ and $\alpha_0=\max_{i=1,2}\max_{x\in L_i}\alpha_i(x)$.

In Section 4 we assume that $\Omega\subset\mathbb{R}^n$, $n=2,3$, is a bounded simply connected domain with a smooth boundary of class C^l, which is divided into three components S_0,S_1 and S_2, such that $S_1\cap S_2=\phi$.

In Sections 7,8 we assume that $\Omega \subset \mathbb{R}^n$, n=2,3, is a bounded non-simply connected domain with a smooth boundary of class C^l which consists of at least one part S_1.

In a neighbourhood of S_1 we introduce a curvilinear system of coordinates. In a neighbourhood $\mathcal{U}(p)$ of an arbitrary point $p \in S_1$ weassume the existence of an ortogonal curvilinear system of coordinates $(\tau_1(x), \dots, \tau_{n-1}(x), n(x))$, n=2,3, $x \in \mathcal{U}(p)$, such that $\mathcal{U}(p) \ni x \to (\tau_1(x), \dots, \tau_{n-1}(x), n(x))$ is a C^l mapping. The surface S_1 is determined by the equation $n(x)=0$. Hence for $n(x)=0$, τ_1, $\dots$, τ_{n-1} are coordinates on S_1. Let us denote by $(\bar{\tau}_1, \dots, \bar{\tau}_{n-1}, \bar{n})$ the ortonormal basis corresponding to the coordinate system introduced above such, that for $x \in S_1$, $\bar{\tau}_1(x), \dots, \bar{\tau}_{n-1}(x)$ are vectors tangent to S_1 and $\bar{n}(x)$ is the outward unit vector normal to S_1. The s derivative, s=0,1,...,l, of the transformations $x \to (\tau_1(x), \dots, \tau_{n-1}(x), n(x))$, $x \to (\bar{\tau}_1(x), \dots, \bar{\tau}_{n-1}(x), \bar{n}(x))$ are bounded by constants α_1, α_2, respectively.

Now we shall introduce some spaces.
By $W_p^l(\Omega)$ and $L_p(\Omega)$ we denote the Sobolev space and the space of integrable functions with norms

$$(2.1) \qquad \|u\|_{l,p,\Omega} = \Big(\sum_{|\alpha| \le l} \int_\Omega |D^\alpha u|^p dx \Big)^{1/p},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ is multiindex and $\Omega \subset \mathbb{R}^n$, and

$$(2.2) \qquad \|u\|_{p,\Omega} = \Big(\int_\Omega |u|^p dx \Big)^{1/p},$$

respectively. For functions defined on the boundary $\partial\Omega$ of Ω we introduce the Slobodetzki-Besov space with the norm

$$(2.3)\quad \|u\|_{l-1/p,p,\partial\Omega} = \|u\|_{l-1,p,\partial\Omega} + \Big(\sum_{|\alpha|=l-1}\int_{\partial\Omega}dx'\int_{\partial\Omega}dy'\,\frac{|D^{\alpha}u(x')-D^{\alpha}u(y')|^{p}}{|x'-y'|^{1+(n-1)p}}\Big)^{1/p}.$$

For p=2 the Sobolev space $W_2^l(\Omega)$ we shall denote by $H^l(\Omega)$ and their norms by $\|\ \|_{l,\Omega}$,hence $\|\ \|_{l,2,\Omega}=\|\ \|_{l,\Omega}$,and the norm in $L_2(\Omega)$ by $\|\ \|_{\Omega}$.

Let B be a Banach space,k be a non-negative integer and T be some positive constant.By $L_r^k(0,T;B)$ we denote the Banach space of functions f(t) on [0,T] which have the values in B for fixed $t\in[0,T]$ and whose k derivative with respect to $t\in[0,T]$ is r-summable withrespect to $t\in[0,T]$ in the B-topology We shall denote the norm in the $L_r(0,T;B)$ space by

$$(2.4)\quad \|u\|_{B,r(0,T)} = \Big(\int_0^T \|u(t)\|_B^r\,dt\Big)^{1/r},$$

and for $r=\infty$ we have

$$(2.5)\quad \|u\|_{B,\infty(0,T)} = \sup_{t\in[0,T]}\|u(t)\|_B ,$$

where $\|\ \|_B$ denotes the norm in the Banach space B.Using the above definitions we introduce the following spaces:

$$\Pi_{l,p}^k(\Omega^T) = \bigcap_{i=l}^{k} L_p^{k-i}(0,T;H^i(\Omega))$$

with the norm

$$(2.6)\quad |u|_{k,l,p,\Omega^T} = \sum_{i=l}^{k}\|D_t^{k-i}u\|_{i,\Omega,p(0,T)} ,$$

where k, l, $k\geq l$, are integers, $\Omega^T=\Omega\times[0,T]$ and $1\leq p\leq\infty$;

$$\Pi_{l,p,q}^k(\Omega^T) = \bigcap_{i=l}^{k} L_p^{k-i}(0,T;W_q^i(\Omega))$$

with the norm

$$(2.7)\quad |u|_{k,l,p,q,\Omega^T} = \sum_{i=l}^{k} \| D_t^{k-i} u \|_{i,q,\Omega,p(0,T)} .$$

For p=2 we shall denote the norm (2.6) by $|u|_{k,l,\Omega^T}$ .In the case of the Besov-Slobodetzki spaces instead of spaces (2.6) and (2.7) we define the following spaces

$$\Pi^{k'}_{l,p}(\partial\Omega^T) = \bigcap_{i=l}^{k} L_p^{k-i}(0,T;H^{i-1/2}(\partial\Omega)) \quad , k'=k-1/2 ,$$

and

$$\Pi^{k'}_{l,p,q}(\partial\Omega^T) = \bigcap_{i=l}^{k} L_p^{k-i}(0,T;W_q^{i-1/q}(\partial\Omega)), \quad k'=k-1/q ,$$

where $l \geqslant 1$,with the norms respectively

$$(2.8)\quad |u|_{k',l,p,\partial\Omega^T} = \sum_{i=l}^{k} \| D_t^{k-i} u \|_{i-1/2,\partial\Omega,p(0,T)} ,$$

$$(2.9)\quad |u|_{k',l,p,q,\partial\Omega^T} = \sum_{i=l}^{k} \| D_t^{k-i} u \|_{i-1/q,q,\partial\Omega,p(0,T)} .$$

For the spaces defined by (2.6) $\div$ (2.9) we can formulate a similar result as the trace theorem.

Lemma 2.1

Let $u \in \Pi^{k}_{l,p,q}(\Omega^T)$,$l \geqslant 1$,then $u|_{\partial\Omega} \in \Pi^{k'}_{l,p,q}(\partial\Omega^T)$ and the following estimate is valid

$$(2.10)\quad |u|_{\partial\Omega}|_{k',l,p,q,\partial\Omega^T} \leqslant c|u|_{k,l,p,q,\Omega^T} .$$

The inverse formulation is also valid.Thus let $u \in \Pi^{k'}_{l,p,q}(\partial\Omega^T)$, $l \geqslant 1$,then there exists a function $\tilde{u} \in \Pi^{k}_{l,p,q}(\Omega^T)$ such that

$$(2.11)\quad |\tilde{u}|_{k,l,p,q,\Omega^T} \le c\,|u|'_{k,l,p,q,\partial\Omega^T} .$$

Proof. The proof follows from the known trace theorems for the Sobolev spaces [Ad, 1], [Be, 1].

Moreover, we introduce the Banach space $\Gamma^k_{l,p}(\Omega)$ with the norm

$$(2.12)\quad |u|_{k,l,p,\Omega} = \sum_{i=l}^{k} \|D_t^{k-i}u\|_{i,p,\Omega} ,$$

and for functions on the boundary $\partial\Omega$ we define the space $\Gamma^{k'}_{l,p}(\partial\Omega)$, $l \geqslant 1$, with the following norm

$$(2.13)\quad |u|'_{k,l,p,\partial\Omega} = \sum_{i=l}^{k} \|D_t^{k-i}u\|_{i-1/p,p,\partial\Omega} , \quad k' = k - 1/p .$$

For functions in spaces $\Gamma^k_{l,p}(\Omega)$ and $\Gamma^{k'}_{l,p}(\partial\Omega)$ we can define the lemma similar to Lemma 2.1. For p=2 the norms of spaces $\Gamma^k_{l,2}(\Omega)$ and $\Gamma^{k'}_{l,2}(\partial\Omega)$ we will denote by $|\cdot|_{k,l,\Omega}$ and $|\ |_{k,l,\partial\Omega}$, respectively.

In this paper we will omit the symbol Ω in symbols of norms where it should appear.

Now we introduce some notations and results which will be used in those parts of this paper where problems connected with corners or dihedral angles are considered.

By $V^k_{p,\mu}(G;F)$, $0 \le k$-integer, $1 \le p$-real, $0 \le \mu$-real we denote the completion of functions $u \in C^k(G)$ with respect to the norm

$$(2.14)\quad \|u\|_{V^k_{p,\mu}(G;F)} = \Big(\sum_{0 \le |\alpha| \le k} \int_G |D^\alpha u|^p \varrho(x)^{p[\mu-(k-|\alpha|)]} dx \Big)^{1/p} ,$$

where $\varrho(x)$ =dist(x,F), $G,F \subset \mathbb{R}^n$ with Euclidean norm. Moreover, we shall denote $L_{p,\mu}(G;F) = V^0_{p,\mu}(G;F)$ and $H^k_\mu(G;F) = V^k_{2,\mu}(G;F)$. For functions determined in d_ϑ we introduce the space $\mathcal{E}_\mu(d_\vartheta)$ with the norm

$$(2.15)\qquad \|u\|_{\mathcal{E}^k_\mu(d_\vartheta)} = \left(\sum_{j=0}^{k} |\xi|^{2(k-j)} \|u\|_{H^j_\mu(d_\vartheta;0)} \right)^{1/2},$$

where $\xi \in \mathbb{R}^1$ is a parameter. Using the Parseval identity we get the equality

$$(2.16)\qquad \|u\|^2_{H^k_\mu(\mathcal{D}_\vartheta;F)} = \sum_{s=0}^{k} \int_{-\infty}^{\infty} d\xi \, \|\tilde{u}\|^2_{\mathcal{E}^s_\mu(d_\vartheta)},$$

where $\tilde{u}(x',\xi) = \int_{\mathbb{R}^1} dz\, u(x',z) e^{i\xi z}$ is the Fourier transformate of $u(x',z)$.

We shall use the following Hardy inequality [Be, 1]: if $\int_0^\infty \left|\frac{\partial f}{\partial x}\right|^p x^\alpha dx < \infty$, $\alpha < p-1$, $p \geqslant 1$, $\lim_{x\to 0} f(x) = 0$ then

$$(2.17)\qquad \int_0^\infty |f(x)|^p x^{\alpha-p} dx \leq \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty \left|\frac{\partial f}{\partial x}\right|^p x^\alpha dx .$$

Using the Hardy inequality we formulate a few lemmas.

Lemma 2.2

Let $f(x)$ be a given function for $x > 0$, $x \in \mathbb{R}^1$, such that $\int_0^\infty x^\alpha |f^{(k)}(x)|^p dx < \infty$, $\alpha < p-1$, $\lim_{x\to 0} f^{(i)}(x) = 0$, $i=0,1,\dots,k-1$, $p \geqslant 1$, where $f^{(k)} = \frac{\partial^k f}{\partial x^k}$, then

$$(2.18)\qquad \int_0^\infty x^{\alpha-pk} |f|^p dx \leqslant \left[\frac{p}{(p-1-\alpha)(2p-1-\alpha)\dots(kp-1-\alpha)}\right]^p \int_0^\infty x^\alpha |f^{(k)}(x)|^p dx .$$

Proof. The proof follows from (2.17).

Lemma 2.3

Let $f(x)$ be a given function for $x>0$ such, that $\int\limits_0^\infty x^{\alpha}|f^{(1)}(x)|^p dx < \infty$, $\alpha < p-1$, $p \geq 1$, then $f(x)$ is continuous.

Proof. We consider

$$|f(x_1)-f(x_2)| = \int\limits_{x_1}^{x_2} f^{(1)}(s)\,ds \leq \int\limits_{x_1}^{x_2} |f^{(1)} x^{\frac{\alpha}{p}}| \frac{dx}{x^{\alpha/p}} \leq$$

$$\leq \Big(\int\limits_{x_1}^{x_2} |f^{(1)}|^p x^{\alpha} dx\Big)^{1/p} \Big(\int\limits_{x_1}^{x_2} x^{-q\frac{\alpha}{p}} dx\Big)^{1/q} \leq C\Big[\frac{1}{1-\varepsilon}\big(x_2^{1-\varepsilon}-x_1^{1-\varepsilon}\big)\Big]^{1/q}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\varepsilon=\frac{q\alpha}{p}<1$, because $\alpha<p-1$. This ends the proof.

Lemma 2.4

Let $p>2$, $f\in W_p^l(d_\vartheta)$ and has a compact support, then $f-P_f^{l-1}\in V_{p,0}^l(d_\theta)$ and

$$(2.19)\qquad \|f-P_f^{l-1}\|_{V_{p,0}^l(d_\vartheta)} \leq C\|f\|_{l,p,d_\vartheta} ,$$

where $P_f^{l-1}(x)=\sum\limits_{|\alpha|\leq l-1}\frac{1}{\alpha!}D_{x'}^{\alpha}f|_{x'=0}\,x'^{\alpha}$, α is multiindex.

Proof. If $f\in W_p^l(d_\vartheta)$ then $\int\limits_0^\infty r\left|\frac{\partial^l f}{\partial r^l}\right|^p dr<\infty$ for almost all $\varphi\in[0,\vartheta]$. Then from Lemma 2.3 for $p>2$, derivatives $\frac{\partial^s f}{\partial r^s}$, $s\leq l-1$, are continuous and we can construct the polynomial $P_f^{l-1}(r)=\sum\limits_{s\leq l-1}\frac{1}{s!}\frac{\partial^s f}{\partial r^s}\Big|_{r=0} r^s$. The function $\hat f=f-P_f^{l-1}(x)$ satisfies the assumptions of Lemma 2.2, so we have

$$(2.20)\qquad \int\limits_0^\infty r^{1-ps}\left|\frac{\partial^{l-s}}{\partial r^{l-s}}\hat f\right|^p dr \leq C\int\limits_0^\infty r\left|\frac{\partial^l}{\partial r^l}f\right|^p dr ,$$

where $s\leq l-1$. Integrating (2.20) over φ and summing over all $s=0,1,\dots,l-1$ we obtain (2.19). This ends the proof.

For $p=2$ we shall use the following Hardy inequality [Kon, 1] :

$$(2.21)\qquad \int_0^\infty r^{2\mu-3-2s}\,|f-P_f^s(r)|^2\,dr \leq C\int_0^\infty r^{2\mu-1}\left|\frac{\partial^{s+1}f}{\partial r^{s+1}}\right|^2 dr\,,$$

where the integral on the right-hand side is bounded, $s \geqslant 0$, $0 \leq \mu \leq 1$, and f has a compact support.

Lemma 2.5

Let $f_i \in H^s(d_\vartheta;0)$. Then the problem

$$\Delta' e_i = f_i\,,\quad i=1,2\,,$$

$$(2.22)\qquad e_1\big|_{\gamma_0}=0\,,\quad (e_1\cos\vartheta+e_2\sin\vartheta)\big|_{\gamma_\vartheta}=0\,,$$

$$\frac{\partial e_2}{\partial\varphi}\Big|_{\gamma_0}=0\,,\quad \left(\frac{\partial e_1}{\partial\varphi}\sin\vartheta-\frac{\partial e_2}{\partial\varphi}\cos\vartheta\right)\Big|_{\gamma_\vartheta}=0\,,$$

has a unique solution $e_i \in H^{s+2}(d_\vartheta;0)$, $i=1,2$, if

$$(2.23)\qquad \frac{\pi}{\vartheta} > s+2\,.$$

Proof. For the homogeneous problem (2.22) the poles [Kon, 1] are $\frac{\pi k}{\vartheta}-1$, k is a natural number. Then from [Kon, 1] it follows that (2.23) is the necessary condition for the existence of a solution of (2.22) in $H^{s+2}(d_\vartheta;0)$. Moreover, similarly to [Kon, 1] we can prove the existence. This concludes the proof.

3. Statement of the leakage problem for an incompressible fluid

In this paper we consider the Euler equations for an incompressible fluid in $\Omega^T = \Omega \times [0,T]$:

$$v_t + v^k v_{x^k} + \nabla p = f\,, \tag{3.1}$$

$$\operatorname{div} v = 0\,, \tag{3.2}$$

with initial values

$$v|_{t=0} = a(x)\,, \qquad \operatorname{div} a = 0\,. \tag{3.3}$$

Following [Kaz; 1,2,3,4], [Za; 1,2,3,6] we see that the initial boundary value problem for the Euler equations can be well posed for different boundary conditions. We shall consider the following cases of boundary conditions:

$$\begin{aligned} &\omega|_{S_1} = \chi \quad \text{and} \quad v\cdot\bar{n}|_{\partial\Omega} = b \quad \text{, where} \quad v_n|_{S_1} = b|_{S_1} \equiv b_1 \leqslant 0, \\ &v_n|_{S_2} = b|_{S_2} = b_2 \geqslant 0, \quad v_n|_{S_0} = 0 \quad \text{and} \quad v_n = v\cdot\bar{n}, \quad -b_1 \equiv d \geqslant 0, \end{aligned} \tag{3.4}$$

where $\omega = \operatorname{rot} v$ is a vorticity vector,

$$v|_{S_1} = \eta, \text{ such that } -d \equiv \eta\cdot\bar{n} \leqslant 0, \quad p|_{S_2} = \pi \quad \text{and} \quad v\cdot\bar{n}|_{S_0} = 0, \tag{3.5}$$

and

$$(3.6)\qquad v|_{S_1} = \eta\,,\ \text{such that}\ \ \eta\cdot\bar{n} \leq 0\,,\ v\cdot\bar{n}|_{S_2} = b \geq 0\,,\ v\cdot\bar{n}|_{S_c} = 0\,.$$

From (3.2) and (3.4_2) it follows that

$$(3.7)\qquad \int_{\partial\Omega} b(s)ds = 0\,.$$

To prove that the above initial boundary value problems are well posed we reduce these problems to a few well posed problems,respectively.

3.1 .Boundary conditions (3.4)

Applying the operator rot to (3.1) and (3.3),and using (3.4_1) we arrive at the following problem

$$(3.8)\qquad \omega_t + v^k\omega_{x^k} - \omega^k v_{x^k} = F \equiv \operatorname{rot} f\,,$$

$$(A)\ (3.9)\qquad \omega|_{t=0} = \omega_0 \equiv \operatorname{rot} a\,,$$

$$(3.10)\qquad \omega|_{S_1} = \chi\,.$$

Suppose that ω is a solution of the problem (A) .Then v is determined bythe vorticity ω ,as a solution of the following overdetermined but well posed elliptic problem (B):

$$(B)\qquad \begin{aligned} &\operatorname{rot} v = \omega\,,\\ &\operatorname{div} v = 0\,,\\ &v_n|_{\partial\Omega} = b\,. \end{aligned}$$

Let (ω, v) be a solution of the problem (A, B) then v is a solution of the problem (3.1), (3.2), (3.3), (3.4).

Lemma 3.1

Let $v \in C^{1,\alpha}(\Omega^t)$, $F \in C^{\alpha}(\Omega^t)$ then there exists a unique solution of the problem (A), such that $\omega \in C^{1,\alpha}(\Omega^t)$.

Proof. For a given vector field $v(x,t)$ we construct the characteristic curves of (3.8) determined by the equations

$$\frac{dy}{ds} = v(y(x,t;s),s),$$

(3.11)

$$y(x,t;t) = x,$$

where s is a parameter, $0 \leq s \leq t$. We classify these curves into two disjoined sets, such that

(a) $y(x,t;s) \in \Omega$ for each $s \in [0,t]$,

(b) there exists a moment $t_*(x,t) \in [0,t)$ such that $y(x,t;t_*(x,t)) \in S_1$.

Using the characteristic curves we can rewrite (3.8) in the form

$$(3.12) \quad \frac{d}{ds}\,\omega(y(x,t;s),s) - \omega^k(y(x,t;s),s)\, v_{y_k}(y(x,t;s),s) = F(y(x,t;s),s) .$$

For given characteristic curves (3.11), equations (3.12) consist of a family of ordinary differential equations. The initial values for equations (3.12) on the curves belonging to the set (a) and to the set (b) are determined by (3.9) and (3.10), respectively. Hence the boundary condition (3.10) is necessary to prove the uniqueness of a solution of the problem (A). The results of Section A complete the proof.

In general the problem (A,B) has no solution, therefore we have the result:

Lemma 3.2

Let (ω, v) be a solution of the problem (A, B). If

(3.13) $\chi = \eta + \pi \bar{n}$,

where $\eta \in TS_1$ then π satisfies the linear partial differential equation on S_1:

$$(3.14)\quad \pi_t + \sum_{\mu=1}^{2} \left[v_\mu \pi_{,\tau_\mu} + \pi \left(v_{\mu,\tau_\mu} + v_\mu \operatorname{rot}_n(\bar{n} \times \bar{\tau}_\mu) \right) \right] = \varphi(x', t), \quad x' \in S_1,$$

where $\eta_\mu = \eta \cdot \bar{\tau}_\mu$, $F_n = F \cdot \bar{n}$, $v_\mu = v \cdot \bar{\tau}_\mu$, $\bar{\tau}_1, \bar{\tau}_2, \bar{n}$ is the ortonormal basis determined in a neighbourhood of S_1 and introduced in Section 2 and

$$(3.15)\quad \varphi(x', t) = \sum_{\mu=1}^{2} \left[(\eta_\mu b_1)_{,\tau_\mu} + b_1 \eta_\mu \operatorname{rot}_n(\bar{n} \times \bar{\tau}_\mu) \right] + F_n .$$

Moreover, the initial values on π follow from (3.9):

$$(3.16)\quad \pi\big|_{t=0} = \bar{n} \cdot \operatorname{rot} a\big|_{S_1} = \bar{n} \cdot \omega_0\big|_{S_1} .$$

For $\pi = 0$, instead of (3.14), we have the following restriction on η:

$$(3.17)\quad \varphi(x', t) \equiv \sum_{\mu=1}^{2} \left[(\eta_\mu b_1)_{,\tau_\mu} + b_1 \eta_\mu \operatorname{rot}_n(\bar{n} \times \bar{\tau}_\mu) \right] + F_n = 0 .$$

Proof. We shall demonstrate that the vector $\chi = \eta + \pi \bar{n}$ appearing in (3.10) must be subjected to some restrictions. Applying the operator div to (3.8) we obtain the equation

$$\left(\frac{\partial}{\partial t} + v^k \frac{\partial}{\partial x^k} \right) \operatorname{div} \omega = 0 ,$$

which implies that div ω =const on characteristic curves (3.11). On the curves of the family (a) the initial values imply that div ω =0 because div $\omega|_{t=0}$ =div rot a =0. On the curves of family (b) we have that div ω = div $\omega|_{t=t_*(x,t)}$= div $\omega|_{S_1}$. But the problem (B) implies that div ω =0 in the domain Ω. Therefore we get the following restriction:

$$\text{(3.18)} \qquad \operatorname{div} \omega|_{S_1} = 0 .$$

We shall demonstrate that (3.18) impose a restriction on the vector $\chi = \eta + \pi \bar{n}$. In such a case, because (3.18) contains the derivative normal to the boundary, the equation (3.18) makes sense only on solutions of the problem (A,B). In order to show this we have to rewrite (3.18) more explicitly using the curvilinear system of coordinates introduced in Section 2. Using (3.10) and the curvilinear coordinates in (3.18) we obtain

$$\text{(3.19)} \qquad \sum_{\mu=1}^{2} \left(\eta_{\mu,\tau_\mu} + \eta_\mu \operatorname{div} \bar{\tau}_\mu \right) + \omega_{n,n}|_{S_1} + \pi \operatorname{div} \bar{n} = 0,$$

where $\omega_n = \omega \cdot \bar{n}$, and $\eta_{,\tau}$, $\omega_{,n}$ are derivatives of η and ω in the direction tangent and normal to S_1, respectively. The unknown quantity $\omega_{n,n}|_{S_1}$ can be calculated from (3.8). Indeed, applying (3.10) to the normal component of (3.8) on S_1 we obtain

$$\text{(3.20)} \qquad \begin{aligned} &\pi_t + \sum_{\mu=1}^{2} \left(b_1 \eta_\mu \mathcal{L}_\mu + v_\mu \bar{\tau}_\mu \cdot \nabla \pi - \pi v_\mu \mathcal{L}_\mu - \eta_\mu \bar{\tau}_\mu \cdot \nabla b_1 \right) + b_1 \omega_{n,n}|_{S_1} + \\ &- \pi v_{n,n}|_{S_1} = F_n \qquad \text{on } S_1 \end{aligned}$$

where $v|_{S_1} = v_\mu \bar{\tau}_\mu + b_1 \bar{n}$, $\mathcal{L}_\mu(x) = (\bar{n} \cdot \nabla \bar{\tau}_\mu) \cdot \bar{n}$.

In the new system of curvilinear coordinates, the equation div v =0 takes the following form

$$(3.21)\quad \sum_{\mu=1}^{2} (v_{\mu,\tau_\mu} + v_\mu \operatorname{div} \bar{\tau}_\mu) + v_{n,n}\big|_{S_1} + b_1 \operatorname{div} \bar{n} = 0 .$$

Eliminating the quantities $v_{n,n}|_{S_1}$ and $\omega_{n,n}|_{S_1}$ from (3.19), (3.20) and (3.21) weget the equation (3.14) on Π . This ends the proof.

Lemma 3.3

The equation (3.14) does not depend on an extension of vectors $\bar{\tau}_1(x)$, $\bar{\tau}_2(x)$, $\bar{n}(x)$ on a neighbourhood of S_1.

Proof. To show this we introduce a curvilinear system of coordinates $q_i = q_i(x)$, i=1,2,3, such that determinates of matricies $\frac{\partial(x)}{\partial(q)}$ and $\frac{\partial(q)}{\partial(x)}$ are nonzero, where $x=(x_1,x_2,x_3)$, $q=(q_1,q_2,q_3)$. Using the notations of [Ko,2] we introduce the Lami's coefficients H_i , i=1,2,3,: $x_{q_i} = H_i e_i$, where e_i are unit vectors, and $\nabla q_i = \frac{1}{H_i} e_i$, i=1,2,3, where we assumed that the curvilinear coordinates are ortonormal, i.e. $e_i \cdot e_j = \delta_{ij}$, $i,j=1,2,3$. Moreover, the rotation operator in these coordinates has the form

$$(\operatorname{rot} a)_i = \frac{H_i}{H_1 H_2 H_3} \varepsilon_{ijk}\, \partial_{q_j}(a_k H_k) \quad , \quad i,j,k = 1,2,3.$$

In our case $q_i = \tau_i$, i=1,2, $q_3 = n$, and $a(\tau_1,\tau_2,n) = \bar{n}(\tau_1,\tau_2,n) \times \bar{\tau}_\mu(\tau_1,\tau_2,n)$, therefore $\bar{n}\cdot \operatorname{rot} a = \frac{1}{H_1 H_2}\Big[\frac{\partial}{\partial \tau_1}(a_2 H_2) - \frac{\partial}{\partial \tau_2}(a_1 H_1)\Big]$, hence $\bar{n}\cdot\operatorname{rot}(\bar{n}\times\bar{\tau}_\mu)$ does not depend on any extension on a neighbourhood of S_1. This completes the proof.

Lemma 3.2 implies that the problem (3.14), (3.16) must be added to problems (A), (B). Hence the function Π must be also treated as an unknown function similarly to v and ω .

Therefore it arises the question about well posedness of the problem (3.14) , (3.16) .The following lemma will answer to this question

Lemma 3.4

Let Ω be a domain with smooth boundary.Theproblem (3.14) (3.16) is well posed for $t \in [0, \frac{a_1}{c})$ if

$$a \cdot \bar{\sigma}\big|_{x \in L_1} \geqslant a_1 > 0 , \tag{3.22}$$

where a_1 is a constant, $\bar{\sigma}$ is the unit outward vector normal to L_1 and tangent to $\partial\Omega$,and $|v_t| \leq C$ .If the condition (3.22) is not satisfied the problem (3.14) , (3.16) is not well posed and we must prescribe additionally that

$$\pi\big|_{L_1} = g(x'', t) , \quad \text{for} \quad x'' \in L_1 . \tag{3.23}$$

Let Ω be a domain with edges.Then the problem (3.14), (3.16) is well posed if the dihedral angle between S_1 and S_0 is not less than $\pi/2$ and for the angle less than $\pi/2$ we must add the condition (3.23) and then the problem (3.14), (3.16), (3.23) is well posed.

The problem (3.14), (3.16) or (3.14), (3.16), (3.23) will be denoted by (C).

Proof.To formulate the boundary conditions for the equation (3.14) we introduce the characteristic curves for (3.14), which are generated by the vector field $\tilde{v} = \sum_{\mu=1}^{2} v_\mu \bar{\tau}_\mu |_{S_1}$ tangent to S_1 and can be described as follows

$$\frac{d\bar{\tau}_\mu}{ds} = v_\mu(\bar{\tau}_1(\tau, t; s), \bar{\tau}_2(\tau, t; s); s) ,$$

$$\bar{\tau}_\mu(\tau_1, \tau_2, t; t) = \tau_\mu , \tag{3.24}$$

where $\mu = 1,2$, and s is a parameter, $0 \leqslant s \leqslant t$. Along the characteristic curves (3.24), the equation (3.14) can be written as an ordinary differential equation

$$(3.25) \qquad \frac{d\Pi}{ds} + \sum_{\mu=1}^{2} \left(v_{\mu,\tau_\mu} + v_\mu \operatorname{rot}(\bar{n} \times \bar{\tau}_\mu)\right)\Pi = \Psi(x',t) \quad , x' \in S_1 .$$

Similarly as in the case of the equation (3.8), the direction of $\tilde{v}(x,t)$ in points $x \in \partial S_1 = S_1 \cap S_0 = L_1$ decide about the boundary conditions for (3.14).

For domains with smooth boundary the direction of velocity vector given in the initial moment $a(x)|_{L_1}$ can decide only about the direction of $\tilde{v}(x,t)|_{L_1}$. Therefore if (3.22) is satisfied and $|v_t| \leqslant C$, C is a constant, we have $v \cdot \bar{\delta}|_{x \in L_1} \geqslant a_1 - Ct$, so for $t < \frac{a_1}{C}$ we get $v \cdot \bar{\delta}|_{x \in L_1} > 0$. But if (3.22) is not satisfied we do not know the direction of $\tilde{v}|_{x \in L_1}$ so we must add the condition (3.23). This completes the proof for domains with smooth boundary.

For domains with edges we have the following relation

$$(3.26) \qquad -v(x) \cdot \bar{\delta}(x) = d(x) \operatorname{ctg} \alpha(x) \qquad \text{for } x \in L_1 ,$$

where $\alpha(x)$, $x \in L_1$, is the angle between surfaces S_1 and S_0, and $d(x) = -b_1(x) \geqslant 0$. For $\alpha(x) \geqslant \frac{\pi}{2}$: $v \cdot \bar{\delta} \geqslant 0$ and in this case the problem (3.14), (3.16) is well posed. For $\alpha(x) < \frac{\pi}{2}$ this problem will be well posed if we add the condition (3.23). This ends the proof.

At last we derive some necessary restrictions on the initial and boundary conditions. From (3.3) and (3.4) we get that

$$(3.27) \qquad a \cdot \bar{n}|_{\partial\Omega} = b|_{t=0} .$$

From (3.3), (3.10), (3.13) and (3.16) we have

$$\operatorname{rot} a \cdot \bar{\tau}_{\mu}\big|_{S_1} = \eta \cdot \bar{\tau}_{\mu}\big|_{t=0} , \tag{3.28}$$

where $\mu = 1,2$ and $\bar{\tau}_{\mu}$ are vectors tangent to S_1. Moreover, from (3.8), (3.9), (3.10) and (3.16) we have the following restriction

$$\sum_{\mu=1}^{2} \eta_{\mu,t} \cdot \bar{\tau}_{\mu} + \bar{n} \cdot \omega_0 \bar{n} + a^k \omega_{0,x^k} - \omega_0^k a_{,x^k} = F , \tag{3.29}$$

for $t=0$ and $x \in S_1$. At last from (3.16) and (3.23) we obtain

$$\varrho(x,0) = \omega_0(x) \cdot \bar{n} \tag{3.30}$$

for $x \in L_1$.

3.2 Boundary conditions (3.5)

To show that the set of the initial and boundary conditions (3.3), (3.5) is sufficient to determined the unique solution of equations (3.1), (3.2) we consider the following two problems: the first one, in which we assume that p is a given function

$$v_t + v^k v_{x^k} = -\nabla p + f , \tag{3.31}$$

(D)
$$v\big|_{t=0} = a , \tag{3.32}$$

$$v\big|_{S_1} = \eta \quad , \quad v_n\big|_{S_0} = 0 , \tag{3.33}$$

and the second one, in which we assume that v is a given function

$$\Delta p = \operatorname{div} f - v^k_{x^i} v^i_{x^k}, \tag{3.34}$$

$$p|_{S_2} = \pi(x',t), \quad x' \in S_2, \tag{3.35}$$

The problem (3.34), (3.35) is not well posed unless we determine the missing boundary conditions. This will be subject to the following lemma

Lemma 3.5

Let v be a given function of class $C^{1,\alpha}(\Omega^t)$, $\eta \in C^{1,\alpha}(S_1^t)$, vectors $\bar{\tau}_1$, $\bar{\tau}_2$, $\bar{n}$ belong to C^1 in a neighbourhood of S_1 and the equation (3.2) is satisfied on S_1. Then the initial and boundary conditions (3.3) and (3.5), and the equations (3.1), (3.2) determine the following well posed elliptic problem for p:

$$\Delta p = \operatorname{div} f - v^k_{x^i} v^i_{x^k}, \tag{3.34}$$

$$p|_{S_2} = \pi(x',t), \quad x' \in S_2, \tag{3.35}$$

(E) $$\frac{\partial p}{\partial n}\Big|_{S_1} = f_n|_{S_1} - \eta_{n,t} + \sum_{\mu=1}^{2}\big(\eta_n \eta_{\mu,\tau_\mu} + \eta_n \eta_\mu \operatorname{div} \bar{\tau}_\mu + \eta_n \eta_\mu \bar{\tau}_\mu \cdot \bar{n}_{,n} - \eta_\mu \eta_{n,\tau_\mu}\big) + \sum_{\mu,\nu=1}^{2} \eta_\mu \eta_\nu \bar{\tau}_\nu \cdot \bar{n}_{,\tau_\mu} + \eta_n^2 \operatorname{div} \bar{n} \equiv g(\eta, \bar{\tau}, \bar{n}), \tag{3.36}$$

$$\frac{\partial p}{\partial n}\Big|_{S_0} = - v^k v \cdot \bar{n}_{,x^k} + f \cdot \bar{n}|_{S_0}. \tag{3.37}$$

Proof. The equations (3.36), (3.37) must be derived only. Multiplying (3.1) by $\bar{n}$, projecting the result on S_1 and using the curvilinear coordinates we obtain

$$\frac{\partial p}{\partial n}\Big|_{S_1} = (f\cdot\bar{n} - v_t\cdot\bar{n} - v^k v_{x^k}\cdot\bar{n})\Big|_{S_1} = f_n\Big|_{S_1} - \eta_{n,t} - \eta_n v_{n,n}\Big|_{S_1} + \tag{3.38}$$
$$- \sum_{\mu=1}^{2} \eta_\mu \eta_{n,\tau_\mu} + \eta^k \eta\cdot\bar{n}_{,x^k} ,$$

where $\eta = \sum_{\mu=1}^{2}\eta_\mu \bar{\tau}_\mu + \eta_n \bar{n}$ .In (3.38) there appears an unknown quantity $v_{n,n}|_{S_1}$ .To calculate it we apply the operator div to (3.1) and using (3.34) we obtain

$$\left(\frac{\partial}{\partial t} + v^k \frac{\partial}{\partial x^k}\right) \operatorname{div} v = 0 , \tag{3.39}$$

which implies only that div v =const on characteristic curves (3.11) .However,according to (3.2),div v =0 in Ω .The equation (3.2) is satisfied on characteristic curves of the family (a) (see the proof of Lemma 3.1) because the initial values are such,that div $v|_{t=0}$ =div a =0.The initial values for the curves of the family (b) are given by boundary values on which (3.2) imposes the following restriction

$$\operatorname{div} v\big|_{t=t_*(x,t)} = \operatorname{div} v\big|_{S_1} = 0 . \tag{3.40}$$

Using the curvilinear coordinate system, (3.40) yields

$$v_{n,n}\big|_{S_1} = - \sum_{\mu=1}^{2}(\eta_{\mu,\tau_\mu} + \eta_\mu \operatorname{div}\bar{\tau}_\mu) - \eta_n \operatorname{div}\bar{n} . \tag{3.41}$$

Using (3.41) in (3.38) we get

$$\frac{\partial p}{\partial n}\Big|_{S_1} = f_n\Big|_{S_1} - \eta_{n,t} + \sum_{\mu=1}^{2}(\eta_n \eta_{\mu,\tau_\mu} + \eta_\mu \eta_n \operatorname{div}\bar{\tau}_\mu - \eta_\mu \eta_{n,\tau_\mu}) + \tag{3.42}$$
$$+ \eta_n^2 \operatorname{div}\bar{n} + \eta^k \eta\cdot\bar{n}_{,x^k} ,$$

which implies (3.36). Using that $v \cdot \bar{n}|_{S_0} = 0$, from (3.1) we obtain (3.37). This completes the proof.

Let v, p be a solution of the problem (D,E) then it is easy to see that v, p is a solution of the problem (3.1), (3.2), (3.3), (3.5) also.

The problem (3.1), (3.2), (3.3), (3.6) was considered in [Kaz, 4].

4.The leakage problem in domain with smooth boundary

In this case we can prove the existence of solutions in the case of boundary conditions (3.4) for π equal to zero and η is tangent to S_1.We will prove the theorem:

Theorem 4.1

Let us assume that:

(a) $r > n, n=2,3, r$ is a real number, $\partial\Omega$ is of class C^3, $\Omega \subset \mathbb{R}^n$,

(b) $a \in W_r^2(\Omega)$, $\operatorname{div} a = 0$,

(c) $b \in L_\infty(0,T; W_r^{2-1/r}(\partial\Omega))$, $\int\limits_{\partial\Omega} b(s)\,ds = 0$, $d = -b|_{S_1} \geq 0$,

(d) the following expressions are bounded : $\max\limits_{t\in[0,T]} \max\limits_{S_1} d \leq B_1$,

$\max\limits_{t\in[0,T]} \int\limits_{S_1} d^{1-r}(|\eta|^r + |\eta_t|^r + |\eta_\pi|^r)\,ds \leq B_2$, $\max\limits_{t\in[0,T]} \int\limits_{S_1} d^{1-r}|F|^r\,ds \leq B_3$,

(e) $F \in L_\infty(0,T; W_r^1(\Omega)) \cap L_\infty^1(0,T; L_r(\Omega))$,

where B_i, $i=1,2,3$, are constants. Then for $t\in[0,T]$, where

$$(4.1)\quad T < t^* = \frac{1}{C(B_1,B_2,B_3)}\left[\|a\|_{2,r}^r + \|a\|_{2,r}^{2r} + \|F(0)\|_r^r + 1\right]^{-1}$$

the problem (A, B) has the unique solution such that

$$(4.2)\quad v \in \Pi^2_{1,\infty,r}(\Omega^T), \quad \omega \in \Pi^1_{0,\infty,r}(\Omega^T)$$

Proof.To prove this theorem we use the method of successive approximations.Therefore,instead of problems (A), (B) we consider the following system of problems:

$$(4.3)\quad \overset{m+1}{\omega}_t + \overset{m}{v}{}^k \overset{m+1}{\omega}_{x^k} - \overset{m}{v}_{x^k} \overset{m+1}{\omega}{}^k = F ,$$

$$(A_m)\ (4.4)\quad \overset{m+1}{\omega}\Big|_{t=0} = \omega_0 ,$$

$$(4.5)\quad \overset{m+1}{\omega}\Big|_{S_1} = \eta ,$$

where $\overset{m}{v}$ is a given function; and

$$(B_m)\quad \begin{aligned} &\operatorname{rot} \overset{m}{v} = \overset{m}{\omega} ,\\ &\operatorname{div} \overset{m}{v} = 0 ,\\ (4.6)\quad &\overset{m}{v}\cdot\bar{n}\big|_{\partial\Omega} = b , \end{aligned}$$

where $\overset{m}{\omega}$ is a given function, $m=0,1,\dots,$ and $\overset{0}{\omega} = \omega_0$. To obtain the estimate on $\overset{m}{v}$ and $\overset{m}{\omega}$ we assume that $\overset{m}{v} \in C^3(\Omega^T)$ and $\overset{m}{\omega} \in C^2(\Omega^T)$. From [La, 1] it follows that the problem (B_m) has a unique solution and the following estimate is valid

$$(4.7)\qquad |\overset{m}{v}|_{2,1,r} \le c\left(|\overset{m}{\omega}|_{1,0,r} + |b|_{2,1,r,\partial\Omega} \right) ,$$

where we have assumed that $\overset{m}{\omega} \in W^1_r(\Omega)$. Lemma A.1 implies that the problem (A_m) has a unique solution, therefore we restrict ourselve to obtain an apriori estimate of sequences $\{\overset{m}{v}\}$, $\{\overset{m}{\omega}\}$ by constants independent of m. From (4.3) and its derivatives with respect to x^i, $i=1,2,3,$ and t we obtain the equality

$$\sum_{i+|j|=0}^{1} \sum_{s=1}^{n} \int_\Omega D_t^i D_x^j \Big(\overset{m+1}{\omega}{}^s_t + \overset{m}{v}{}^k \overset{m+1}{\omega}{}^s_{x^k} - \overset{m}{v}{}^s_{x^k} \overset{m+1}{\omega}{}^k + $$
$$ - F^s \Big) D_t^i D_x^j \overset{m+1}{\omega}{}^s \left| D_t^i D_x^j \overset{m+1}{\omega} \right|^{r-2} dx = 0 ,$$

where $j = (j_1, \dots, j_n)$, is multiindex, $n=2,3,$ and $D_x^j = D_{x^1}^{j_1} \dots D_{x^n}^{j_n}$.

This equality leads us to

$$\frac{1}{r}\frac{d}{dt}|\overset{m+1}{\omega}|^{r}_{1,0,r} \leq -\frac{1}{r}\int_{\partial\Omega}\overset{m}{v}\cdot\bar{n}\left(|\overset{m+1}{\omega}|^{r}+|\overset{m+1}{\omega}_{x}|^{r}+|\overset{m+1}{\omega}_{t}|^{r}\right)ds+\max_{\Omega}|\overset{m}{v}_{x}|\,\|\overset{m+1}{\omega}\|^{r}_{r}+$$

$$+2\max_{\Omega}|\overset{m}{v}_{x}|\,\|\overset{m+1}{\omega}\|^{r}_{r}+\max_{\Omega}|\overset{m+1}{\omega}|\,\|\overset{m}{v}_{xx}\|_{r}\,\|\overset{m+1}{\omega}\|^{r-1}_{r}+\max_{\Omega}|\overset{m}{v}|\,\|\overset{m+1}{\omega}_{x}\|_{r}\|\overset{m+1}{\omega}_{t}\|^{r-1}_{r}+$$

$$(4.8)\qquad +\max_{\Omega}|\overset{m}{v}_{x}|\,\|\overset{m+1}{\omega}_{t}\|^{r}_{r}+\max_{\Omega}|\overset{m+1}{\omega}|\,\|\overset{m}{v}_{xt}\|_{r}\|\overset{m+1}{\omega}_{t}\|^{r-1}_{r}+\|F\|_{1,r}\|\overset{m+1}{\omega}\|^{r-1}_{1,r}+$$

$$+\|F_{t}\|_{r}\|\overset{m+1}{\omega}_{t}\|^{r-1}_{r} \leq -\frac{1}{r}\int_{\partial\Omega}\overset{m}{v}\cdot\bar{n}\left(|\overset{m+1}{\omega}|^{r}+|\overset{m+1}{\omega}_{x}|^{r}+|\overset{m+1}{\omega}_{t}|^{r}\right)ds+$$

$$+C|\overset{m}{v}|_{2,1,r}|\overset{m+1}{\omega}|^{r}_{1,0,r}+|F|_{1,0,r}|\overset{m+1}{\omega}|^{r-1}_{1,0,r},$$

where we have used the theorems of imbedding for $r>n, n=2,3$. Using (4.6) and (3.8) we estimate the surface integral appearing in the last inequality as follows:

$$(4.9)\quad -\int_{\partial\Omega}\overset{m}{v}\cdot\bar{n}\left(|\overset{m+1}{\omega}|^{r}+|\overset{m+1}{\omega}_{x}|^{r}+|\overset{m+1}{\omega}_{t}|^{r}\right)ds \leq \int_{S_1}d\left(|\overset{m+1}{\omega}|^{r}+|\overset{m+1}{\omega}_{x}|^{r}+|\overset{m+1}{\omega}_{t}|^{r}\right)ds \equiv I.$$

Using in (4.9) the curvilinear coordinates introduced in Section 2, we obtain

$$I \leq \int_{S_1}d\left(|\overset{m+1}{\omega}|^{r}+|\overset{m+1}{\omega}_{,\tau}|^{r}|\tau_{x}|^{r}+|\overset{m+1}{\omega}_{,n}|^{r}|n_{x}|^{r}+|\overset{m+1}{\omega}_{t}|^{r}\right)ds \leq$$

$$\leq C(\alpha_1)\int_{S_1}d\left(|\overset{m+1}{\omega}|^{r}+|\overset{m+1}{\omega}_{t}|^{r}+|\overset{m+1}{\omega}_{,\tau}|^{r}+|\overset{m+1}{\omega}_{,n}|^{r}\right)ds \equiv I'$$

The normal derivative of $\overset{m+1}{\omega}$, which appears in the above expression, can be calculated from (4.3) as follows:

$$\overset{m+1}{\omega}_{,n}\Big|_{S_1} = d^{-1}\left[\eta_{t}+\overset{m}{v}_{\mu}\eta_{,\tau_{\mu}}-\eta_{\mu}\overset{m}{v}_{,\tau_{\mu}}+F\right].$$

Using this expression for $\overset{m+1}{\omega}_{,n}\big|_{S_1}$ and (4.5) in I' we obtain

$$I' \leq C(\alpha_1) \int_{S_1} d\,[|\eta|^r + |\eta_{,\tau}|^r + |\eta_t|^r + d^{-r}(|\eta_t|^r + |\overset{m}{v}|^r |\eta_{,\tau}|^r + |\overset{m}{v}_{,\tau}|^r |\eta|^r + |F|^r)]\,ds.$$

Therefore, we have

$$(4.10) \qquad I' \leq C_1 + C_2 \|\overset{m}{v}\|^r_{2,r} ,$$

where $C_1 = C_1(\alpha_1, B_1, B_2, B_3)$ and $C_2 = C_2(\alpha_1, B_2)$ are constants. Using (4.10) in (4.8) we obtain that

$$(4.11) \qquad \frac{d}{dt} |\overset{m+1}{\omega}|^r_{1,0,r} \leq C_1 + C_2 \|\overset{m}{v}\|^r_{2,r} + C_3 |\overset{m}{v}|_{2,1,r} |\overset{m+1}{\omega}|^r_{1,0,r} + C_4 |\overset{m+1}{\omega}|^{r-1}_{1,0,r} .$$

From (4.7) and (4.11), assuming that $\overset{m}{y} = |\overset{m}{\omega}|_{1,0,r}$ and using the Hölder inequality, we obtain

$$\frac{d}{dt} \overset{m+1}{y} \leq C_5 + C_6 \overset{m}{y} + (C_7 \overset{m}{y}^{1/r} + C_8) \overset{m+1}{y} \equiv \alpha(\overset{m}{y}) \overset{m+1}{y} + \beta(\overset{m}{y}) ,$$

$$(4.12) \qquad \overset{m+1}{y}\Big|_{t=0} = y(0) = \|\omega_0\|^r_{1,r} + \|-a^k (\mathrm{rot}\, a)_{x^k} + a_{x^k} (\mathrm{rot}\, a)^k + F(0)\|^r_r ,$$

$$\overset{\circ}{y}(t) = y(0) ,$$

where α, β are monotonically increasing functions of their arguments. Using the method of induction we will show that $\overset{m}{y}(t)$ are bounded independently of m. Assuming that $\overset{m}{y}(t) \leq \varrho \overset{\circ}{y}$, $\varrho > 1$ (for m=0 it is evident) we will show that $\overset{m+1}{y}(t) \leq \varrho \overset{\circ}{y}$ for a sufficiently small time interval which depends on ϱ. Indeed, from (4.12) we have

$$(4.13) \qquad \frac{d}{dt} \overset{m+1}{y} \leq \alpha(\overset{m}{y}) \overset{m+1}{y} + \beta(\overset{m}{y}) \leq \alpha(\varrho \overset{\circ}{y}) \overset{m+1}{y} + \beta(\varrho \overset{\circ}{y}) \leq \gamma(\varrho \overset{\circ}{y}) (\overset{m+1}{y} + 1) ,$$

where $\gamma(\varrho \overset{\circ}{y}) = \max\{\alpha(\varrho \overset{\circ}{y}), \beta(\varrho \overset{\circ}{y})\}$.

Integrating (4.13) we obtain $\overset{m+1}{y}(t) \leq (\overset{\circ}{y} + 1) e^{\gamma t} - 1$ and demanding that $\overset{m+1}{y}(t) \leq \varrho \overset{\circ}{y}$ we get the restriction for time t :

$$\leq t'(\varrho) = \frac{1}{C(\varrho \overset{\circ}{y}+1)} \ln \frac{\varrho \overset{\circ}{y}+1}{\overset{\circ}{y}+1}$$

the function $t'(\varrho)$ has a maximum for $\varrho = e + \frac{e-1}{\overset{\circ}{y}}$, so we can define

$$t^* = \max_{\varrho} t'(\varrho) = \frac{1}{Ce(\overset{\circ}{y}+1)} , \tag{4.15}$$

where C is equal to the maximum of constants appearing in (4.12). Therefore, we obtained the following estimate

$$\overset{m}{y}(t) \leq e\overset{\circ}{y} + e - 1 . \tag{4.16}$$

Hence, using the fact $C^{3,2}(\Omega^T)$, $C^2(\Omega^T)$ are dense in $L_\infty(0,T;W_r^2(\Omega))$ and $L_\infty(0,T;W_r^1(\Omega))$, respectively, we have proved that $\sup_{t\in[0,T]} \|\overset{m}{v}(t)\|_{2,r}$, $\sup_{t\in[0,T]} \|\overset{m}{v}_t(t)\|_{1,r}$, $\sup_{t\in[0,T]} \|\overset{m}{\omega}(t)\|_{1,r}$ and $\sup_{t\in[0,T]} \|\overset{m}{\omega}_t(t)\|_r$ are bounded by constants independent of m.
Now we shall prove convergence of the sequence $\{\overset{m}{v}\}$ in the norm $L_\infty(0,T;W_r^1(\Omega))$ and the sequence $\{\overset{m}{\omega}\}$ in the norm of $L_\infty(0,T;L_r(\Omega))$. Introducing

$$\overset{m}{\vartheta} = \overset{m}{v} - \overset{m-1}{v} , \quad \overset{m}{\Omega} = \overset{m}{\omega} - \overset{m-1}{\omega} ,$$

we get from problems (A_m) , (B_m)

$$\begin{aligned} &\overset{m+1}{\Omega}_t + \overset{m}{v}^k \overset{m+1}{\Omega}_{,x^k} + \overset{m}{\vartheta}^k \overset{m}{\omega}_{x^k} - \overset{m+1}{\Omega}^k \overset{m}{v}_{x^k} - \overset{m}{\omega}^k \overset{m}{\vartheta}_{x^k} = 0, \\ &\overset{m+1}{\Omega}\Big|_{t=0} = 0, \ \overset{m+1}{\Omega}\Big|_{S_1} = 0 , \end{aligned} \tag{4.17}$$

and

(4.18) $\quad \operatorname{rot} \overset{m}{\vartheta} = \overset{m}{\Omega}, \ \operatorname{div} \overset{m}{\vartheta} = 0, \ \overset{m}{\vartheta}\cdot\bar{n}\big|_{\partial\Omega} = 0.$

Multiplying (4.17) by $\overset{m+1}{\Omega}|\overset{m+1}{\Omega}|^{r-2}$, integrating the result over Ω, we obtain

$$\frac{1}{r}\frac{d}{dt}\|\overset{m+1}{\Omega}\|^{r}_{r,\Omega} = -\int_{\partial\Omega}\overset{m}{v}\cdot\bar{n}|\overset{m+1}{\Omega}|^{r}ds - \int_{\Omega}\Big(\overset{m}{\vartheta}{}^{k}\overset{m}{\omega}_{x^k} - \overset{m+1}{\Omega}{}^{k}\overset{m}{v}_{x^k} + \tag{4.19}$$
$$- \overset{m}{\omega}{}^{k}\overset{m}{\vartheta}_{x^k}\Big)\cdot\overset{m+1}{\Omega}|\overset{m+1}{\Omega}|^{r-2}dx = 0.$$

Using the boundary conditions, the Hölder inequality and theorems of imbedding, (4.19) implies

(4.20) $\quad \frac{d}{dt}\|\overset{m+1}{\Omega}\|^{r}_{r,\Omega} \le C_1\|\overset{m}{\omega}\|_{1,r,\Omega}\|\overset{m}{\vartheta}\|_{1,r,\Omega}\|\overset{m+1}{\Omega}\|^{r-1}_{r,\Omega} + C_2\|\overset{m}{v}\|_{1,r,\Omega}\|\overset{m+1}{\Omega}\|^{r}_{r,\Omega}.$

Mreover, from (4.18) we have

(4.21) $\quad \|\overset{m}{\vartheta}\|_{1,r,\Omega} \le C\|\overset{m}{\Omega}\|_{r,\Omega}.$

Introducing the notation $\overset{m}{\sigma} = \|\overset{m}{\Omega}\|^{r}_{r,\Omega}$ and using the estimate (4.16), from (4.20) and (4.21) we obtain

(4.22) $\quad \frac{d}{dt}\overset{m+1}{\sigma} \le \alpha\overset{m+1}{\sigma} + \beta\overset{m}{\sigma},$

where $\overset{m}{\sigma}(0) = 0$, $\overset{0}{\sigma} = \|\overset{0}{\Omega}\|^{r}_{r,\Omega} = \|\omega_0\|^{r}_{r,\Omega} \equiv \sigma_0$, α, β are constants. Integrating (4.22) we have $\overset{m}{\sigma} \le e^{\alpha t}\sigma_0\frac{(\beta t)^m}{m!}$, hence the series $\sum_{m=0}^{\infty}\overset{m}{\sigma}$ converges. It means that the sequence $\{\overset{m}{\omega}\}$ converges in $L_\infty(0,T;L_r(\Omega))$ and from (4.21) it follows that $\{\overset{m}{v}\}$ converges in $L_\infty(0,T;W^1_r(\Omega))$. Therefore there exist limit functions v, ω in these spaces and from the estimate (4.16) it follows that they belong to spaces $\Pi^{1}_{0,\infty,r}(\Omega^{T})$ and $\Pi^{2}_{1,\infty,r}(\Omega^{T})$,respectively.To show that the limit functions are solutions of the problem (A,B) we shall consider the following integral identities

$$
(4.23)\quad \begin{aligned}
&-\int_{\Omega^T} \overset{m+1}{\omega} \varphi_t \, dx\, dt + \int_{\Omega} \omega_0(x)\varphi(x,0)\, dx - \int_{S_1^T} d_\eta \, \varphi(x',t)\, dx'\, dt \, + \\
&-\int_{\Omega^T} \left(\overset{m}{v}{}^k \overset{m+1}{\omega} \varphi_{x^k} + \overset{m+1}{\omega}{}^k \overset{m}{v}_{x^k} \varphi\right) dx\, dt = \int_{\Omega^T} F\varphi \, dx\, dt,
\end{aligned}
$$

where $\overset{m}{v} = \mathcal{B}(\overset{m}{\omega}, b)$,

for arbitrary continuously differentiable functions φ ,such that $\varphi(x,T)=0$ and $\varphi|_{S_2}=0$. From the properties of functions $\overset{m}{v}$ and $\overset{m}{\omega}$ it follows that all integrals in (4.23) are bounded. Therefore,from the Lebesque theorem we can converge with m to infinity in (4.23) and we obtain the same identities for the limit functions v and ω. Therefore,the limit functions are solutions of the problem (A,B).

5. The leakage problem in domains with edges

We shall consider two classes of domains: with dihedral angles π/n and with arbitrary dihedral angles.

5.1 The boundary condition (3.4), for $\pi = 0$

Similarly as it was done in the case of Theorem 4.1, we can prove the following theorem:

Theorem 5.1

Let us assume that:

(a) $r > 3$, r is a real number, S_ν, $\nu = 0,1,2$, are of class C^3,

(b) $a \in W_r^2(\Omega)$, $\operatorname{div} a = 0$,

(c) $b \in L_\infty(0,T; W_r^{2-1/r}(\partial\Omega))$, $\int_{\partial\Omega} b(s)\,ds = 0$,

(d) $-b|_{S_1} = d \geq d_0 > 0$, d_0 is a constant,

(e) $\eta \in \Pi^1_{0,\infty,r}(S_1^T)$;

(f) $F \in L_\infty(0,T; W_r^1(\Omega))$,

(g) the domain Ω has only dihedral angles π/n.

Moreover, we assume the compatibility conditions

(h) $D_\tau^\sigma a \cdot \bar{n}|_{\partial\Omega} = D_\tau^\sigma b|_{t=0}$,

where $0 \leq \sigma \leq 1$, D_τ denotes a derivative tangent to $\partial\Omega$, and

(k) $\omega(0)|_{S_1} = \eta|_{t=0}$.

Then for $t \in [0,T]$, where

$$(5.1) \quad T < t^* = \frac{1}{C}\left[\|a\|_{2,r}^{2r} + \|F(0)\|_r^r + 1\right]^{-1}$$

the problem (A,B) has a solution such that

$$v \in \Pi^2_{1,\infty,r}(\Omega^T), \quad \omega \in \Pi^1_{0,\infty,r}(\Omega^T).$$

Proof. The proof of this theorem differs from the proof of Theorem 4.1 in estimating of the surface integral I .The difference follows from the assumption (d) of this theorem.This concludes the proof.

Now we shall prove the regularity theorem for smoother initial boundary values.To do this we consider the following system of problems:

$$(5.2) \quad D_t^s(\overset{m+1}{\omega}_t + \overset{m}{v}^k \overset{m+1}{\omega}_{x^k} - \overset{m+1}{\omega}^k \overset{m}{v}_{x^k}) = D_t^s F,$$

$$(A_m^s)(5.3) \quad D_t^s \overset{m+1}{\omega}\Big|_{t=0} = \begin{cases} \omega_0 & \text{for } s=0, \\ D_t^{s-1}(F + \overset{m+1}{\omega}^k \overset{m}{v}_{x^k} - \overset{m}{v}^k \overset{m+1}{\omega}_{x^k})\Big|_{t=0} & \text{for } s \geq 1, \end{cases}$$

$$(5.4) \quad D_t^s \overset{m+1}{\omega}\Big|_{S_1} = D_t^s \eta,$$

where $\overset{m}{v}$ is a given function;and

$$(B_m^s) \quad \begin{aligned} &\operatorname{rot} D_t^s \overset{m}{v} = D_t^s \overset{m}{\omega}, \\ &\operatorname{div} D_t^s \overset{m}{v} = 0, \\ &\bar{n} \cdot D_t^s \overset{m}{v}\Big|_{\partial\Omega} = D_t^s b, \end{aligned}$$

where $\overset{m}{\omega}$ is a given function, $m=0,1,\ldots$, $s=0,1,\ldots,l$, and $\overset{0}{\omega} = \omega_0 = \operatorname{rot} a$ .The problem (A_m^s) is linear evolution problem for $D_t^s \overset{m+1}{\omega}$ and has a unique solution, which follows from Lemma 3.1. At first we shall obtain apriori estimates for solutions of problems (A_m^s), (B_m^s), $s=0,\dots,l$, where l is an arbitrary positive integer.

Lemma 5.1

Let us assume that

(a) $r > \frac{3}{l}$, $l \geq 1$ (b) $\overset{m}{v}(t) \in \Gamma_{1,r}^{l+1}(\Omega)$,

(c) $\overset{m}{v} \cdot \bar{n}|_{S_1} = b$, $-b|_{S_1} = d \geq d_0 > 0$, $b|_{S_0} = 0$, $b|_{S_2} \geq 0$, $\int_{\partial\Omega} b(s)ds = 0$.

Then an arbitrary solution $\overset{m+1}{\omega}(t) \in \Gamma_{0,r}^{l}(\Omega) \cap C^l(\bar{\Omega})$ of the problems (B_m^s), $s=0,\dots,l$, satisfies the following differential inequality:

$$(5.5) \quad \frac{d}{dt}|\overset{m+1}{\omega}|_{l,0,r}^{r} \leq I + C(r,l,\Omega)|\overset{m}{v}|_{l+1,1,r}|\overset{m+1}{\omega}|_{l,0,r}^{r} + |F|_{l,0,r}|\overset{m+1}{\omega}|_{l,0,r}^{r-1},$$

where I is a surface integral of the following form:

$$(5.6) \quad I = \sum_{s=0}^{l} \sum_{|\alpha| \leq l-s} \int_{S_1} d\,|D_t^s D_x^\alpha \overset{m+1}{\omega}|^r ds .$$

Proof. From (5.2) and from its derivatives of the order $|\alpha|$, $|\alpha| = 0,\dots,l-s$, with respect to x^i, $i=1,2,3$, we have the equality

$$\sum_{|\alpha|=0}^{l-s} \int_{\Omega} D_x^\alpha D_t^s (\overset{m+1}{\omega}{}^{\sigma}_{t} + \overset{m}{v}{}^k \overset{m+1}{\omega}{}^{\sigma}_{x^k} - \overset{m+1}{\omega}{}^k \overset{m}{v}{}^{\sigma}_{x^k} - F^\sigma) D_x^\alpha D_t^s \overset{m+1}{\omega}{}^{\sigma} |D_x^\alpha D_t^s \overset{m+1}{\omega}|^{r-2} dx = 0,$$

where α is a multiindex. We recall that if $r > \frac{3}{l}$, $l \geq 1$, then $W_r^l(\Omega)$ is an algebra for the pointwise multiplication of functions [Bo, 1], [Ad, 1]. Therefore, the above equality leads us to

$$\frac{1}{r}\frac{d}{dt}\|D_t^s \overset{m+1}{\omega}\|_{l-s,r}^{r} \leq -\frac{1}{r}\sum_{|\alpha|=0}^{l-s} \int_{\partial\Omega} \overset{m}{v}\cdot\bar{n}\,|D_x^\alpha D_t^s \overset{m+1}{\omega}|^r ds +$$

$$+\,C(l,s)\Big[\sum_{|\alpha|=1}^{l-s}\sum_{i=1}^{\alpha}\sum_{p=0}^{s}\Big|\int_{\Omega} D_x^i D_t^p \overset{m}{v}{}^k D_x^{\alpha-i} D_t^{s-p} \overset{m+1}{\omega}{}^{\sigma}_{x^k} D_x^{\alpha} D_t^{s} \overset{m+1}{\omega}{}^{\sigma} \,|D_x^{\alpha} D_t^{s} \overset{m+1}{\omega}|^{r-2}\,dx\Big| +$$

$$+\sum_{|\alpha|=0}^{l-s}\sum_{i=0}^{\alpha}\sum_{p=0}^{s}\Big|\int_{\Omega} D_x^i D_t^p \overset{m+1}{\omega}{}^k D_x^{\alpha-i} D_t^{s-p} \overset{m}{v}{}^{\sigma}_{x^k} D_x^{\alpha} D_t^{s} \overset{m+1}{\omega}{}^{\sigma} \,|D_x^{\alpha} D_t^{s} \overset{m+1}{\omega}|^{r-2}\,dx\Big|\Big] +$$

(5.7)

$$+\,\|D_t^s F\|_{l-s,r}\,\|D_t^s \overset{m+1}{\omega}\|_{l-s,r}^{r-1} \le -\frac{1}{r}\sum_{|\alpha|=0}^{l-s}\int_{\partial\Omega} \overset{m}{v}\cdot\bar{n}\,|D_x^{\alpha} D_t^{s} \overset{m+1}{\omega}|^{r}\,ds\; +$$

$$+\,C(l,s)\sum_{p=0}^{s}\Big[\max_{\Omega}|D_x^{l-s} D_t^p \overset{m}{v}|\,\|D_t^{s-p}\overset{m+1}{\omega}\|_{l-s,r}\,\|D_t^s \overset{m+1}{\omega}\|_{l-s,r}^{r-1} +$$

$$+\,\max_{\Omega}|D_t^p \overset{m+1}{\omega}|\,\|D_t^{s-p}\overset{m}{v}\|_{l+1-s,r}\,\|D_t^s \overset{m+1}{\omega}\|_{l-s,r}^{r-1} +$$

$$+\,\|D_t^p \overset{m}{v}\|_{l+1-s,r}\,\|D_t^{s-p}\overset{m+1}{\omega}\|_{l-s,r}\,\|D_t^s \overset{m+1}{\omega}\|_{l-s,r}^{r-1}\Big] + \|D_t^s F\|_{l-s,r}\,\|D_t^s \overset{m+1}{\omega}\|_{l-s,r}^{r-1}\,.$$

Therefore, we get

$$\frac{1}{r}\frac{d}{dt}\|D_t^s \overset{m+1}{\omega}\|_{l-s,r}^{r} \le -\frac{1}{r}\sum_{|\alpha|=0}^{l-s}\int_{\partial\Omega} \overset{m}{v}\cdot\bar{n}\,|D_x^{\alpha} D_t^{s} \overset{m+1}{\omega}|^{r}\,ds\; +$$

(5.8)

$$+\,C(r,l,s,\Omega)\sum_{p=0}^{s}\|D_t^{s-p}\overset{m}{v}\|_{l+1-s,r}\,\|D_t^{p}\overset{m+1}{\omega}\|_{l-s,r}^{r} + \|D_t^s F\|_{l-s,r}\,\|D_t^s \overset{m+1}{\omega}\|_{l-s,r}^{r-1}\,.$$

Summing (5.8) over s, s=0,...,l, and using the assumption (c) of the lemma we obtain (5.5). This concludes the proof.

Now we shall calculate an estimate for the surface integral (5.6). This will be the subject of the following lemma.

Lemma 5.2

Let us assume that

(a) $r > \frac{3}{l}$; $l \ge 1$, (b) $d \ge d_0 > 0$, $d \in \Gamma^{l}_{0,r}(S_1)$, (c) $\eta \in \Gamma^{l}_{0,r}(S_1)$,

(d) $F \in \Gamma^{l-1}_{0,r}(S_1)$, (e) S_1 is of class C^{l} .

Then for $\overset{m}{v} \in \Gamma^{l+1}_{1,r}(\Omega)$ the following estimate for the surface integral I is valid

$$(5.9)\quad I \leq \sum_{s=0}^{l} C_s |\overset{m}{v}|^s_{l+1,r} ,$$

where $C_s = C_s(d_0, |d|_{l,\rho,r,S_1}, |\eta|_{l,0,r,S_1}, |F|_{l-1,0,r,S_1}, \mathcal{L}_1)$.

Proof. Introducing curvilinear coordinates the term I is estimated by

$$(5.10)\quad I \leq C(\mathcal{L}_1) \sum_{s=0}^{l} \sum_{|\alpha|=0}^{l-s} \int_{S_1} d|D^{\alpha}_{(\tau,n)} D^s_t \overset{m+1}{\omega}|^r ds = C(\mathcal{L}_1) \sum_{s+\varsigma+|\alpha|\leq l} I^{s,\varsigma,\alpha} ,$$

where $I^{s,\varsigma,\alpha} = \int_{S_1} d|D^{\alpha}_{(\tau)} D^s_t D^{\varsigma}_n \overset{m+1}{\omega}|^r ds$ and $D^{\alpha}_{(\tau,n)}$, $D^{\alpha}_{(\tau)}$ denote all possible derivatives of the order α with respect to τ_1, τ_2, n and τ_1, τ_2, respectively. Now we shall estimate particular terms in the right-hand side of (5.10). At first we shall obtain the form of $D^s_n \overset{m+1}{\omega}$. From (5.2) for $s=0$ we calculate

$$(5.11)\quad \overset{m+1}{\omega}_{,n} = -\overset{m}{v}_n^{-1}\left[\overset{m+1}{\omega}_t + \overset{m}{v}_\mu \overset{m+1}{\omega}_{,\tau_\mu} - \overset{m+1}{\omega}{}^k \overset{m}{v}_{x^k} - F\right]$$

therefore, its s normal derivative has the form

$$(5.12)\quad D^s_n \overset{m+1}{\omega}_{,n} = -\sum_{\varsigma=0}^{s} D^{\varsigma}_n(\overset{m}{v}_n^{-1}) D^{s-\varsigma}_n \left[\overset{m+1}{\omega}_t + \overset{m}{v}_\mu \overset{m+1}{\omega}_{,\tau_\mu} - \overset{m+1}{\omega}{}^k \overset{m}{v}_{x^k} - F\right],$$

where $s=0,\dots,l-1$. Moreover, the following expression will be useful

$$(5.13)\quad \overset{m+1}{\omega}_{,n\tau_\nu} - \overset{m+1}{\omega}_{,\tau_\nu n} = \overset{m+1}{\omega}_{,\tau_\mu} \mathcal{L}_{\mu\nu}(x) + \overset{m+1}{\omega}_{,n} \beta_\nu(x),$$

where $\mathcal{L}_{\mu\nu}(x) = [\bar{\tau}_\nu, \bar{n}]\cdot\bar{\tau}_\mu$, $\beta_\mu(x) = [\bar{\tau}_\mu, \bar{n}]\cdot\bar{n}$.

From (5.11) we see that $\overset{m+1}{\omega}_{,n}$ is the polynomial of degree one with respect to $\overset{m}{v}$, $\overset{m}{v}_x$ and with coefficients dependent linearly on $\overset{m+1}{\omega}$, $\overset{m+1}{\omega}_{,\tau}$, $\overset{m+1}{\omega}_{,t}$ .Using (5.11) in (5.12) for s=1 we have that $D_n^2 \overset{m+1}{\omega}$ is a polynomial of degree two with respect to $\overset{m}{v}$, $\overset{m}{v}_x$, $\overset{m}{v}_{xx}$, $\overset{m}{v}_{xt}$, $\overset{m}{v}_t$ and with coefficient dependent on $D^{\sigma}_{(\tau,t)} \overset{m+1}{\omega}$, $\sigma \leq 2$, $D^{\varkappa}_{(x,t)} F$, $\varkappa \leq 1$.Moreover,the coefficients of the highest derivatives $D^1_{(x,t)} \overset{m}{v}_x$, $D^2_{(\tau,t)} \overset{m+1}{\omega}$ are equal $\frac{\overset{m+1}{\omega}}{\overset{m}{v}_n}$, $\frac{\overset{m}{v}}{\overset{m}{v}_n}$,respectively.

Now we formulate the inductive assumption.The derivative $D_n^s \overset{m+1}{\omega}$ is a polynomial of degree s with respect to $\overset{m}{v}$ and its derivatives up to $D^{s-1}_{(x,t)} \overset{m}{v}_x$ with coefficients dependent on $D^{\sigma}_{(\tau,t)} \overset{m+1}{\omega}$, $D^{\sigma-1}_{(x,t)} F$, $D^{\sigma-1}_{(\tau,t)} \overset{m}{v}_n$ \ ,$0 \leq \sigma \leq$ s.Moreover,in the expression of $D_n^s \overset{m+1}{\omega}$ the following highest derivatives: $D^{s-1}_{(x,t)} \overset{m}{v}_x$, $D^{s}_{(\tau,t)} \overset{m+1}{\omega}$, $D^{s-1}_{(x,t)} F$, $D^{s-1}_{(\tau,t)} \overset{m}{v}_n$ appear linearly and the first three of these with coefficients equal to one of $\frac{\overset{m}{v}}{\overset{m}{v}_n}$, $\frac{\overset{m+1}{\omega}}{\overset{m}{v}_n}$,what follows from the rules of differentiation.To find a qualitative form of the expression $D_n^{s+1} \overset{m+1}{\omega}$ we consider (5.12).From (5.12) we see that there appears a derivative $D_x^{s+1} \overset{m}{v}$ with coefficient $\frac{\overset{m+1}{\omega}}{\overset{m}{v}_n}$ .Moreover,by the inductive assumption the expression $(D_n^s \overset{m+1}{\omega})_{,t}$ follows the derivative $D_t^s \overset{m}{v}_x$,so terms $D^s_{(x,t)} \overset{m}{v}_x$ with coefficients $\frac{\overset{m}{v}}{\overset{m}{v}_n}$, $\frac{\overset{m+1}{\omega}}{\overset{m}{v}_n}$,arise.In addition,by the inductive assumption,derivatives $(D_n^s \overset{m+1}{\omega})_{,t}$, $\overset{m}{v}_{\mu} D_n^s \overset{m+1}{\omega}_{,\tau_{\mu}}$, $D_n^s F$ follow the derivatives $D^{s+1}_{(\tau,t)} \overset{m+1}{\omega}$, $D^{s}_{(\tau,t)} \overset{m}{v}_n$, $D^s_{(x,t)} F$. Furthermore, $D_n^{s+1} \overset{m+1}{\omega}$ is the polynomial of degree s+1 with respect to $\overset{m}{v}$ and its derivatives up to $D^s_{(x,t)} \overset{m}{v}_x$,this results from the expressions $\overset{m}{v}_{\mu} D_n^s \overset{m+1}{\omega}_{,\tau_{\mu}}$, $D_n^s \overset{m+1}{\omega}{}^k \overset{m}{v}_{xk}$ and the inductive assumption.Therefore,we have shown that $D_n^{s+1} \overset{m+1}{\omega}$ is a polynomial of degree s+1 with respect to $\overset{m}{v}$ and its derivatives up to $D^s_{(x,t)} \overset{m}{v}_x$ with coefficients dependent on $D^{\sigma}_{(\tau,t)} \overset{m+1}{\omega}$, $D^{\sigma-1}_{(x,t)} F$, $D^{\sigma-1}_{(\tau,t)} \overset{m}{v}_n$,$0 \leq \sigma \leq$ s+1.Moreover,in the expression

$D_m^{s+1}\overset{m+1}{\omega}$ the highest derivatives $D^s_{(x,t)}\overset{m}{v}_x$, $D^{s+1}_{(\tau,t)}\overset{m+1}{\omega}$, $D^s_{(x,t)}F$, $D^s_{(\tau,t)}\overset{m}{v}_n$ appear linearly and the first three of these have coefficients equal to one of $\overset{m}{v}/\overset{m}{v}_n$, $\overset{m+1}{\omega}/\overset{m}{v}_n$. Therefore, using that $W^l_r(\Omega)$ for $l>\frac{3}{r}$ is an algebra and (5.4) we have the estimate

$$I^s = \sum_{\sigma+|\alpha|=0}^{l-s} I^{s,\sigma,\alpha} \leq \sum_{\sigma=0}^{s} c_\sigma |\overset{m}{v}|^\sigma \quad , s=0,\dots,l, \tag{5.14}$$

where $c_\sigma = c_\sigma(d_0, \alpha_1, |d|_{l,0,r,s_1}, |\eta|_{l,0,r,s_1}, |F|_{l-1,0,r,s_1})$.
At last summing I^s over all s, s=0,...,l, we conclude the proof.

To obtain an estimate for $|\overset{m+1}{\omega}(t)|_{l,0,r}$ from (5.5) we have to know an estimate for $|\overset{m+1}{\omega}(0)|_{l,0,r}$, which will result from the following lemma.

Lemma 5.3

Let us assume that

(a) $a \in W_r^{l+1}(\Omega)$, (b) $F(0) \in \Gamma_D^{l-1}{}_{,r}(\Omega)$, (c) $b(0) \in \Gamma^{l+1-1/r}_{1,2\ ,r}(\partial\Omega)$.

Then the following estimate is valid

$$\begin{aligned} \|D_t^s \overset{m+1}{\omega}(0)\|_{l-s,r} \leq C\Big(\|a\|^{s+1}_{l+1,r} + \sum_{i+j_1+\dots+j_{s-1}+k_1+\dots+k_{s-1}\leq s} \|a\|^i_{l+1,r}\|F(0)\|^{j_1}_{l-1,r}\dots \\ \dots\|D_t^{s-2}F(0)\|^{j_{s-1}}_{l+1-s,r}\|D_t b(0)\|^{k_1}_{l-1/r,r}\dots\|D_t^{s-1}b(0)\|^{k_{s-1}}_{l+2-s-1/r,r}\Big) + \|D_t^{s-1}F(0)\|_{l-s,r} , \end{aligned} \tag{5.15}$$

where C is a constant. Summing (5.15) over s, s=0,...,l, we obtain

$$|\overset{m+1}{\omega}(0)|_{l,0,r} \leq C\Big(\|a\|^{l+1}_{l+1,r} + \sum_{i+j+k\leq l} \|a\|^i_{l+1,r}|F(0)|^j_{l-1,0,r}|b(0)|^k_{l+1,2,r,\partial\Omega}\Big). \tag{5.16}$$

Proof. To prove this lemma we use the method of induction to obtain the form of $D_t^s \overset{m+1}{\omega}\big|_{t=0}$, s=1,...,l. At first we have

$$(5.17)\quad \overset{m+1}{\omega}_t(0) = F(0) + \mathrm{rot}^k a\, a_{x^k} - a^k(\mathrm{rot}\,a)_{x^k} ,$$

which is a sum of $F(0)$ and the polynomial of second degree with respect to a and its derivatives up to the second order. To calculate the higher derivatives we consider the expression

$$(5.18)\quad D_t^{s+1}\overset{m+1}{\omega}(0) = D_t^s F(0) + \sum_{\sigma=0}^{s}\left(D_t^\sigma \overset{m+1}{\omega}{}^k(0)\, D_t^{s-\sigma}\overset{m}{v}_{x^k}(0) - D_t^\sigma \overset{m+1}{\omega}_{x^k}(0)\, D_t^{s-\sigma}\overset{m}{v}{}^k(0)\right).$$

To express (5.18) in the dependence of the known quantities we calculate $D_t^s \overset{m}{v}(0)$ from the problem (B_m^s) in the form

$$(5.19)\quad D_t^s \overset{m}{v}(0) = \mathcal{F}\left(D_t^s \overset{m}{\omega}(0), D_t^s b(0)\right),$$

where $\mathcal{F}$ is a linear functional which represents a solution of the problem (B_m^s). Using (5.17) and (5.19) in (5.18) for s=1 we have that $\overset{m+1}{\omega}_{tt}(0)$ is the sum of $D_t F(0)$, the polynomial of degree three with respect to a and its derivatives up to $D_x^3 a$ and the polynomial of degree two with respect to $F(0)$, $D_{(x,t)}^i b(0)$, $i \le 1$, $D_x^j a$, $j \le 2$. Therefore, we can formulate the inductive assumption. Let $D_t^s \overset{m+1}{\omega}(0)$ be a sum of $D_t^{s-1} F(0)$, a polynomial of degree s+1 with respect to a and its derivatives up to $D_x^{s+1} a$ and a polynomial of degree s with respect to $D_{(x,t)}^k F(0)$, $D_{(x,t)}^l b(0)$, $D_x^j a$, where $k \le s-2$, $l \le s-1$, $j \le s$. Then, from (5.18) we see that $D_t^{s+1}\overset{m+1}{\omega}(0)$ contains $D_t^s F(0)$ and two polynomial also. From the term $\overset{m}{v}(0)\, D_t^s \overset{m+1}{\omega}_x(0)$ and the induction assumption it follows that $D_t^{s+1}\overset{m+1}{\omega}(0)$ contains a polynomial of degree s+2 with respect to a and its derivatives up to $D_x^{s+2}a$ .Moreover, $D_t^s \overset{m}{v}_x = J_x(D_t^s \overset{m}{\omega}, D_t^s b)$ implies that $D_t^{s+1} \overset{m+1}{\omega}(0)$ contains $D_t^s b(0)$ and by the induction assumption $D_t^{s+1} \overset{m+1}{\omega}(0)$ contains $D_t^{s-1} F_x(0)$,so also $D_{(x,t)}^{s-1} F_x(0)$. At last,from the nonlinearity of the expression (5.16) it follows that $D_t^{s+1} \overset{m+1}{\omega}(0)$ has a polynomial form of degree one greater than $D_t^s \overset{m+1}{\omega}(0)$ .Therefore,we proved that $D_t^{s+1} \overset{m+1}{\omega}(0)$ is the sum of $D_t^s F(0)$,a polynomial of degree s+2 with respect to a and its derivatives up to $D_x^{s+2}a$ and a polynomial of degree s+1 with respect to $D_{(x,t)}^k F(0)$, $D_{(x,t)}^l b(0)$, $D_x^j a$,where $k \leq s-1, l \leq s, j \leq s+1$.Hence,from the inductive considerations,the same form have all derivatives $D_t^s \overset{m+1}{\omega}(0)$,s=1,...,l.Using the form of $D_t^s \overset{m+1}{\omega}(0)$ we obtain (5.15) and (5.16).This ends the proof.

Using the above lemmas we shall prove the theorem:

Theorem 5.2

Let us assume that the following conditions hold:

(a) $r \geq 1, r > \frac{3}{l}$,$l \geq 1$,l is a natural number,$S_\nu \in C^{l+2}$, $\nu = 0,1,2$,

(b) $a \in W_r^{l+1}(\Omega)$, $\operatorname{div} a = 0$,

(c) $\eta \in \Pi^l_{0,\infty,r}(S_1^T)$,

(d) $b \in \Pi^{l+1-1/r}_{0,\infty,r}(\partial\Omega^T)$, $-b|_{S_1} = d \geq 0$, $b|_{S_0} = 0$, $b|_{S_2} \geq 0$,
$\int_{\partial\Omega} b(s)ds = 0$, $\partial\Omega = S_0 \cup S_1 \cup S_2$,

(e) $d \geq d_0 > 0$, d_0 is a constant ,

(f) $F \in \Pi^l_{0,\infty,r}(\Omega^T)$.

In the case of arbitrary dihedral angles the following condition must also be satisfied (see Theorems D.2 and D.4) :

(g) $\frac{\pi}{\mathcal{L}_0} > l+2$ and $r=2, l \geq 2$,

because in this case we know the proof of the existence of the problem (B) for r=2 only, and $\mathcal{L}_0$ is the maximal dihedral angle. Moreover, we must assume that the following compatibility conditions are satisfied:

(h) $D_\tau^\sigma a \cdot \bar{n}\big|_{\partial\Omega} = D_\tau^\sigma b\big|_{t=0}$,

where $0 \leq \sigma \leq l+1-l', l' > \frac{3}{r}$, the symbol D_τ denotes a derivative tangent to $\partial\Omega$, and

(k) $D_\tau^\sigma (D_t^s \omega)(0)\big|_{S_1} = D_\tau^\sigma D_t^s \eta\big|_{t=0}$,

where $0 \leq s \leq l, 0 \leq \sigma \leq l-s-l'$ and D_τ denotes a derivative tangent to S_1. Then for $t \in [0,T]$, where

(5.20) $$T < t^* = \frac{1}{C_\varepsilon (Y(0)+1)}$$

and $Y(0)$ is estimated by (5.16), the problem (A,B) has a solution such that

(5.21) $$v \in \Pi^{l+1}_{1,\infty,r}(\Omega^T), \quad \omega \in \Pi^{l}_{0,\infty,r}(\Omega^T) .$$

Proof. From (5.5) and (5.9) we have

(5.22) $$\frac{d}{dt} |\overset{m+1}{\omega}|^r_{l,0,r} \leq W(|\overset{m}{v}|_{l+1,1,r}) + c|\overset{m}{v}|_{l+1,1,r} |\overset{m+1}{\omega}|^r_{l,0,r} + |F|_{l,0,r} |\overset{m+1}{\omega}|^{r-1}_{l,0,r} ,$$

where $W(\cdot)$ is the polynomial of degree l .For problems (B^s_m) $s=0,\dots,l$,we have the estimate (see Theorems C.1,C.2,D.2,D.4)

$$(5.23)\qquad |\overset{m}{v}|_{l+1,1,r} \leq c\left(|\overset{m}{\omega}|_{l,0,r} + |b|_{l+1-1/r,1-1/r,\partial\Omega}\right),$$

and the existence for domains with arbitrary dihedral angles if additionally the condition (g) is satisfied.From (5.22) and (5.23) using the Hölder inequality,we obtain

$$(5.24)\qquad \begin{aligned} \frac{d}{dt}|\overset{m+1}{\omega}|^r_{l,0,r} &\leq (c_1|\overset{m}{\omega}|_{l,0,r}+c_2)|\overset{m+1}{\omega}|^r_{l,0,r} + W(c_1|\overset{m}{\omega}|_{l,0,r}+c_2) \leq \\ &\leq \gamma(|\overset{m}{\omega}|_{l,0,r})(|\overset{m+1}{\omega}|^r_{l,0,r}+1), \end{aligned}$$

where $\gamma(y)=\max\{c_1y+c_2, W(c_1y+c_2)\}$, $\overset{m+1}{y}(0)=|\overset{m+1}{\omega}(0)|_{l,0,r}=Y(0)$ which is estimated by (5.16).The equation (5.24) has the same form as (4.13) ,so the other considerations are the same as in Theorem 4.1.This ends the proof.

5.2 The boundary condition 3.5

To prove the existence and uniqueness of solutions of this problem we use the method of successive approximations.Therefore,instead of problems (D),(E) we consider the following system of problems:

$$(5.25)\qquad (D^s_t\overset{m+1}{v})_{,t} + \overset{m}{v}\cdot\nabla D^s_t\overset{m+1}{v} = -\sum_{j=0}^{s-1} D^{s-j}_t\overset{m}{v}\cdot\nabla D^j_t\overset{m+1}{v} - D^s_t(\nabla\overset{m}{p}-f),$$

$$(D^s_m)\,(5.26)\qquad D^s_t\overset{m+1}{v}\big|_{t=0} = \begin{cases} a(x) & \text{for } s=0, \\ -D^{s-1}_t(\overset{m}{v}\cdot\nabla\overset{m+1}{v}+\nabla\overset{m}{p}-f)\big|_{t=0} & \text{for } s\geq 1, \end{cases}$$

$$(5.27)\qquad D^s_t\overset{m+1}{v}\big|_{S_1} = D^s_t\eta,$$

$$\Delta D_t^s \overset{m}{p} = D_t^s(\operatorname{div} f - \overset{m}{v}{}^i_{x^k} \overset{m}{v}{}^k_{x^i}),$$

$$(E_m^s) \qquad D_t^s \overset{m}{p}\big|_{S_2} = D_t^s \pi(x',t), \ x' \in S_2,$$

$$\frac{\partial}{\partial n} D_t^s \overset{m}{p}\big|_{S_1} = D_t^s\, g(\eta, \bar{\tau}, \bar{n}),$$

$$\frac{\partial}{\partial n} D_t^s \overset{m}{p}\big|_{S_0} = D_t^s(\overset{m}{v}{}^k \overset{m}{v}_{x^k} \cdot \bar{n} + f \cdot \bar{n})\big|_{S_0},$$

where $m=0,1,\ldots,$ $\overset{0}{v} = a$ $, s=0,\ldots,l-1$ and

$$(5.28) \quad g(\eta, \bar{\tau}, \bar{n}) = f_n\big|_{S_1} - \eta_{n,t} + \sum_{\mu=1}^{2} (\eta_n \eta_{\mu,\tau_\mu} + \eta_\mu \eta_n \operatorname{div} \bar{\tau}_\mu - \eta_\mu \eta_{n,\tau_\mu}) + \eta_n^2 \operatorname{div} \bar{n} + \eta^k \eta \cdot \bar{n}_{x^k}.$$

From Sections C,D we know that the problems (E_m^s) have unique solutions. We summarize these results in the following form:

Lemma 5.4

Let us assume that $r > \frac{3}{l}$ $, l \geqslant 1$, $f \in \Gamma_{1,r}^{l+1}(\Omega)$, $\overset{m}{v} \in \Gamma_{1,r}^{l+1}(\Omega)$, $\pi \in \Gamma_{2,r}^{l+2-1/r}(S_2)$, $\eta \in \Gamma_{0,r}^{l+2-1/r}(S_1)$ and the smooth parts of boundary are of class C^{l+3}. Then for $t \in [0,T]$ there exists a unique solution $\overset{m}{p} \in \Gamma_{2,r}^{l+2}(\Omega)$ of problems (E_m^s) , $s=0,\ldots,l$, such that

$$(5.29) \quad |\overset{m}{p}|_{l+2,2,r} \leq c_1 + c_2 |\overset{m}{v}|^2_{l+1,1,r},$$

where c_1 depends on l, r, Ω , $|f|_{l+1,1,r}$, $|\pi|_{l+2-1/r,1,r,S_2}, |\eta|_{l+2-1/r,0,r,S_1}$ and on bounds of $l+3$-th derivative of the boundary, c_2 depends on r, Ω .

Proof. The existence of solutions of the problem (E_m^s) for a domain with edges is shown in Sections C,D . The existence of solutions of this problem for domains with smooth boundaries is well known. Moreover, the following estimate is valid

$$|\overset{m}{p}|_{l+2,2,r} \leq C\big(|\operatorname{div} f - \overset{m}{v}{}^{i}_{x^k}\overset{m}{v}{}^{k}_{x^i}|_{l,\Omega,r} + |\Pi|_{l+2-1/r,2,r,S_2} +$$

$$(5.30)\quad + |\overset{m}{v}{}^{k}\overset{m}{v}\cdot\bar{n}_{x^k} + f\cdot\bar{n}|_{l+1-1/r,1,r,S_0} + |\eta_{n,t} + \eta_\mu\eta_{n,\tau_\mu} - \eta_n(\eta_{\nu,\tau_\nu} + \eta_\nu \operatorname{div}\bar{\tau}_\nu +$$

$$+ \eta_n \operatorname{div}\bar{n}) - \eta^k\eta\cdot\bar{n}_{x^k} - f\cdot\bar{n}|_{l+1-1/r,1,r,S_1}\big)$$

where C depends on l, r, Ω. The equation (5.29) follows (5.30). This concludes the proof.

Now we shall prove the existence and uniqueness of solutions of the problems (D_m^s).

Lemma 5.5

Let the initial data functions satisfy the restriction

$$(5.31)\qquad a\cdot\bar{n}|_{S_2} \geq a_0 = \mathrm{const} > 0 ,$$

and $\overset{m}{p} \in C^{2,\alpha}(\bar{\Omega}^T)$, $f \in C^{1,\alpha}(\bar{\Omega}^T)$, $a \in C^{1,\alpha}(\bar{\Omega})$, $\eta \in C^{1,\alpha}(S_1^T)$, $\overset{m}{v} \in C^{1,\alpha}(\bar{\Omega}^T)$ where $\bar{\Omega}^T = \bar{\Omega}\times[0,T]$. If $|\overset{m}{v}_t| \leq C = \mathrm{const}, m > 0$, there exists a unique solution of the problem (D_m^0) for $t \in [0,T_1]$, where

$$(5.32)\qquad T_1 = \frac{a_0}{C}$$

such that $\overset{m+1}{v} \in C^{1,\alpha}(\bar{\Omega}^{T_1})$.

Moreover, if $D_t^\sigma \overset{m}{p} \in C^{2,\alpha}(\bar{\Omega}^T)$, $D_t^\sigma f \in C^{1,\alpha}(\bar{\Omega}^T)$, $D_t^\sigma \eta \in C^{1,\alpha}(S_1^T)$, $D_t^\sigma \overset{m}{v} \in C^{1,\alpha}(\bar{\Omega}^T)$, $D_t^\sigma \overset{m+1}{v} \in C^{1,\alpha}(\bar{\Omega}^T)$, $\sigma = 0,\dots,s-1$, and (5.32) is satisfied, there exists a solution of the problem (D_m^s) for $t \in [0,T_1]$ such that $D_t^s \overset{m+1}{v} \in C^{1,\alpha}(\bar{\Omega}^{T_1})$.

Proof. At first we have to show that the fluid leaves the domain Ω through S_2. From the assumptions of the lemma $\overset{m}{v}_n|_{S_2} \geq a_0 - Ct$, so for $t \leq T_1$ $\overset{m}{v}_n|_{S_2} \geq 0$. To show that the problem (D_m^0) is well posed, we introduce the characteristic curves of

(5.25) determined by the equations

$$\frac{dy}{ds} = \overset{m}{v}(y(x,t;s),s),$$

(5.33) $$y(x,t;t) = x,$$

where s is a parameter, $0 \leq s \leq t$. We classify these curves into two disjoined sets (a), (b) (see the proof of Lemma 3.1). Then the equation (5.25) can be written in the following form

(5.34) $$\frac{d}{ds}\overset{m+1}{v}(y(x,t;s),s) = -\nabla_y \overset{m}{p}(y(x,t;s),s) + f(y(x,t;s),s),$$

For each characteristic curve (5.33), the equation (5.34) represents a corresponding ordinary differential equation. The initial values for the equation (5.34) on characteristic curves belonging to (a) or to (b) are determined by (5.26) or (5.27), respectively. This shows that the problem (D^0_m) is well posed. The existence and differential properties of the solution of the problem (D^0_m) result from the expression obtained by integration (5.34) with respect to s using the initial values determined by (5.26) and (5.27). Similar considerations are valid for problems (D^s_m). This concludes the proof.

Now we obtain some apriori estimates of solutions $D^s_t \overset{m+1}{v} \in W^{l-s}_r(\Omega)$, $D^s_t \overset{m}{p} \in W^{l+1-s}_r(\Omega)$, $r > \frac{3}{l-1}$, $l \geq 2$, $s=0,1,\ldots,l-1$, $m=0,\ldots$, of problems (D^s_m), (E^s_m), respectively. We distinguish the case $l=2$, because then the solutions belong to the largest admissable Sobolev space. Moreover, in this case calculations can be done explicitly.

Lemma 5.6

Let us assume that:

(a) $T \leq T_1$, $r > 3$, $d \geq d_0 > 0$,

(b) $\eta \in \Gamma^2_{0,r}(S_1)$, $f \in \Gamma^2_{1,r}(\Omega)$,

(c) $S_1 \in C^3$,

(d) $\overset{m-1}{p}, \overset{m}{p} \in \Gamma^3_{2,r}(\Omega)$, $\overset{m}{v} \in W^2_r(\Omega)$,

(e) (5.31) is satisfied

Let $\overset{m+1}{v}$ be a solution of the problem (D^c_m), then for $t \in [0,T]$ the following differential inequality is valid

$$
\begin{aligned}
\frac{d}{dt}\|\overset{m+1}{v}\|^r_{2,r} \leq {} & C_1 + C_2\|\overset{m}{p}\|^r_{3,r} + C_3\|\overset{m}{p}_t\|^r_{2,r} + C_4\|\overset{m}{p}\|^r_{3,r}\|\overset{m-1}{p}\|^r_{3,r} + \\
(5.35) \qquad & + C_5\|\overset{m}{v}\|_{2,r}\|\overset{m+1}{v}\|^r_{2,r} + (\|\overset{m}{p}\|_{3,r} + C_6)\|\overset{m+1}{v}\|^{r-1}_{2,r} ,
\end{aligned}
$$

where $C_1 = C_1(d_0, S_1, |\eta|_{2,0,r,S_1}, |f|_{2,1,r,\Omega})$, $C_2 = C_2(d_0, S_1, |\eta|_{2,1,r,S_1})$, $C_3 = C_3(d_0, S_1, \|\eta\|_{2,r,S_1})$, $C_4 = C_4(S_1)$, $C_5 = C_5(r,\Omega)$, $C_6 = \|f\|_{2,r}$.

Proof. From the equation (5.25) of the problem (D^0_m) we have the following equality

$$
(5.36) \qquad \sum_{s=1}^{3}\sum_{|j|\leq 2}\int_\Omega D^j_x(\overset{m+1}{v}{}^s_t + \overset{m}{v}{}^k\overset{m+1}{v}{}^s_{x_k} + \nabla_s\overset{m}{p} - f^s)\, D^j_x\overset{m+1}{v}{}^s\, |D^j_x\overset{m+1}{v}|^{r-2} = 0,
$$

where $D^j_x = \dfrac{\partial^{j_1}}{\partial x_1^{j_1}}\dfrac{\partial^{j_2}}{\partial x_2^{j_2}}\dfrac{\partial^{j_3}}{\partial x_3^{j_3}}$, $j_1+j_2+j_3 = |j|$.

From (5.36) we have

$$\frac{1}{r}\frac{d}{dt}\|\overset{m+1}{v}\|^r_{2,r} \le -\frac{1}{r}\int_{\partial\Omega}\overset{m}{v}\cdot\bar n\,(|\overset{m+1}{v}|^r+|\overset{m+1}{v}_x|^r+|\overset{m+1}{v}_{xx}|^r)ds+\max_{\Omega}|\overset{m}{v}_x|\,\|\overset{m+1}{v}_x\|^r_r+$$

(5.37)
$$+\frac{1}{r}\max_{\Omega}|\mathrm{div}\,\overset{m}{v}|\,\|\overset{m+1}{v}\|^r_{2,r}+\max_{\Omega}|\overset{m+1}{v}_x|\,\|\overset{m}{v}_{xx}\|_r\|\overset{m+1}{v}_{xx}\|^{r-1}_r+2\max_{\Omega}|\overset{m}{v}_x|\,\|\overset{m+1}{v}_{xx}\|^r_r+$$
$$+(\|\overset{m}{\rho}\|_{3,r}+\|f\|_{2,r})\|\overset{m+1}{v}\|^{r-1}_{2,r}.$$

Using the curvilinear coordinates introduced in Section 2 and the boundary conditions (3.5) the surface integral in (5.37) is estimated by

$$-\int_{\partial\Omega}\overset{m}{v}\cdot\bar n\,(|\overset{m+1}{v}|^r+|\overset{m+1}{v}_x|^r+|\overset{m+1}{v}_{xx}|^r)ds \le \int_{S_1} d\,(|\overset{m+1}{v}|^r+|\overset{m+1}{v}_x|^r+|\overset{m+1}{v}_{xx}|^r)ds \le$$

$$\le C(S_1)\int_{S_1} d\,(|\eta|^r+|\eta_{,\tau}|^r+|\eta_{,\tau\tau}|^r+|\overset{m+1}{v}_{,\tau n}|^r+|\overset{m+1}{v}_{,nt}|^r+|\overset{m+1}{v}_{,nn}|^r+|\overset{m+1}{v}_{,n}|^r)ds \equiv I.$$

From (5.25) restricted to S_1 for s=0 we calculate

$$\overset{m+1}{v}_{,n}\Big|_{S_1}=d^{-1}[\eta_t+\eta_\mu\eta_{,\tau_\mu}+\nabla\overset{m}{\rho}-f]\equiv d^{-1}A,\ \text{so}\ \overset{m+1}{v}_{,n\tau_\nu}\Big|_{S_1}=-d^{-2}d_{,\tau_\nu}A+d^{-1}A_{,\tau_\nu},$$

$$\overset{m+1}{v}_{,nt}\Big|_{S_1}=-d^{-2}d_tA+d^{-1}A_t,$$

$$\overset{m+1}{v}_{,nn}\Big|_{S_1}=\overset{m}{v}_n^{-2}\,\overset{m}{v}_{n,n}\Big[\overset{m+1}{v}_t+\overset{m}{v}_\mu\overset{m+1}{v}_{,\tau_\mu}+\nabla\overset{m}{\rho}-f\Big]-\overset{m}{v}_n^{-1}\Big[\overset{m+1}{v}_{,nt}+$$

(5.38)
$$+\overset{m}{v}_{\mu,n}\overset{m+1}{v}_{,\tau_\mu}+\overset{m}{v}_\mu\overset{m+1}{v}_{,n\tau_\mu}+\nabla\overset{m}{\rho}_{,n}-f_{,n}+\overset{m}{v}_\mu(\overset{m+1}{v}_{,\tau_\nu}a_{\nu\mu}+\overset{m+1}{v}_{,n}b_\mu)\Big]$$

Knowing that $\overset{m}{v}_{,n}\big|_{S_1}=d^{-1}(\eta_t+\eta_\mu\eta_{,\tau_\mu}+\nabla\overset{m-1}{\rho}-f)$ we obtain

$$\overset{m+1}{v}_{,nn}\Big|_{S_1}=g_1(\eta,\eta_t,\eta_\tau,\eta_{t\tau},\eta_{\tau\tau},f_t,f_\tau,f)+g_2(\eta,\eta_t,\eta_\tau,f)\Big[\overset{m}{\rho}_x\overset{m-1}{\rho}_x+\overset{m}{\rho}_x+\overset{m}{\rho}_{xx}+\overset{m}{\rho}_{xt}\Big],$$

where $d=-\eta_n=-\eta\cdot\bar n$, $a_{\nu\mu}(x)=[\bar n,\bar\tau_\nu]\cdot\bar\tau_\mu$, $b_\nu(x)=[\bar n,\bar\tau_\nu]\cdot\bar n$ and $[\bar\tau_\nu,\bar\tau_\mu]$ is the commutator of vector fields $\bar\tau_\mu$, $\bar\tau_\nu$.

Substituting (5.38) into I we obtain the estimate

$$(5.39)\quad I \le C_1 + C_2\|\overset{m}{p}\|^r_{3,r} + C_3\|\overset{m}{p}_t\|^r_{2,r} + C_4\|\overset{m}{p}\|^r_{3,r}\,\|\overset{m-1}{p}\|^r_{3,r},$$

where C_i, $i=1,\dots,4$, are the same constants as in (5.35). Using the Sobolev imbeddings in (5.37) and using (5.39) we obtain (5.35). This ends the proof.

In the estimate (5.35) the norm $\|\overset{m}{p}_t\|_{2,r}$ appears, so we have to consider the problem (E^1_m) and consequently the problem (D^1_m), also. Therefore, we formulate

Lemma 5.7

Let the assumptions of Lemma 5.6 be satisfied, then for $t\in[0,T]$ the following differential inequality for solutions $\overset{m}{v}$, $\overset{m+1}{v}\in\Gamma^2_{1,r}(\Omega)$ of the problem (D^1_m) is valid

$$(5.40)\quad \begin{aligned}\frac{d}{dt}\|\overset{m+1}{v}_t\|^r_{1,r} \le{}& C_7 + C_8\|\overset{m}{p}\|^r_{2,r} + C_9\|\overset{m}{p}_t\|^r_{2,r} + C_{10}\big(\|\overset{m}{v}\|_{2,r}\|\overset{m+1}{v}_t\|^r_{1,r} +\\ &+\|\overset{m}{v}_t\|_{1,r}\|\overset{m+1}{v}\|_{2,r}\|\overset{m+1}{v}_t\|^{r-1}_{1,r}\big) + \big(\|\overset{m}{p}_t\|_{2,r} + C_{11}\big)\|\overset{m+1}{v}_t\|^{r-1}_{1,r},\end{aligned}$$

where $C_7 = C_7(d_0,r,\Omega,S_1,|\eta|_{2,0,r,S_1},|f|_{2,1,r,\Omega})$, $C_8 = C_8(d_0,r,\Omega,S_1,|\eta|_{2,1,r,S_1})$, $C_9 = C_9(d_0,r,S_1,\|\eta\|_{2,r,S_1})$, $C_{10} = C_{10}(r,\Omega)$, $C_{11} = \|f_t\|_{1,r}$.

Proof. From (5.25) of the problem (D^1_m) we have

$$(5.41)\quad \sum_{s=1}^{3}\sum_{|j|\le 1}\int_\Omega D^j_x\big(\overset{m+1}{v}{}^s_t + \overset{m}{v}{}^k\overset{m+1}{v}{}^s_{x^k} + \nabla_s\overset{m}{p} - f^s\big)_{,t}\, D^j_x\overset{m+1}{v}{}^s_t\,|\overset{m+1}{v}_{xt}|^{r-2}dx = 0,$$

which after applying the Hölder inequality implies

$$(5.42)\quad \begin{aligned}\frac{1}{r}\frac{d}{dt}\|\overset{m+1}{v}_t\|^r_{1,r} \le{}& -\frac{1}{r}\int_{\partial\Omega}\overset{m}{v}\cdot\bar{n}\,\big(|\overset{m+1}{v}_t|^r + |\overset{m+1}{v}_{xt}|^r\big)ds + \frac{1}{r}\max_\Omega|\operatorname{div}\overset{m}{v}|\,\|\overset{m+1}{v}_t\|^r_{1,r} +\\ &+\max_\Omega|\overset{m+1}{v}_x|\,\|\overset{m}{v}_t\|_r\|\overset{m+1}{v}_t\|^{r-1}_r + \max_\Omega|\overset{m+1}{v}_x|\,\|\overset{m}{v}_{xt}\|_r\|\overset{m+1}{v}_{xt}\|^{r-1}_r +\\ &+\max_\Omega|\overset{m}{v}_t|\,\|\overset{m+1}{v}_{xx}\|_r\|\overset{m+1}{v}_{xt}\|^{r-1}_r + \max_\Omega|\overset{m}{v}_x|\,\|\overset{m+1}{v}_{xt}\|^r_r +\end{aligned}$$

$$+ (\|\overset{m}{p}_t\|_{2,r} + \|f_t\|_{1,r}) \|\overset{m+1}{v}_t\|_{1,r}^{r-1} .$$

Using the curvilinear coordinates and (5.38), the surface integral in (5.42) is estimated by

$$- \int_{\partial\Omega} \overset{m}{v}\cdot\bar{n}(|\overset{m+1}{v}_t|^r + |\overset{m+1}{v}_{xt}|^r) ds \le C(S_1)\int_{S_1} d(|\eta_t|^r + |\eta_{\tau t}|^r + |\overset{m+1}{v}_{,nt}|^r) ds \le$$

(5.43)

$$\le C_7 + C_8 \|\overset{m}{p}\|_{2,r}^r + C_9 \|\overset{m}{p}_t\|_{1,r}^r ,$$

where constants C_7, C_8, C_9 are described in (5.40). From (5.42) and (5.43) we obtain (5.40). This concludes the proof.

Lemmas 5.4, 5.6, 5.7 imply an apriori bounds on elements of sequences $\{\overset{m}{v}\}$, $\{\overset{m}{v}_t\}$, $\{\overset{m}{p}\}$, $\{\overset{m}{p}_t\}$ of the solutions of problems (D_m^s), (E_m^s), s=0,1. These bounds are local in time and independent of m, what is a consequence of the following lemma:

Lemma 5.8

Let us assume that $S_i \in C^4$, i=0,1,2, $\eta \in \Pi^{3-1/r}_{4,\infty,r}(S_1^T)$, $\pi \in \Pi^{3-1/r}_{2,\infty,r}(S_2^T)$, $f \in \Pi^2_{1,\infty,r}(\Omega^T)$ and

(5.44) $$T \le \max_{g>1} \min\left\{ \frac{a_0}{(g y_0)^{1/r}}, t'(g)\right\} ,$$

where $y_0 = \|a\|_{2,r}^r + \|-a^k a_{x_k} + \nabla p(0) + f(0)\|_{1,r}^r \le \|a\|_{2,r}^r + \|a\|_{2,r}^{2r} + \|p(0)\|_{2,r}^r + \|f(0)\|_{1,r}^r \le C(\|a\|_{2,r}, \|f(0)\|_{2,r}, \|\pi(0)\|_{2-1/r,r,S_2}, |\eta(0)|_{2-1/r,1,r,S_1})$, $t'(g) = \frac{1}{\gamma(g y_0)} \ln\frac{g y_0+1}{y_0+1}$ and $\gamma(y) = C(y^4+1)$. Moreover, we denote a value for which the function $T(g) = \min\left\{\frac{a_0}{(g y_0)^{1/r}}, t'(g)\right\}$ has a maximum by $g_0 > 1$. Then the solutions of problems (D_m^s), (E_m^s), s=0,1, are estimated in the following manner

(5.45) $$|\overset{m}{v}|_{2,1,r} \le (g_0 y_0)^{1/r} ,$$

Moreover, from Lemma 5.4 we have the estimate of $|\overset{m}{p}|_{3,2,r}$

$$(5.46)\qquad |\overset{m}{p}|_{3,2,r} \leq C_1' + C_2'(\varrho_0 y_0)^{1/r}$$

where C_i', i=1,2, are explained in (5.29).

Proof. From inequalities (5.35), (5.40) we have for $m \geq 1$

$$(5.47)\qquad \begin{aligned} \frac{d}{dt}|\overset{m+1}{v}|^r_{2,1,r} \leq\; & C_1^1 + C_2^1|\overset{m}{p}|^r_{3,2,r} + C_3^1|\overset{m}{p}|^r_{3,2,r}|\overset{m-1}{p}|^r_{3,2,r} + \\ & + C_4^1|\overset{m}{v}|_{2,1,r}|\overset{m+1}{v}|^r_{2,1,r} + (C_5^1|\overset{m}{p}|_{3,2,r} + C_6^1)|\overset{m+1}{v}|^{r-1}_{2,1,r}, \end{aligned}$$

where C_i^1, i=1,...,6, are constants dependent on C_k, k=1,...,11. Moreover, Lemma 5.4 implies

$$(5.48)\qquad |\overset{m}{p}|_{3,2,r} \leq C_1^2 + C_2^2|\overset{m}{v}|^2_{2,1,r},$$

where C_i^2, i=1,2, are described in (5.29). Assuming that $\overset{m}{y}(t) = |\overset{m}{v}(t)|^r_{2,1,r}$, from (5.47), (5.48) we get for $m \geq 1$

$$(5.49)\qquad \frac{d}{dt}\overset{m+1}{y} \leq \gamma(\overset{m}{y}, \overset{m-1}{y})(\overset{m+1}{y} + 1),$$

where $\gamma(\overset{m}{y}, \overset{m-1}{y}) = C(\overset{m}{y}^4 + \overset{m-1}{y}^4 + 1)$ and C is the upper bound of all constants appearing in (5.47) and (5.48).

Using the method of induction, we will show that $\overset{m}{y}(t)$ is bounded independently of m. We have to obtain a differential inequality similar to (5.49) for the function $\overset{1}{v}$, knowing that $\overset{0}{v} = a$. From (5.25) for m=0 and s=0,1, similarly as in Lemmas 5.6, 5.7 we obtain

$$\frac{d}{dt}|\overset{1}{v}|^{r}_{2,1,r} \le \int_{S_1} d\,(|\overset{1}{v}|^{r}+|\overset{1}{v}_x|^{r}+|\overset{1}{v}_t|^{r}+|\overset{1}{v}_{xx}|^{r}+|\overset{1}{v}_{xt}|^{r})\,ds\; + \tag{5.50}$$
$$+\,C'_{12}\,\|a\|_{2,r}\,|\overset{1}{v}|^{r}_{2,1,r}+(|\overset{\circ}{p}|_{3,1,r}+C'_{13})\,|\overset{1}{v}|^{r-1}_{2,1,r}\,,$$

where $C'_{12}=C'_{12}(r,\Omega)$, $C'_{13}=|f|_{1,0,r}$ .To estimate the surface integral in (5.50) we consider

$$\overset{1}{v}_{,n}=-a_n^{-1}[\overset{1}{v}_t+a_\mu\overset{1}{v}_{,\tau_\mu}+\nabla\overset{\circ}{p}-f],\ \text{so}\ \ \overset{1}{v}_{,n}|_{S_1}=-a_n^{-1}[\eta_t+a_\mu\eta_{,\tau_\mu}+\nabla\overset{\circ}{p}-f], \tag{5.51}$$

where $a_n=-d(0)$ and then $\overset{1}{v}_{,nt}$, $\overset{1}{v}_{,n\tau_\nu}$ can be calculated.At last we can get

$$\overset{1}{v}_{,nn}=a_n^{-2}a_{n,n}[\overset{1}{v}_{,t}+a_\mu\overset{1}{v}_{,\tau_\mu}+\nabla\overset{\circ}{p}-f]-a_n^{-1}[\overset{1}{v}_{,nt}+a_{\mu,n}\overset{1}{v}_{,\tau_\mu}+a_\mu\overset{1}{v}_{,\tau_\mu n}+\nabla\overset{\circ}{p}_{,n}-f_{,n}]. \tag{5.52}$$

Using (5.51), (5.52) ,the surface intgral in (5.50) is estimated in the following way

$$I \le g_1(d_0,\|a\|_{1,r,S_1},|\eta|_{2,0,r,S_1},|f|_{1,0,r,S_1})\; + \tag{5.53}$$
$$+\,g_2(d_0,\|a\|_{1,r,S_1},|f|_{1,0,r,S_1})(|\overset{\circ}{p}|^{2r}_{3,2,r}+|\overset{\circ}{p}|^{r}_{3,2,r})\,.$$

Now the functions $\overset{\circ}{p}$, $\overset{\circ}{p}_t$ we calculate from the problems (E°_{o}), (E^{1}_{o}):

$$\begin{aligned} &\Delta\overset{\circ}{p}=\operatorname{div}f-a^{i}_{x^k}a^{k}_{x^i}\,,\\ (E^{\circ}_{o})\qquad &\overset{\circ}{p}|_{S_2}=\pi(x',t)\,,\quad \frac{\partial\overset{\circ}{p}}{\partial n}\Big|_{S_1}=g(\eta,\bar n,\bar\tau),\\ &\frac{\partial\overset{\circ}{p}}{\partial n}\Big|_{S_0}=(f_n+a^k a\cdot\bar n_{x^k})|_{S_0}\,, \end{aligned}$$

and

$$(E_0^1)\qquad \begin{aligned} &\Delta \overset{\circ}{p}_t = \operatorname{div} f_t\,, \\ &\overset{\circ}{p}_t\big|_{S_2} = \pi_t(x,t),\quad \frac{\partial \overset{\circ}{p}_t}{\partial n}\Big|_{S_1} = g(\eta_t, \bar{n}, \bar{\tau}),\quad \frac{\partial \overset{\circ}{p}_t}{\partial n}\Big|_{S_0} = f_t\cdot\bar{n}\big|_{S_0}\,. \end{aligned}$$

From problems (E_0^s), $s=0,1$, we have the following estimate

$$(5.54)\qquad |\overset{\circ}{p}|_{3,2,r} \leq C(\|a\|_{2,r}, |f|_{2,1,r}, |\pi|_{3-1/r,2,r,S_2}, |\eta|_{3-1/r,1,r,S_1})\,.$$

Therefore, from (5.50), (5.53), (5.54) we have

$$(5.55)\qquad \frac{d}{dt}|\overset{1}{v}|^r_{2,1,r} \leq c_{12} + c_{13}|\overset{1}{v}|^r_{2,1,r}\,,$$

where c_{12}, c_{13} depend on $\|a\|_{2,r}$, $|f|_{2,1,r}$, $|\pi|_{3-1/r,2,r,S_2}$, $|\eta|_{3-1/r,1,r,S_1}$. Integrating (5.55) with respect to time we obtain

$$(5.56)\qquad |\overset{1}{v}|^r_{2,1,r} \leq (1+y_0)\,e^{(c_{12}+c_{13})t} - 1\,.$$

Demanding that $\overset{1}{y} \leq \varrho y_0$, $\varrho > 1$, we obtain the following restriction for t

$$(5.57)\qquad t \leq t_1(\varrho) = \frac{1}{c_{12}+c_{13}} \ln \frac{\varrho y_0 + 1}{y_0+1}\,.$$

Now we shall obtain the required estimate for $m \geq 1$. Assuming that $\overset{m}{y}(t) \leq \varrho y_0$, $\overset{m-1}{y}(t) \leq \varrho y_0$, (for m=1 it has been shown above) we shall show that $\overset{m+1}{y}(t) \leq \varrho y_0$ for a sufficiently small time interval which depends on ϱ. Indeed, from (5.49) we obtain $\overset{m+1}{y}(t) \leq (1+y_0)e^{\gamma(\varrho y_0, \varrho y_0)t} - 1$ and demanding that $\overset{m+1}{y} \leq \varrho y_0$ we get the following restriction for t

$$(5.58)\quad t \leq t_2(\varrho) = \frac{1}{\gamma(\varrho y_0, \varrho y_0)} \ln \frac{\varrho y_0 + 1}{y_0 + 1} .$$

The function $t_2(\varrho)$ has a maximum for $\varrho = \varrho_*$, where ϱ_* is a solution of the equation: $1 = 4\varrho^3 y_0^3 \frac{1+\varrho y_0}{1+\varrho^4 y_0^4} \ln \frac{\varrho y_0 + 1}{y_0 + 1}$.
Let $t'(\varrho) = \min\{t_1(\varrho), t_2(\varrho)\}$, then assuming that $t \leq t'(\varrho)$ and that (5.32) is satisfied, we conclude the proof.

From Lemma 5.8 in the same way as it was done in the proof of Theorem 4.1 we can prove the following theorem:

Theorem 5.3

Let us assume that:

(a) $r > 3$, S_ν, $\nu = 0,1,2$, are of class C^3,

(b) $a \in W_r^2(\Omega)$, $\operatorname{div} a = 0$, $a \cdot \bar{n}|_{S_2} \geq a_0 > 0$, a_0 is a constant,

(c) $\eta \in \Pi^{3-1/r}_{1,\infty,r}(S_1^T)$, $-\eta \cdot \bar{n}|_{S_1} = d \geq d_0 > 0$, d_0 is a constant,

(d) $\pi \in \Pi^{3-1/r}_{2,\infty,r}(S_2^T)$, (e) $f \in \Pi^2_{1,\infty,r}(\Omega^T)$,

(f) the domain Ω has only dihedral angles π/n,

$$(5.59)\quad T \leq \max_{\varrho} \min\left(\frac{a_0}{(\varrho y_0)^{1/r}}, t'(\varrho)\right),$$

where $y_0 = \|a\|^r_{2,r} + \|-a^k a_{x^k} - \nabla p(0) + f(0)\|^r_{1,r} \leq C(\|a\|_{3,r}, \|f(0)\|_{2,r}, \|\pi(0)\|_{2-1/r,r,S_2}, |\eta(0)|_{2-1/r,1,r,S_1})$

and $t'(\varrho) = \frac{1}{\gamma(\varrho y_0)} \ln \frac{\varrho y_0 + 1}{y_0 + 1}$, $\varrho > 1$, $\gamma(y) = C(y^4 + y^2 + 1)$, $C = \text{const}$.

Moreover, we have to assume the compatibility conditions

(g) $D_\tau^{\sigma}(D_t^s v)(0)\big|_{S_1} = D_\tau^{\sigma} D_t^s \eta\big|_{t=0}$,

where $0 \le s \le 1, 0 \le \sigma \le 2-s-1', 1' > \frac{3}{r}$, s, σ are natural numbers, D_τ denotes a derivative tangent to S_1, and

(h) $D_\tau^{\sigma} p(0)\big|_{S_2} = D_\tau^{\sigma} \pi\big|_{t=0}$,

where $0 \le \sigma \le 2$, σ is natural number and D_τ denotes a derivative tangent to S_2.

Then the problem (D,E) has a solution v , p for $t \in [0,T]$ such that

$$v \in \Pi^2_{1,\infty,r}(\Omega^T), \quad p \in \Pi^3_{2,\infty,r}(\Omega^T).$$

Now we shall consider an apriori estimate for solutions of problems (D_m^s) , $s=0,\dots,l$, where $l \geq 2$ is an arbitrary integer.

Lemma 5.9

Let us assume that:

(a) $r > \frac{3}{l-1}$, $d \geq d_0 > 0$, $l \geq 2$,

(b) $\eta \in \Gamma^l_{0,r}(S_1)$, $f \in \Gamma^l_{1,r}(\Omega)$,

(c) S_1 is of class C^{l+1},

(d) $\overset{m}{p} \in \Gamma^{l+1}_{2,r}(\Omega)$, $\overset{m}{v} \in \Gamma^l_{1,r}(\Omega)$

Then for $t \in [0,T]$ an arbitrary solution $\overset{m+1}{v} \in \Gamma^l_{1,r}(\Omega)$ of the problem (D_m^s) , $s=0,\dots,l-1$, satisfies the following differential inequality:

$$(5.60)\quad \frac{d}{dt}|\overset{m+1}{v}|^{r}_{l,1,r} \leq I + C(r,l,\Omega)\,|\overset{m}{v}|_{l,1,r}|\overset{m+1}{v}|^{r}_{l,1,r} + (|\overset{m}{p}|_{l+1,2,r}+|f|_{l,1,r})|\overset{m+1}{v}|^{r-1}_{l,1,r},$$

where I is the following surface integral

$$(5.61)\quad I = \sum_{\substack{i+j\leq l\\ j\leq l-1}} \int_{S_1} d\,|D^j_t D^i_x \overset{m+1}{v}|^r ds,$$

where $D^i_x = \frac{\partial^{i_1}}{\partial x_1^{i_1}}\frac{\partial^{i_2}}{\partial x_2^{i_2}}\frac{\partial^{i_3}}{\partial x_3^{i_3}}$, $i=i_1+i_2+i_3$, $D_t = \frac{\partial}{\partial t}$.

Proof. To prove this and next lemmas we assume that all derivatives of all functions which appear in the problem are continuous. From problems (D^s_m), $s=0,\dots,l-1$ we have

$$(5.62)\quad \sum_{\substack{i+j\leq l\\ j\leq l-1}}\sum_{s=1}^{3}\int_\Omega D^j_t D^i_x(\overset{m+1}{v}{}^s_t + \overset{m}{v}{}^k \overset{m+1}{v}{}^s_{x^k} + \nabla_s \overset{m}{p} - f^s)\, D^j_t D^i_x \overset{m+1}{v}{}^s\, |D^j_t D^i_x \overset{m+1}{v}|^{r-2} dx = 0.$$

From (5.62), using the Hölder inequality and Sobolev imbeddings theorems, we conclude the proof.

Now we shall estimate the surface integral (5.61) which appears in Lemma 5.9.

Lemma 5.10

Let us assume that assumptions (a), (b), (c) of Lemma 5.9 are satisfied, and $a \in W^l_r(\Omega)$. Then the surface integral (5.61) has the following estimate

$$(5.63)\quad I \leq C(S_1)\sum_{i=0}^{l} I^i$$

where

$$(5.64)\quad I^i \leq \sum_{i_1+\dots+i_{i'}\leq i'} C_{i_1\dots i'}\, |\overset{m}{p}|^{r i_1}_{l+1,2,r}\cdots|\overset{m-i'}{p}|^{r i_{i'}}_{l+1,2,r},$$

where $i' = \min(i, \min(m, l))$ and

$$C_{i_1 \dots i_{i'}} = C_{i_1 \dots i_{i'}}(d_0, \|a\|_{l,r}, |\eta|_{l,0,r,S_1}, |f|_{l,1,r}).$$

Proof. Using the curvilinear coordinates, from (5.61) we have

$$\text{(5.65)}\qquad \begin{aligned} I &\leq C(S_1) \sum_{i=0}^{l} \sum_{\substack{|\alpha|+j \leq l-i \\ j \leq l-1}} \int_{S_1} d\, |D_{(\tau)}^{\alpha} D_t^{j} D_n^{i} \overset{m+1}{v}|^{r} ds \equiv \\ &\equiv C(S_1) \sum_{i=0}^{l} I^{i}, \end{aligned}$$

where α is a multiindex and $D_{(\tau)}^{\alpha}$ denotes all possible derivatives of order α, with respect to τ_1, τ_2; $D_n = \frac{\partial}{\partial n}$. To estimate (5.65) we have to consider the form of $D_n^{s} \overset{m+1}{v}$, $s \leq l$. From (5.25) of (D_m^0) we have

$$\text{(5.66)}\qquad D_n \overset{m+1}{v} = -\overset{m}{v}_n^{-1}\left[\overset{m+1}{v}_t + \overset{m}{v}_\mu \overset{m+1}{v}_{,\tau_\mu} + \nabla \overset{m}{p} - f\right].$$

Differentiating (5.66) with respect to n we obtain

$$\text{(5.67)}\qquad \begin{aligned} D_n^2 \overset{m+1}{v} &= \overset{m}{v}_n^{-2} \overset{m}{v}_{n,n}\left[\overset{m+1}{v}_t + \overset{m}{v}_\mu \overset{m+1}{v}_{,\tau_\mu} + \nabla \overset{m}{p} - f\right] - \overset{m}{v}_n^{-1}\left[\overset{m+1}{v}_{,nt} + \overset{m}{v}_{\mu,n} \overset{m+1}{v}_{,\tau_\mu} + \right. \\ &\left. + \overset{m}{v}_\mu \overset{m+1}{v}_{,n\tau_\mu} + \nabla \overset{m}{p}_{,n} - f_{,n} + \overset{m}{v}_\mu (\overset{m+1}{v}_{,\tau_\nu} a_{\nu\mu} + \overset{m+1}{v}_{,n} b_\mu)\right], \end{aligned}$$

where $a_{\nu\mu} = [\bar{n}, \bar{\tau}_\nu] \cdot \bar{\tau}_\mu$, $b_\mu = [\bar{n}, \bar{\tau}_\mu] \cdot \bar{n}$ and $\overset{m}{v}_{,n}$ is described by (5.25) of (D_{m-1}^0), hence it has the following form

$$\text{(5.68)}\qquad \overset{m}{v}_{,n} = -\overset{m-1}{v}_n^{-1}\left[\overset{m}{v}_t + \overset{m-1}{v}_\mu \overset{m}{v}_{,\tau_\mu} + \nabla \overset{m-1}{p} - f\right].$$

At last the s+1 derivative with respect to n has the form

$$\text{(5.69)}\qquad D_n^{s+1} \overset{m+1}{v} = -\sum_{i \leq s} D_n^{i} \overset{m}{v}_n^{-1} D_n^{s-i}\left[\overset{m+1}{v}_t + \overset{m}{v}_\mu \overset{m+1}{v}_{,\tau_\mu} + \nabla \overset{m}{p} - f\right].$$

To obtain the form of $D_n^{s+1}\overset{m+1}{v}$ we shall use inductive considerations. From (5.66) we see that $D_n\overset{m+1}{v}$ has the polynomial form of degree two with respect to $D_t^i D_{(\tau)}^j \overset{m}{v}$, $D_t^i D_{(\tau)}^j \overset{m+1}{v}$ $i+j\leq 1$, and that it is linear with respect to $D_t^i D_{(\tau)}^j \overset{m+1}{v}$, $i+j\leq 1$, $\overset{m}{p}_x$, f. From (5.67) it follows that $D_n^2\overset{m+1}{v}$ is the polynomial of degree four with respect to $D_t^i D_{(\tau)}^j \overset{m+1-\sigma}{v}$, where $\sigma=0,1,2$, $i+j\leq 2$, and that it is the polynomial of degree two with respect to $D_t^i D_{(x)}^j \overset{m-\varsigma}{p}$, $D_t^k D_{(x)}^l f$, where $i+j\leq 2$, $i\leq 1$, $\varsigma=0,1$, $k+l\leq 1$. We see that $D_n^2\overset{m+1}{v}$ depends linearly on $D_t^i D_{(\tau)}^j \overset{m+1}{v}$, where $i+j\leq 2$.

Let us introduce the inductive assumption. Let $D_n^s\overset{m+1}{v}$ be a polynomial with respect to $D_t^i D_{(\tau)}^j \overset{m+1-\sigma}{v}$, $D_t^i D_{(x)}^j \overset{m-\varsigma}{p}$ and $D_t^k D_{(x)}^l f$, where $i+j\leq s$, $\sigma=0,\dots,s$, $\varsigma=0,\dots,s-1$, $k+l\leq s-1$. Then, from (5.69) we see that $D_n^{s+1}\overset{m+1}{v}$, comparing with $D_n^s\overset{m+1}{v}$, contains additional terms $\overset{m-s}{v}$, $\overset{m-s}{p}$, and the order of all derivatives increase by one. Moreover, the degree of the polynomial with respect to $\overset{m+1}{v},\dots,\overset{m-s}{v}$ and $\overset{m}{p},\dots,\overset{m-s}{p}$, f increase by two and one, respectively. At last, if $s>m$, then in $D_n^{s+1}\overset{m+1}{v}$ there appear the functions $D_{(x)}^{\varkappa} a$, $0\leq\varkappa\leq s$, in the power up to $s-m$.

From the inductive assumptions we obtain, that $D_n^{s+1}\overset{m+1}{v}$ is a polynomial of degree $2(s+1)$ and $s+1$ with respect to $D_t^i D_{(\tau)}^j \overset{m+1-\sigma}{v}$ $i+j\leq s+1$, $\sigma=0,\dots,s+1$, and $D_t^i D_{(x)}^j \overset{m-\varsigma}{p}$, $D_t^k D_{(x)}^l f$, $i+j\leq s+1$, $i\leq s$, $\varsigma=0,\dots,s$, $k+l\leq s$, respectively. We have to underline, that $\overset{m+1}{v}$ with all its derivatives appears there linearly.

Hence, using the fact that $W_r^l(\Omega)$, $r>\frac{3}{l-1}$ is an algebra, we obtain (5.63), (5.64), so we have proved the lemma.

Now we shall estimate the initial conditions $D_t^s\overset{m+1}{v}\big|_{t=0}$ appearing in problems (D_m^s), $s=0,1,\dots,l-1$, $l\geq 2$.

Lemma 5.11

Let us assume that $a\in W_r^l(\Omega)$, $f(0)\in\Gamma_{l-s,r}^{l-1}(\Omega)$, $\Pi(0)\in\Gamma_{l+2-s,r}^{l+1-1/r}(S_2)$,

$g(0)\in \Gamma^{l-1/r}_{l+1-s,r}(S_2)$, $r>\frac{3}{l-1}$, $l\geq 2$, then for solutions of problems (D^s_m), (E^s_m) , $s=0,1,\dots,l-1$, the following estimate is valid

$$(5.70)\quad \| D^s_t \overset{m+1}{v}|_{t=0}\|_{l-s,r} \leq F^s\big(\|a\|_{l,r}, |f(0)|_{l-1,l-s,r}, |\Pi(0)|_{l+1-1/r,l+2-s,r,S_2}, |g(0)|_{l-1/r,l+1-s,r,S_2}\big)$$

where F^s is a polynomial of degree $s+1$ with respect to its arguments, $s=1,\dots,l-1$, and for $s=0$ $\|\overset{m+1}{v}|_{t=0}\|_{l,r} = \|a\|_{l,r}$.

Proof. To obtain the form of $D^s_t \overset{m+1}{v}|_{t=0}$, $s=0,\dots,l-1$, let us use the inductive considerations. For $s=0$ we have $\overset{m+1}{v}|_{t=0}=a$. For $s=1$ we have $D_t\overset{m+1}{v}|_{t=0} = -a\cdot\nabla a - \nabla \overset{m}{p}(0)+f(0)$, where $\overset{m}{p}(0)$ is a solution of the problem (E^0_m) for $t=0$. Therefore, we have

$$\overset{m}{p}(0) = \mathcal{F}\big(\operatorname{div} f(0) - a^i_{x_k} a^k_{x_i}, g(\eta(0),\bar{\tau},\bar{n}), (f(0)\cdot\bar{n} + a^k a\cdot\bar{n})|_{S_0}, \Pi(\alpha,0)\big),$$

where $\mathcal{F}$ is a linear functional, which represents a solution of the problem (E^0_m) . Hence, the function $D_t\overset{m+1}{v}|_{t=0}$ is the polynomial of degree two with respect to $D^i_x a$, $i\leq 1$, and it depends on $f(0)$, $g(0)$, $\Pi(0)$. For $s=2$ we have:

$$D^2_t\overset{m+1}{v}|_{t=0} = -D_t\overset{m}{v}|_{t=0}\cdot\nabla a - a\cdot\nabla D_t\overset{m+1}{v}|_{t=0} - \nabla D_t\overset{m}{p}|_{t=0} + D_t f|_{t=0} ,$$

where $D_t\overset{m}{v}|_{t=0}$, $D_t\overset{m+1}{v}|_{t=0}$ are calculated from the previous step (for $s=1$) , and $D_t\overset{m}{p}|_{t=0}$, calculated from the problem (E^1_m), has the form

$$D_t\overset{m}{p}|_{t=0} = \mathcal{F}\big(D_t(\operatorname{div} f - \overset{m}{v}{}^i_{x_k}\overset{m}{v}{}^k_{x_i})|_{t=0}, D_t g|_{t=0}, D_t(f\cdot\bar{n} + \overset{m}{v}{}^k\overset{m}{v}\cdot\bar{n}_{x_k})|_{S_0\cap t=0}, D_t\Pi|_{t=0}\big).$$

Therefore, $D^2_t\overset{m+1}{v}|_{t=0}$ is a polynomial of degree three with respect to $D^i_{(x)} a$, $0\leq i\leq 2$. Moreover, it depends linearly on $D^i_t D^j_{(x)} f|_{t=0}$, $i+j\leq 1$, $D_t\overset{m}{p}_x|_{t=0}$, $\overset{m-1}{p}_x(0)$. At last, the first and third argument of $D_t \overset{m}{p}(0)|_{t=0}$ constitute a polynomial of degree three with respect to $D^i_{(x)} a$, $0 \leq i \leq 2$, and they depend linearly on $D^i_t D^j_{(x)} f|_{t=0}$, $i+j \leq 1$, $D^i_{(x)} \overset{m-1}{p}|_{t=0}$, $i=1,2$.

Now we shall consider the s-th-derivative of $\overset{m+1}{v}$:

$$(5.71)\quad D^s_t \overset{m+1}{v}\Big|_{t=0} = -\sum_{j=0}^{s-1} \left(D^j_t \overset{m}{v} \cdot \nabla D^{s-1-j}_t \overset{m+1}{v}\right)\Big|_{t=0} - \nabla D^{s-1}_t \overset{m}{p}\Big|_{t=0} + D^{s-1}_t f\Big|_{t=0}.$$

Knowing, that $\mathcal{F}$ is a linear functional with respect to its arguments, we can treat $D^{s-1}_t \overset{m+1}{v}$ as a polynomial with respect to a , $f(0)$, $\pi(0)$, $g(0)$ and their derivatives. Moreover, the derivatives of f , π , g with respect to time for t=0 have to be considered also. The expression $D^s_t \overset{m+1}{v}|_{t=0}$, comparing with $D^{s-1}_t \overset{m+1}{v}|_{t=0}$, additionally contains the derivatives $D^{s-1}_t f|_{t=0}$ and $D^{s-1}_t \overset{m}{p}|_{t=0}$, where $D^{s-1}_t \overset{m}{p}|_{t=0}$ depends on $D^{s-1}_t \pi|_{t=0}$, $D^{s-1}_t g|_{t=0}$. In the end, because of the bilinear and linear differential operator of the first order appearing in (5.71), the expression $D^s_t \overset{m+1}{v}|_{t=0}$ has a polynomial form (with respect to a , $f(0)$, $\pi(0)$, $g(0)$ and their derivatives) of the degree greater than $D^{s-1}_t \overset{m+1}{v}|_{t=0}$ by one. The order of derivatives of a , f , π , g appearing in $D^s_t \overset{m+1}{v}|_{t=0}$, is greater by one, also. Therefore, from the inductive considerations, $D^s_t \overset{m+1}{v}|_{t=0}$ is a polynomial of degree s+1 with respect to a ,..., $D^s_{(x)} a$, $D^i_{(x)} D^j_t f(0)$, $D^i_{(x)} D^j_t \pi(0)$, $D^i_{(x)} D^j_t g(0)$ for $i+j \leq s-1$.

At last, using Lemma 5.4 and the fact that W^{l-1}_r , $r > \frac{3}{l-1}$ is an algebra, we conclude the proof.

Using Lemmas 5.9, 5.10, 5.11 we shall prove the theorem:

Theorem 5.4

Let us assume that:

(a) $r > \frac{3}{l-1}$, $l \geq 2$, S_ν , $\nu = 0,1,2$, are of class C^{l+2},

(b) $a \in W_r^l(\Omega)$, $\operatorname{div} a = 0$, $a \cdot \bar{n}|_{S_2} \geq a_0 > 0$,

(c) $\eta \in \Pi_{1,r}^{l+1-1/r}(S_1^T)$, $-\eta \cdot \bar{n}|_{S_1} = d \geq d_0 > 0$,

(d) $f \in \Pi_{1,r}^{l}(\Omega^T)$, (e) $\pi \in \Pi_{2,r}^{l+1-1/r}(S_2^T)$.

In the case of arbitrary dihedral angles we have to assume additionally that:

(g) $\frac{\pi}{\alpha_0} > l$ and $r = 2$, $l \geq 3$,

where α_0 is the maximal dihedral angle in Ω .Let

(5.72) $$T \leq \max_{\varrho} \min \left\{ \frac{a_0}{(\varrho y_0)^{1/r}}, t'(\varrho) \right\},$$

where y_0 is described in the assumptions of Theorem 5.3 and

(5.73) $$t'(\varrho) = \frac{1}{\gamma(\varrho Y(0))} \ln \frac{\varrho Y(0)+1}{Y(0)+1} \quad , \varrho > 1,$$

where $\gamma(\varrho Y(0)) = \max \{ C(r,l,\Omega)[\varrho Y(0)+1], W(\varrho Y(0),\dots,\varrho Y(0)) \}$,
and

$$W(a_1,\dots,a_i) = \sum_{i_1+\dots+i_i \leq 2i} C'_{i_1\dots i_i} a_1^{r i_1} \dots a_i^{r i_i},$$

$i=\min(m,l)$, where

$$C'_{i_1\dots i_i} = C'_{i_1\dots i_i}(l,r,\Omega,d_0,|\eta|_{l+1-1/r,1,r,S_1^T}, |f|_{l,1,r,\Omega^T}, |\pi|_{l+1-1/r,2,r,S_2^T}, \|a\|_{l,r}).$$

In the end $Y(0) = \sum_{s=0}^{l-1} \| D_t^s \bar{v}|_{t=0} \|_{l-s,r}^r$,which is estimated in Lemma 5.11. Moreover, the following compatibility conditions are satisfied

$$D_\tau^\sigma (D_t^s v)(0)|_{S_1} = D_\tau^\sigma D_t^s \eta|_{t=0} ,$$

where $0 \le s \le l-1, 0 \le \sigma \le l-s-l', l' > \frac{3}{r}$, s, σ are natural numbers, D_τ is a derivative tangent to S_1, and

$$D_\tau^\sigma (D_t^s p)(0)|_{S_2} = D_\tau^\sigma D_t^s \pi|_{t=0} ,$$

where $0 \le s \le l-2, 0 \le \sigma \le l+1-s-l'$, s, σ are natural numbers and D_τ is a derivative tangent to S_2.

Then there exists a solution of the problem (D,E) for $t \in [0,T]$ such that

$$v \in \Pi^{l}_{1,\infty,r}(\Omega^T), \quad p \in \Pi^{l+1}_{2,\infty,r}(\Omega^T).$$

Proof. Using the Young inequality in (5.60) we obtain

$$\frac{d}{dt} |\overset{m+1}{v}|^r_{l,1,r} \le I + \frac{2^r}{r}\left(|\overset{m}{p}|^r_{l+1,2,r} + |f|^r_{l,1,r}\right) + \tag{5.74}$$
$$+ \left[C(r,l,\Omega) |\overset{m}{v}|_{l,1,r} + \frac{r-1}{r}\right] |\overset{m+1}{v}|^r_{l,1,r} .$$

Using (5.29) and (5.63) in (5.74) we have

$$\frac{d}{dt} |\overset{m+1}{v}|^r_{l,1,r} \le W(|\overset{m}{v}|_{l,1,r}, \dots, |\overset{m-i}{v}|_{l,1,r}) + C(r,l,\Omega)(|\overset{m}{v}|_{l,1,r} + 1) |\overset{m+1}{v}|^r_{l,1,r} . \tag{5.75}$$

If $|\overset{i}{v}|_{l,1,r} \le \varrho Y(0)$, $\varrho > 1, i \le m$, then from the assumptions of this theorem we get the estimate

$$(5.76)\quad \frac{d}{dt}\left|\overset{m+1}{v}\right|^{r}_{L,1,r} \leq \gamma(\varrho Y(0))\left[\left|\overset{m+1}{v}\right|^{r}_{L,1,r}+1\right],$$

which is the same as the estimate (4.13).Therefore,the remaining part of the proof of this theorem is the same as in the proof of Theorem 4.1.This concludes the proof.

6. Leakage problem in two-dimensional domain with corners

In the two-dimensional case the problem (B) can be simplified. The second equation of (B) implies that there exists a function φ such that

$$(6.1) \qquad v^1 = -\varphi_{,x^2}, \quad v^2 = \varphi_{,x^1} .$$

Then the boundary conditions gives $-n_1\varphi_{,x^2} + n_2\varphi_{,x^1} = b$, so $\varphi_{,\tau}|_{\partial\Omega} = b(\tau)$ and $\varphi(\tau)|_{\partial\Omega} = \int b(\tau)d\tau \equiv \beta(\tau)$, where τ is a parameter along the boundary and n_1, n_2 are coordinates of the vector normal to the boundary. Therefore, instead of (B) we have

$$(6.2) \qquad \Delta\varphi = \omega, \quad \varphi|_{\partial\Omega} = \beta .$$

Moreover, in this case instead of the problem (A) we have

$$(6.3) \qquad \begin{aligned} &\omega_t + v^k \omega_{x^k} = F \equiv \operatorname{rot} f, \\ &\omega|_{t=0} = \omega_0, \\ &\omega|_{S_1} = \eta . \end{aligned}$$

Similarly as in the case of problems (A), (B) in Section 5 we can prove the following theorem:

Theorem 6.1

Let us assume that:

(a) $r > \frac{2}{l}$, $l \geq 1$, $-v\cdot\bar{n}|_{S_1} = d \geq d_0 > 0$, d_0 is a constant,

(b) S_ν, $\nu = 1,2,3,4$, are of class C^{l+2},

(c) $a \in W_r^{l+1}(\Omega)$, $\operatorname{div} a = 0$,

(d) $\eta \in \Pi^{l}_{0,\infty,r}(S_1^T)$,

(e) $\operatorname{rot} f \in \Pi^{l}_{0,\infty,r}(\Omega^T)$,

(f) $\beta \in \Pi^{l+2-1/r}_{2,\infty,r}(\partial\Omega^T)$,

and $T \le \max_{\xi} t'(\xi)$,where $t'(\xi) = \frac{1}{\gamma(\xi y_0)} \ln \frac{\xi y_0 + 1}{y_0 + 1}$, $\xi > 1$,
γ is determined in the proof of Theorem 5.2 and $y_0 = \sum_{s=0}^{l} \|D_t^s \omega(0)\|^r_{l-s,r}$
can be estimated similarly as in Lemma 5.3.Moreover,the following compatibility conditions must be satisfied

$$\frac{\partial^\sigma}{\partial\tau^\sigma}\left(\frac{\partial^s}{\partial t^s}\omega\right)(0)\Big|_{S_1} = \frac{\partial^\sigma}{\partial t^\sigma}\frac{\partial^s}{\partial t^s}\eta\Big|_{t=0},$$

where $s \le l-l'$, $l' > \frac{2}{r}$, $\sigma \le l-l'-s$, $\frac{\partial}{\partial\tau}$ is a derivative tangent to S_1; and

$$\frac{\partial^s}{\partial\tau^s} a\cdot\bar{n}\Big|_{\partial\Omega} = \frac{\partial^s}{\partial\tau^s} b\Big|_{t=0},$$

where $s \le l-l'$ and $\frac{\partial}{\partial\tau}$ is a derivative tangent to $\partial\Omega$.

Then in the case of arbitrary corners for

(6.4) $$\frac{\pi}{\alpha_0} > 2+l-\frac{2}{r},$$

where α_0 is the maximal angle,there exists a local solution of the problem (6.2),(6.3) such that

$$v \in \Pi^{l+1}_{1,\infty,r}(\Omega^T),\quad \omega \in \Pi^{l}_{0,\infty,r}(\Omega^T).$$

Proof. The proof is similar to the proof of Theorem 5.2. The difference appears in the case of domains with arbitrary angles, because in this case we use different results (Theorems D.2, D.4 for the three-dimensional case and Theorem E.2 in the two-dimensional case) . These results give different conditions on angles see the condition (g) of Theorem 5.2 and (6.4) . This ends the proof.

In the case of the problem (D), (E) we obtain the theorem almost the same as Theorem 5.4. We formulate it in the following manner:

Theorem 6.2

Let us assume that:

(a) $r > \frac{2}{l-1}$, $l \geqslant 2$, $-v\cdot\bar{n}|_{S_1} = d \geqslant d_0 > 0$,

(b) S_ν, $\nu = 1,2,3,4$, are of class C^{l+2},

(c) $a \in W_r^l(\Omega)$, $\operatorname{div} a = 0$, $a\cdot\bar{n}|_{S_3} \geqslant a_0 > 0$,

(d) $\eta \in \Pi_{0,\infty,r}^{l+1-1/r}(S_1^T)$,

(e) $\pi \in \Pi_{2,\infty,r}^{l+1-1/r}(S_3^T)$, (f) $f \in \Pi_{0,\infty,r}^{l}(\Omega^T)$,

and $T \leq \max_{g} \min\left(\frac{a_0}{(g y_0)^{1/r}}, t^1(g)\right)$, where $y_0 = \sum_{s=0}^{l-1} \|D_t^s v(0)\|_{l-s,r}^r$,
$t^1(g) = \frac{1}{\gamma(g y_0)} \ln \frac{g y_0 + 1}{y_0 + 1}$, $g > 1$, γ is described in the assumptions of Theorem 5.4 and additionally the condition must be satisfied

(6.5) $\frac{\pi}{\alpha_0} > l + 1 - \frac{2}{r}$

in the case of arbitrary corners, where $\measuredangle_0$ is the maximal angle. Moreover, the following compatibiliy conditions must be satisfied:

$$D_\tau^\sigma (D_t^s v)(0)\big|_{S_1} = D_\tau^\sigma D_t^s \eta\big|_{t=0} ,$$

where $s \leq l-l'$, $\sigma \leq l-l'-s$, $l' > \frac{2}{r}$, D_τ is a derivative tangent to S_1; and

$$D_\tau^\sigma (D_t^s p)(0)\big|_{S_3} = D_\tau^\sigma D_t^s T\big|_{t=0}$$

where $s \leq l-l'$, $\sigma \leq l+1-l'-s$ and D_τ is a derivative tangent to S_3.

Then there exists a local solution of the problem (D,E) such that

$$v \in \Pi^{l}_{1,\infty,r}(\Omega^T), \quad p \in \Pi^{l+1}_{2,\infty,r}(\Omega^T), \quad l \geq 2.$$

Proof. The proof is almost the same as the proof of Theorem 5.4. The distinguish appears in the case of domains with arbitrary angles (see the proof of Theorem 6.1). This concludes the proof.

7. Statement of a mixed problem for a compressible barotropic motion

In this section we shall consider the barotropic motion of a compressible nonviscous fluid in a bounded, simply connected domain $\Omega \subset \mathbb{R}^n$, where n=2,3. We assume that the flow is given by a gas in an isentropic motion (i.e. the entropy is constant). Moreover, the Clapeyron's equation of state with constant specific heats is assumed.

Let us denote the velocity, the density and the fluid pressure in a point x and in a time t by $v(x,t)$, $\rho(x,t)$ and $p(x,t)$, and the density of the external mass forces by $f(x,t)$.

Then, the fluid motion is described by the following equations:

$$(7.1)\qquad \rho(v_t + v\cdot\nabla v - f) = -\nabla p(\rho),$$

$$(7.2)\qquad \rho_t + \operatorname{div}(\rho v) = 0,$$

where $p = A\rho^{\gamma}$, $A>0$, $1 \leq \gamma \leq 2$. Moreover, we assume the following initial and boundary conditions:

$$(7.3)\qquad v|_{t=0} = a(x),$$

$$(7.4)\qquad \rho|_{t=0} = \rho_0(x).$$

$$(7.5)\qquad v\cdot\bar{n}|_{\partial\Omega} = 0.$$

The existence and uniqueness of solutions of this problem was proved by Veiga [Ve, 1]. Moreover, repeating the considerations from [Ve, 1] we have the following compatibility conditions:

$$(7.6) \qquad a\cdot\bar{n}\big|_{\partial\Omega} = 0,$$

$$(7.7) \qquad \frac{p'(\varrho_0)}{\varrho_0}\frac{\partial\varrho_0}{\partial n} = n^i_{,x^j}a^i a^j + f_n\Big|_{t=0} \quad \text{on } \partial\Omega,$$

$$(7.8) \qquad n^j_{,x^i}\left(v^i_t(0)a^j + a^i v^j_t(0)\right) + f_t(0)\cdot\bar{n} + \frac{\partial}{\partial n}\left[\left(a\cdot\frac{\nabla\varrho_0}{\varrho_0} + \operatorname{div} a\right)p'(\varrho_0)\right] = 0 \quad \text{on } \partial\Omega,$$

where $v_t(0) = -a\cdot\nabla a + f\big|_{t=0} - p'(\varrho_0)\frac{\nabla\varrho_0}{\varrho_0}$, $p'(\varrho_0) = \frac{dp}{d\varrho_0}$.
It was assumed also that

$$(7.9) \qquad \varrho_0(x) \geqslant m_0 > 0.$$

The proof in [Ve, 1] is very complicated and basis on the Leray-Schauder's principle. We shall prove the existence of solutions of this problem by our method of successive approximations, which is simpler than presented in [Ve, 1]. To do this, we replace the problem (7.1)$\div$(7.5) by a few well posed problems.

Introducing the new variable $g = \ln\varrho$ we obtain the following form of the above problem:

$$(7.10) \qquad v_t + v\cdot\nabla v - f = -h(g)\nabla g,$$

where $h(g) = p'(\exp g)$, $p'(\varrho) = \frac{dp}{d\varrho} = a_s^2$ and a_s is the sound velocity.

$$(7.11) \qquad g_t + v\cdot\nabla g + \operatorname{div} v = 0,$$

$$(7.12) \qquad v\big|_{t=0} = a(x),$$

$$(7.13)\qquad g|_{t=0} = g_0 \equiv \ln \varrho_0 .$$

Applying the operator div to equations (7.10), from (7.10) and (7.11) we obtain

$$(7.14)\qquad \delta_t + v\cdot\nabla\delta + \operatorname{div}(h(g)\nabla g) = \operatorname{div} f - v^i_{x^k} v^k_{x^i} \equiv F,$$

$$(7.15)\qquad g_t + v\cdot\nabla g + \delta = 0,$$

where $\delta = \operatorname{div} v$. Eliminating δ from (7.14) and (7.15) we get the following problem for g, denoted as (F):

$$(7.16)\qquad Q^2 g - \operatorname{div}(h(g)\nabla g) = \operatorname{div} f - v^i_{x^k} v^k_{x^i} \equiv F,$$

(F)

$$(7.17)\qquad g|_{t=0} = g_0,\quad g_t|_{t=0} = -a\cdot\nabla g_0 - \operatorname{div} a,$$

$$(7.18)\qquad \frac{\partial g}{\partial n}\Big|_{\partial\Omega} = \frac{n^i_{,x^j} v^i v^j + f_n}{h(g)}\Big|_{\partial\Omega},$$

where $Q = \partial_t + v\cdot\nabla$. Moreover, we treat the function $v(x,t)$ appearing in (7.16) and (7.18) as a given function. The equation (7.16) is a strictly hyperbolic one, what will be shown in Section F. Moreover, in this section the existence and uniqueness of solutions of the problem (F) will be shown, using the results of [Ve,1].

Applying the operator rot to (7.10) and (7.12), we obtain the following problem on the vorticity vector $\omega = \operatorname{rot} v$:

$$(7.19)\qquad \omega_t + v^k\omega_{x^k} - \omega^k v_{x^k} + \operatorname{div} v\ \omega = \operatorname{rot} f,$$

(G)

$$(7.20)\qquad \omega|_{t=0} = \omega_0 \equiv \operatorname{rot} a,$$

where v is a given function. At last, a vector $v(x,t)$ we calculate from the following overdetermined elliptic problem:

$$\text{(H)} \qquad \begin{aligned} \operatorname{div} v &= -Qg, \\ \operatorname{rot} v &= \omega \\ v_n|_{\partial\Omega} &= 0, \end{aligned}$$

where $v(x,t)$, $g(x,t)$ and $\omega(x,t)$ in the right-hand sides of equations from (H) are given.

8. The mixed problem for barotropic motion in bounded domain

In this section we shall prove the existence and uniqueness of solutions of the problem (F,G,H). To do this we use the method of successive approximations. Therefore, instead of problems (F), (G), (H) we shall consider the following system of problems. The problem for $\overset{m}{g}$

$$\overset{m-1}{Q}{}^{2}\overset{m}{g} - \operatorname{div}\big(h(\overset{m-1}{g})\nabla\overset{m}{g}\big) = \operatorname{div} f - \overset{m-1}{v}{}^{k}_{x^i}\,\overset{m-1}{v}{}^{i}_{x^k},$$

(F$_m$)
$$\overset{m}{g}\big|_{t=0} = g_0, \quad \overset{m}{g}_t\big|_{t=0} = -a\cdot\nabla g_0 - \operatorname{div} a,$$

$$\frac{\partial\overset{m}{g}}{\partial n}\bigg|_{\partial\Omega} = \frac{n^i_{,x^j}\,\overset{m-1}{v}{}^{i}\,\overset{m-1}{v}{}^{j} + f_n}{h(\overset{m-1}{g})}\bigg|_{\partial\Omega},$$

where $\overset{m-1}{Q} = \partial_t + \overset{m-1}{v}\cdot\nabla$ and $\overset{m-1}{v}, \overset{m-1}{g}$ are given functions. The problem for $\overset{m}{\omega}$

$$\overset{m}{\omega}_t + \overset{m-1}{v}{}^{k}\,\overset{m}{\omega}_{x^k} - \overset{m}{\omega}{}^{k}\,\overset{m-1}{v}_{x^k} + \operatorname{div}\overset{m-1}{v}\,\overset{m}{\omega} = \operatorname{rot} f,$$

(G$_m$)
$$\overset{m}{\omega}\big|_{t=0} = \omega_0 \equiv \operatorname{rot} a,$$

where $\overset{m-1}{v}$ is a given function. At last the problem for $\overset{m}{v}$

$$\operatorname{div}\overset{m}{v} = -\overset{m-1}{Q}\,\overset{m}{g} + \frac{1}{|\Omega|}\int_\Omega \overset{m-1}{Q}\,\overset{m}{g}\,dx,$$

(H$_m$)
$$\operatorname{rot}\overset{m}{v} = \overset{m}{\omega},$$

$$\overset{m}{v}\cdot\bar{n}\big|_{\partial\Omega} = 0,$$

where $\overset{m-1}{v}$, $\overset{m}{g}$ and $\overset{m}{\omega}$ are given functions. The right-hand sides of equations of the problem (H_m) are in such form, that all compatibility conditions are satisfied. Moreover, $\overset{\circ}{v}=a$, $\overset{\circ}{g}=g_0$. The existence of solutions of the problem (F_m) is proved in Section F. To prove the existence of solutions of the problem (G_m) we use the method of characteristic and at last the existence of solutions of the problem (H_m) can be shown by potentials. Introducing the function

$$(8.1)\qquad \overset{m}{\Psi}(t)=\frac{1}{|\Omega|}\int_{\Omega}\overset{m-1}{Q}\,\overset{m}{g}\,dx$$

we shall see that $\lim_{m\to\infty}\overset{m}{\Psi}(t)=0$.

Now we shall prove the following result:

Theorem 8.1

Let us assume that

(1) $\partial\Omega\in C^5$, $a\cdot\bar{n}|_{\partial\Omega}=0$,

(2) $g_0\in H^3(\Omega)$, $a\in H^3(\Omega)$,

(3) $f\in\Pi^3_{1,2}(\Omega^T)$.

Then for $t\le T\le t_0$, where t_0 is determined by (8.11) there exists a unique solution of the problem (F,G,H) such, that $g\in\Pi^3_{0,\infty}(\Omega^t)$, $v\in\Pi^3_{1,\infty}(\Omega^t)$, $\omega\in\Pi^2_{0,\infty}(\Omega^t)$. Moreover, the restriction (8.11), (8.12), (8.22) must be satisfied.

Proof. At first we obtain an apriori estimate for $(\overset{m}{g},\overset{m}{\omega},\overset{m}{v})$. Using Lemma F.1 we have the following estimate for solutions of the problem (F_m).

$$(8.2)\quad |\overset{m}{g}|^2_{3,0,\infty,\Omega^t} \le C\overset{m-1}{\mathcal{L}}\, e^{C\overset{m-1}{\mathcal{L}}{}^2 t}\Big[\|g_0\|^2_{3,2,\Omega}+\|a\|^2_{3,2,\Omega}+\|g_0\|^2_{3,2,\Omega}\|a\|^2_{3,2,\Omega}+ |\operatorname{div} f|^2_{2,1,\Omega^t}+|\overset{m-1}{v}|^2_{3,1,\Omega^t}|\overset{m-1}{v}|^2_{3,1,\infty,\Omega^t}\big(1+|\overset{m-1}{g}|^2_{3,1,\infty,\Omega^t}+|\overset{m-1}{g}|^4_{3,1,\infty,\Omega^t}\big)\Big]$$

where

$$\overset{m-1}{\mathcal{L}} = |\overset{m-1}{v}|^4_{3,1,\infty,\Omega^t} + |h(\overset{m-1}{g})|^2_{3,1,\infty,\Omega^t} + 1 .$$

For solutions of the problem (G_m) we have the estimate

$$(8.3)\quad |\overset{m}{\omega}|^2_{2,0,\infty,\Omega^t} \le C\, e^{Ct|\overset{m-1}{v}|^2_{3,1,\infty,\Omega^t}}\big[|f|^2_{3,1,\Omega^t}+|\omega(0)|^2_{2,0,\Omega}\big]$$

where

$$(8.4)\quad |\omega(0)|^2_{2,0,\Omega} \le \Phi(\|a\|_{3,2,\Omega}, \|g_0\|_{3,2,\Omega}, \|f(0)\|_{2,2,\Omega}) + |f(0)|^2_{2,1,\Omega} .$$

At last for solutions of the problem (H_m) the following estimate is valid

$$(8.5)\quad |\overset{m}{v}|^2_{3,1,\infty,\Omega^t} \le C\big[|\overset{m}{\omega}|^2_{2,0,\infty,\Omega^t}+|\overset{m}{g}|^2_{3,0,\infty,\Omega^t}\big(1+|\overset{m-1}{v}|^2_{3,1,\infty,\Omega^t}\big)\big] .$$

Assuming that

$$(8.6)\quad \overset{m}{\eta}{}^2 = |\overset{m}{g}|^2_{3,0,\infty,\Omega^t} + |\overset{m}{v}|^2_{3,1,\infty,\Omega^t} ,$$

$$(8.7)\quad R^2 = \|g_0\|^2_{3,2,\Omega}+\|a\|^2_{3,2,\Omega}+|f|^2_{3,1,\Omega^t}+|\omega(0)|^2_{2,0,\Omega} ,$$

we have

$$(8.8)\quad |\overset{m}{v}|^2_{3,1,\infty,\Omega^t} \le C\big[e^{Ct\overset{m-1}{\eta}}R^2+|\overset{m}{g}|^2_{3,0,\infty,\Omega^t}(1+\overset{m-1}{\eta}{}^2)\big] ,$$

$$(8.9)\quad |\overset{m}{g}|_{3,0,\infty,\Omega^t} \le C(\overset{m-1}{\eta}{}^4+h^2(\overset{m-1}{\eta})+1)\,e^{C(\overset{m-1}{\eta}{}^8+h^4(\overset{m-1}{\eta})+1)t}\big[R^2+R^4+t\overset{m-1}{\eta}{}^4(1+\overset{m-1}{\eta}{}^4)\big].$$

From (8.8) and (8.9) we have

$$\overset{m}{\eta} \leq C\left[\overset{m-1}{\eta}{}^{4}+h^{2}(\overset{m-1}{\eta})+1\right] e^{C\left[\overset{m-1}{\eta}{}^{8}+h^{4}(\overset{m-1}{\eta})+1\right]t}\left[R^{2}+R^{4}+t\,\overset{m-1}{\eta}{}^{4}(1+\right.$$

$$\left.+\overset{m-1}{\eta}{}^{4})\right](1+\overset{m-1}{\eta}{}^{2}) \equiv G(R,t,\overset{m-1}{\eta}). \tag{8.10}$$

From the form of $G(R,t,\eta)$ we have that $G(0,0,\eta)=0$. Therefore, there exist t_0, R_0 and $\eta_0(R_0,t_0)$ such, that for $t \leq t_0, R \leq R_0$ we have

$$\inf\{\eta>0 : G(R,t,\eta) \leq \eta\} = \eta_0(R_0,t_0). \tag{8.11}$$

Assuming that

$$\overset{\circ}{\eta} \leq \eta_0(R_0,t_0), \tag{8.12}$$

where $\overset{\circ}{\eta}{}^{2} = |\overset{\circ}{g}|^{2}_{3,0,\infty,\Omega^t} + |\overset{\circ}{v}|^{2}_{3,1,\infty,\Omega^t}$, we obtain

$$\overset{m}{\eta} \leq \eta_0(R_0,t_0). \tag{8.13}$$

Now we shall prove a convergence of the sequence $(\overset{m}{g},\overset{m}{\omega},\overset{m}{v})$ in $\Pi^{2}_{0,\infty}(\Omega^t)\times\Pi^{1}_{0,\infty}(\Omega^t)\times\Pi^{2}_{1,\infty}(\Omega^t)$. Introducing $\overset{m}{G}=\overset{m}{g}-\overset{m-1}{g}$, $\overset{m}{\Omega}=\overset{m}{\omega}-\overset{m-1}{\omega}$, $\overset{m}{\vartheta}=\overset{m}{v}-\overset{m-1}{v}$ we consider the following system of problems

$$\overset{m-1}{Q}{}^{2}\overset{m}{G} - \operatorname{div}(h(\overset{m-1}{g})\nabla\overset{m}{G}) = \operatorname{div}(h'(\overset{m-1}{\tilde{g}})\nabla\overset{m-1}{g}\,\overset{m-1}{G}) +$$

$$-\left[2\overset{m-1}{\vartheta}\nabla\tfrac{\partial}{\partial t}+\overset{m-1}{\vartheta}_{t}\nabla+\overset{m-1}{v}\cdot\nabla\overset{m-1}{\vartheta}\nabla+\overset{m-1}{\vartheta}\nabla\overset{m-2}{v}\nabla+(\overset{m-1}{v}{}^{i}\overset{m-1}{\vartheta}{}^{j}+\overset{m-1}{\vartheta}{}^{i}\overset{m-2}{v}{}^{j})\nabla_i\nabla_j\right]\overset{m-1}{g}$$

$$-\overset{m-1}{\vartheta}{}^{i}_{x^k}\overset{m-1}{v}{}^{k}_{x^i}-\overset{m-2}{v}{}^{i}_{x^k}\overset{m-1}{\vartheta}{}^{k}_{x^i}, \tag{8.14}$$

$$\overset{m}{\tilde{G}}\Big|_{t=0}=0,\ \overset{m}{\tilde{G}}_t\Big|_{t=0}=0,$$

$$\frac{\partial \overset{m}{\tilde{G}}}{\partial n}\Big|_{\partial\Omega}=\frac{n^i_{x_j}(\overset{m-1}{v}{}^i\overset{m-1}{\vartheta}{}^j+\overset{m-1}{\vartheta}{}^i\overset{m-1}{v}{}^j)}{h(\overset{m-1}{g})}\Big|_{\partial\Omega}+\tilde{h}'(\tilde{g})\overset{m-1}{G}(n^i_{x_j}\overset{m-2}{v}{}^i\overset{m-2}{v}{}^j+f_n)\Big|_{\partial\Omega},$$

where $\tilde{g}\in[\overset{m-1}{g},\overset{m-2}{g}]$, $\tilde{h}=\frac{1}{h}$, $\tilde{h}'(\xi)=\frac{\partial\tilde{h}}{\partial\xi}$, $h'(\xi)=\frac{\partial h}{\partial\xi}$;

$$\overset{m}{\Omega}_t+\overset{m-1}{v}{}^k\overset{m}{\Omega}_{x^k}+\overset{m-1}{\vartheta}{}^k\overset{m-1}{\omega}_{x^k}-\overset{m}{\Omega}{}^k\overset{m-1}{v}_{x^k}-\overset{m-1}{\omega}{}^k\overset{m-1}{\vartheta}_{x^k}+$$

$$(8.15)\qquad +\operatorname{div}\overset{m-1}{v}\,\overset{m}{\Omega}+\operatorname{div}\overset{m-1}{\vartheta}\,\overset{m-1}{\omega}=0,$$

$$\overset{m}{\Omega}\Big|_{t=0}=0;$$

$$\operatorname{div}\overset{m}{\vartheta}=-(\overset{m}{G}_t+\overset{m-1}{\vartheta}\cdot\nabla\overset{m}{g}+\overset{m-2}{v}\cdot\nabla\overset{m}{G})+\frac{1}{|\Omega|}\int_\Omega(\overset{m}{G}_t+\overset{m-1}{\vartheta}\cdot\nabla\overset{m}{g}+\overset{m-2}{v}\nabla\overset{m}{G})dx,$$

$$(8.16)\qquad \operatorname{rot}\overset{m}{\vartheta}=\overset{m}{\Omega},\qquad \overset{m}{\vartheta}\cdot\bar{n}\big|_{\partial\Omega}=0.$$

Let us denote $\mathcal{L}_0=|\overset{m}{g}|_{3,0,\infty,\Omega^t}+|\overset{m}{v}|_{3,1,\infty,\Omega^t}+|\overset{m}{\omega}|_{2,0,\infty,\Omega^t}$.
Using Lemma F.2 we have

$$(8.17)\qquad |\overset{m}{G}|^2_{2,0,\infty,\Omega^t}\le[C(\mathcal{L}_0)+|f|^2_{3,1,\Omega^t}]t|\overset{m-1}{G}|^2_{2,0,\infty,\Omega^t}+C(\mathcal{L}_0)t|\overset{m-1}{\vartheta}|^2_{3,1,\infty,\Omega^t}$$

From the problem (8.15) we get

$$(8.18)\qquad \frac{d}{dt}|\overset{m}{\Omega}|^2_{1,0,\Omega}\le C(\mathcal{L}_0)(|\overset{m}{\Omega}|^2_{1,0,\Omega}+|\overset{m-1}{\vartheta}|^2_{2,1,\Omega}),$$

hence using that $\overset{m}{\Omega}\big|_{t=0}=0$ we have

$$(8.19)\qquad |\overset{m}{\Omega}|_{1,0,\infty,\Omega^t} \le C(\alpha_0)|\overset{m-1}{\vartheta}|_{2,1,\infty,\Omega^t} .$$

At last the problem (8.16) implies

$$(8.20)\qquad |\overset{m}{\vartheta}|_{2,1,\infty,\Omega^t} \le C(|\overset{m}{G}|_{2,0,\infty,\Omega^t} + |\overset{m}{\Omega}|_{1,0,\infty,\Omega^t}) + C(\alpha_0)|\overset{m-1}{\vartheta}|_{2,1,\infty,\Omega^t} .$$

From (8.17), (8.19) and (8.20) we have

$$(8.21)\qquad \begin{aligned} |\overset{m}{G}|^2_{2,0,\infty,\Omega^t} + |\overset{m}{\vartheta}|^2_{2,1,\infty,\Omega^t} &\le [C(\alpha_0) + |f|^2_{3,1,\Omega^t}]t|\overset{m-1}{G}|^2_{2,0,\infty,\Omega^t} + \\ &+ C(\alpha_0)|\overset{m-1}{\vartheta}|^2_{2,1,\infty,\Omega^t} . \end{aligned}$$

Assuming that

$$(8.22)\qquad (C(\alpha_0) + |f|^2_{3,1,\Omega^t})t + C(\alpha_0) < 1,$$

we obtain that the sequence $\overset{m}{g}, \overset{m}{\omega}, \overset{m}{v}$ converges in $\Pi^2_{0,\infty}(\Omega^t) \times \Pi^1_{0,\infty}(\Omega^t) \times \Pi^2_{1,\infty}(\Omega^t)$, so there exists a limit function (g, ω, v) in these spaces and from the estimate (8.13) the limit function belongs to $\Pi^3_{0,\infty}(\Omega^t) \times \Pi^2_{0,\infty}(\Omega^t) \times \Pi^3_{1,\infty}(\Omega^t)$. To show that (g, ω, v) is a solution of the problem (F,G,H), instead of problems (F) and (G) we consider the following integral identities:

$$(8.23)\qquad \begin{aligned} &\int_{\Omega^T} [\overset{m-1}{Q}{}^2 \overset{m}{g} - \operatorname{div}(h(\overset{m-1}{g})\nabla\overset{m}{g})]\varphi(x,t)\, dx\, dt = \\ &= \int_{\Omega^T} [\operatorname{div} f - \overset{m-1}{v}{}^i_{x^k} \overset{m-1}{v}{}^k_{x^i}]\varphi(x,t)\, dx\, dt , \end{aligned}$$

for every continuously differentiable function $\varphi(x,t)$ and

$$(8.24)\quad \int_{\Omega^T} \Big[\overset{m}{\omega}_t + \big(\overset{m-1}{v}{}^k \overset{m}{\omega}\big)_{,x^k} - \overset{m}{\omega}{}^k \overset{m-1}{v}_{x^k} - \operatorname{rot} f \Big] \Psi(x,t)\, dx\, dt = 0 \;,$$

for every continuously differentiable function $\Psi(x,t)$. The problem (H) we can consider classically because all derivatives of $\overset{m}{v}$ are continuous. Passing in (8.23), (8.24) and (H_m) with m to the infinity we obtain that (g, ω, v) is a solution of the problem (F, G, H), because $\Psi = \lim \overset{m}{\Psi}$ is such, that $\overset{m}{\Psi}_t = 0$ and $\Psi(0) = 0$, because $\overset{m}{g}_t|_{t=0} = -a\nabla g_0 - \operatorname{div} a'$. This ends the proof.

9. The uniqueness problem

In Sections 4,5,6 the existence of solutions of problems (A,B) and (D,E), has been shown only. In this section we shall consider the uniqueness problem

Theorem 9.1

There exists only one solution (v, ω) of the problem (A,B) for $\pi = 0$. The uniqueness is shown in space $L_\infty(0,T;W^1_r(\Omega)) \times L_\infty(0,T;L_r(\Omega))$, $r > n$, $n=2,3$.

Proof. Let $(\overset{s}{v}, \overset{s}{\omega})$, $s=1,2$, be two solutions of the problem (A,B) from $\Pi^2_{1,\infty,r}(\Omega^T) \times \Pi^1_{0,\infty,r}(\Omega^T)$, $r > n$, $n=2,3$ (see Theorems 5.1, 5.2, 6.1). Introducing notations $\vartheta = \overset{1}{v} - \overset{2}{v}$, $\Omega = \overset{1}{\omega} - \overset{2}{\omega}$, from the problem (B) we get

$$
(9.1) \qquad
\begin{aligned}
&\operatorname{rot} \vartheta = \Omega, \\
&\operatorname{div} \vartheta = 0, \\
&\vartheta_n|_{\partial\Omega} = 0,
\end{aligned}
$$

so, we have (see Sections (C), (D)):

$$
(9.2) \qquad \|\vartheta\|_{1,r} \le C\|\Omega\|_r .
$$

Moreover, instead of the problem (A) we have

$$
(9.3) \qquad \Omega_t + \overset{1}{v}{}^k \Omega_{x^k} + \vartheta^k \overset{2}{\omega}_{x^k} - \overset{1}{\omega}{}^k \vartheta_{x^k} - \Omega^k \overset{2}{v}_{x^k} = 0,
$$

$$
(9.4) \qquad \Omega|_{t=0} = 0, \quad \Omega|_{S_1} = 0 .
$$

Multiplying (9.3) by $\Omega|\Omega|^{r-2}$ and integrating the result over Ω we obtain

$$\int_{\Omega}(\Omega_t+\overset{1}{v}^k\Omega_{x^k}+\vartheta^k\overset{2}{\omega}_{x^k}-\overset{1}{\omega}^k\vartheta_{x^k}-\Omega^k\overset{2}{v}_{x^k})\Omega|\Omega|^{r-2}dx=0.$$

From this eqution, using the Hölder inequality we obtain

$$\frac{d}{dt}\|\Omega\|_r^r\leq-\int_{\partial\Omega}\overset{1}{v}_n|\Omega|^r ds+\max_{\Omega}|\vartheta|\,\|\overset{2}{\omega}_x\|_r\|\Omega\|_r^{r-1}+$$

(9.5)

$$+\max_{\Omega}|\overset{1}{\omega}|\,\|\vartheta_x\|_r\|\Omega\|_r^{r-1}+\max_{\Omega}|\overset{2}{v}_x|\,\|\Omega\|_r^r .$$

Knowing that $\overset{s}{v}\in L_\infty(0,T;W_r^2(\Omega))$, $\overset{s}{\omega}\in L_\infty(0,T;W_r^1(\Omega))$, s=1,2, using the boundary conditions (9.4) and relation (9.2) we obtain

(9.6) $\frac{d}{dt}\|\Omega\|_r\leq C\|\Omega\|_r$.

Therefore, from the initial conditions (9.4) we have the uniqueness. This ends the proof.

To consider the uniqueness problem for the problem (D,E) we shall consider the following problem:

(9.7) $v_t+v^k v_{x^k}+\nabla p=f$,

(9.8) $\operatorname{div} v=0$,

(9.9) $v|_{t=0}=a(x)$,

(9.10) $v|_{S_1}=\eta$, $v_n|_{S_0}=0$, $v_n|_{S_2}\geq 0$,

and we ask: what boundary conditions must be added to the problem (9.7) ÷ (9.10) to make it well posed. To answer that question we assume that $\vartheta=\overset{1}{v}-\overset{2}{v}$, $\mathcal{P}=\overset{1}{p}-\overset{2}{p}$, where $(\overset{i}{v},\overset{i}{p})$, i=1,2, are two solution of the problem (9.7) ÷ (9.10) such that

$\overset{i}{v} \in \Pi^2_{1,\infty,r}(\Omega^T)$, $\overset{i}{p} \in \Pi^3_{2,\infty,r}(\Omega^T)$, $r > n$, $n=2,3$ (see Sections 5,6) .Therefore,from (9.7) we obtain

$$(9.11)\quad \frac{d}{dt}\|\vartheta\|_2^2 = -\int_\Omega (\overset{1}{v}{}^k\vartheta_k + \vartheta^k \overset{2}{v}_{x^k})\vartheta\, dx + \int_\Omega \nabla p\cdot\vartheta\, dx .$$

From (9.11) we have

$$(9.12)\quad \frac{d}{dt}\|\vartheta\|_2^2 \leq -\int_{\partial\Omega} \overset{1}{v}_n |\vartheta|^2 ds + \max_\Omega |\overset{2}{v}_x| \, \|\vartheta\|_2^2 + \int_{\partial\Omega} \vartheta\cdot\bar{n}\, p\, ds .$$

From the boundary conditions (9.10) we get

$$(9.13)\quad \vartheta|_{S_1} = 0 ,\ \vartheta\cdot\bar{n}|_{S_0} = 0 ,\ \vartheta\cdot\bar{n}|_{S_2} \geq 0 ,\ \overset{i}{v}_n|_{S_1} \leq 0 ,\ \overset{i}{v}_n|_{S_0} = 0 ,\ \overset{i}{v}_n|_{S_2} \geq 0 ,\ i=1,2.$$

Using (9.13) in (9.12) we have that

$$(9.14)\quad \frac{d}{dt}\|\vartheta\|_2^2 \leq \|\overset{2}{v}\|_{2,r}\|\vartheta\|_2^2 + \int_{S_2} \vartheta_n p\, ds .$$

From (9.14) we see that the problem (9.7) ÷ (9.10) is well posed if we assume on S_2 either

$$(9.15)\quad p|_{S_2} = \pi$$

or

$$(9.16)\quad v_n|_{S_2} = b .$$

Therefore,we can formulate the following result:

Theorem 9.2

Problems (9.7) ÷ (9.10), (9.15) and (9.7) ÷ (9.10), (9.16) have the unique solutions.

10. Conclusions

In this paper we have proved the existence of solutions of some examples of well posed initial boundary value problems for the Euler equations (see Sections 3,4,5,6). We have been interested only in the existence and uniqueness of solutions of these problems in the Sobolev spaces. These problems we divided into two main groups: problems for an incompressible and problems for a compressible fluid.

In the case of an incompressible fluid we have considered domains $\Omega \subset \mathbb{R}^n$, n=2,3, with three kinds of boundaries:

(a) smooth boundary

(b) boundary with dihedral angles π/n, n is a natural number,

(c) boundarywith arbitrary dihedral angles.

For domains with boundaries of type (a) we have proved the existence of unique solutions of the leakage problem with given vorticity vector on S_1 tangent to S_1. As the second boundary condition we assumed the normal component of the velocity vector on the boundary $\partial\Omega$. This problem has been considered in Section 4. In this case we have to assumed that the normal component of velocity is equal zero in points of $S_1 \cap S_0$. Therefore, the assumption (d) of Theorem 4.1 implies that the vorticity vector η and the external force rot$f$ given on S_1 must also vanish in these points. Moreover, the smallest smoothness of these solutions for which we can prove the unique existence, is described by (4.2). From (4.2) and theorems of imbeddings we know that v, v_x and v_t are continuous functions and from the assumption (e) of Theorem 4.1 it follows that f is continuous function also. Therefore, the Euler equations are satisfied in the classical sense. Moreover, from the assumptions (d) and (e) of Theorem 4.1 and from the smoothness of ω expressed by (4.2) it follows that ω and F have greater smoothness in points of S_1, where $d>0$ than it follows from theorems of imbeddings. Therefore, the vorticity vector ω loses smoothness in a neighbourhood of S_1.

For domains with boundaries of type (b), the elliptic problems (B) and (E) have been solved using Green function constructed by the method of reflection (Section C). In the case of boundaries of type (c), the problems (B) and (E) are considered in Section D. To prove the existence of solutions of these problems we must at first consider the Dirichlet and Neumann problems for the Laplace equation in a dihedral angle $\mathcal{D}_\vartheta$ (Section 2). The Dirichlet problem was considered by Mazya and Plamenevsky [M,1,3] in spaces $V^l_{\gamma,\mu}(\mathcal{D}_\vartheta;\Gamma)$ (see 2.14). But in Section D the existence of solutions of the Neumann problem was shown in H^s spaces only. Using these results we have proved the existence of unique solution of problems (A,B) and (D,E) in domains $\Omega\subset\mathbb{R}^3$ with dihedral angles less than $\pi/3$. This restriction follows from the Kondratiev theorem [Kon,1] about solvability of the elliptic boundary problems in corners. For the problem (A,B) its solutions of the smallest smoothness are from spaces (see Theorem 5.2)

$$(10.1)\qquad v\in \Pi^3_{1,\infty,2}(\Omega^T),\qquad \omega\in \Pi^2_{0,\infty,2}(\Omega^T),$$

and for the problem (D,E) from spaces (see Theorem 5.4)

$$(10.2)\qquad v\in \Pi^3_{1,\infty,2}(\Omega^T),\qquad p\in \Pi^4_{2,\infty,2}(\Omega^T).$$

For domains in $\mathbb{R}^3$ with dihedral angles π/n ,n is natural and $n \geqslant 2$,we have proved the existence of solutions of the smallest smoothness from spaces:

$$(10.3) \quad v \in \Pi^2_{1,\infty,r}(\Omega^T), \ \omega \in \Pi^1_{0,\infty,r}(\Omega^T) \ , r>3,$$

for the problem (A,B) (see Theorem 5.1),and from spaces

$$(10.4) \quad v \in \Pi^2_{1,\infty,r}(\Omega^T), \ p \in \Pi^3_{2,\infty,r}(\Omega^T), \ r>3,$$

for the problem (D,E) (see Theorem 5.3) .The distinction between (10.1) and (10.3) or (10.2) and (10.4) follows from the fact that we have proved the existence of solutions of the Neumann problem for the Laplace equation for domains with arbitrary dihedral angles in H^s only.

The twodimensional case is different the above distinction. For the problem (A,B) its solutions of the smallest smoothness belong to spaces

$$(10.5) \quad v \in \Pi^2_{1,\infty,r}(\Omega^T), \ \omega \in \Pi^1_{0,\infty,r}(\Omega^T), \ r>2,$$

and for the problem (D,E) they belong to spaces

$$(10.6) \quad v \in \Pi^2_{1,\infty,r}(\Omega^T), \ p \in \Pi^3_{2,\infty,r}(\Omega^T), \ r>2 .$$

Moreover,for domains with arbitrary corners we have the following restriction on angles:

$$(10.7) \quad \alpha_0 < \frac{\pi}{3-\frac{2}{r}} < \frac{\pi}{2} \quad .$$

In all these cases the Euler equations are satisfied in the classical sense.

For domains with edges we can sassume that the normal component of velocity on S_1 can be different from zero. Moreover, the problem (D,E) can be solved only in domains with edges because we must assume that the normal component of velocity on S_2 is positive (we consider simply connected domains). This is satisfied if the initial velocity vector has the positive normal component on S_2 and the velocity vector is of class C^1 with respect to time.

In the case of problem (D,E) the normal derivative of pressure on $S_1 \cup S_0$ depends on an extension of $\bar{n}$ to some neighbourhood of $S_1 \cup S_0$.

Now we shall show that the problem (A,B,C) has no solutions in Sobolev spaces. Similarly as in the proof of Theorem 4.1 we have

$$\frac{d}{dt}|\omega|^{r}_{1,0,r} \leq I + C|v|_{2,1,r}|\omega|^{r}_{1,0,r} + |F|_{1,0,r}|\omega|^{r-1}_{1,0,r}\,, \tag{10.8}$$

and the surface integral I is estimated by

$$I \leq C_1 + C_2\|v\|^{r}_{2,r} + C_3 J^{r} + C_4\|v\|^{r}_{2,r} J^{r}\,, \tag{10.9}$$

where $J^{r} = \int_{S_1}(|\pi|^{r} + |\pi_t|^{r} + |\pi_{,\tau}|^{r})ds$ and C_i, $i=1,\dots,4$, are constants. An estimate on J we obtain from the problem (C). From (3.20) we have

$$\begin{aligned}\sum_{i+j\leq 1}\sum_{\nu,\mu=1}^{2}\int_{S_1} D_t^{i}D_{\tau_\nu}^{j}\left[\pi_t + v_\mu \pi_{,\tau_\mu} + \pi(v_{\mu,\tau_\mu} + v_\mu \operatorname{rot}_n(\bar{n}\times\bar{\tau}_\mu))\right] D_t^{i}D_{\tau_\nu}^{j}\pi\,|D_\tau^{i}D_\tau^{j}\pi|^{r-2}ds = \\ = \int_{S_1}(\varphi\pi^{r-1} + \varphi_t\pi_t^{r-1} + \varphi_{\tau_\nu}\pi_{\tau_\nu}|\pi_\tau|^{r-2})ds\,.\end{aligned} \tag{10.10}$$

After straighforward calculations ,from (10.10) we get

$$\frac{d}{dt} J^r \le -\int_{L_1} v_\mu \sigma_\mu (|\pi|^r + |\pi_t|^r + |\pi_\tau|^r)\, ds' + C_1(\|v\|_{2,r} + \|v_t\|_{1,r}) J^r +$$

(10.11)

$$+ C_2 \int_{S_1} |v_{xt}||\pi||\pi_t|^{r-1} ds + C_3 \int_{S_1} |v_{xx}||\pi||\pi_{,\tau}|^{r-1} ds + J_\varphi J^{r-1}$$

where $J_\varphi = \int_{S_1} (|\varphi|^r + |\varphi_t|^r + |\varphi_{,\tau}|^r)\, ds$, $L_1 = S_1 \cap S_0$.

From (10.8) it follows that $\omega(t) \in W^1_r(\Omega)$ and $\omega_t(t) \in L_r(\Omega)$, therefore the problem (B) implies that $v \in W^2_r(\Omega)$ and $v_t \in W^1_r(\Omega)$. From the third and fourth terms in the right-hand side of (10.11) it follows that v must belong to $W^2_r(S_1)$ and $v_t \in W^1_r(S_1)$ but it does not occur because the traces theorems imply that $v \in W^{2-1/r}_r(S_1)$ and $v_t \in W^{1-1/r}_r(S_1)$ .This difficulty makes impossible to prove the existence of solutions of the problem (A,B,C) in Sobolev spaces.However,this problem can be solved in the Hölder spaces [Kaz,3] .

Similarly we shall show that the problem (3.1) $\div$ (3.3) and (3.6) can not be solved in our case.From [Kaz,4] it follows that this problem can be transformed into the following system of well posed problems:

$$\text{rot}\, v = \omega,$$

(I) $$\text{div}\, v = 0$$

$$v\cdot \bar{n}|_{\partial\Omega} = \bar{b},$$

where $\bar{b}|_{S_1} = \eta\cdot\bar{n}$, $\bar{b}|_{S_2} = b$, $\bar{b}|_{S_0} = 0$,and

$$\Delta p = \text{div} f - v^k_{x^i} v^i_{x^k},$$

$$\frac{\partial p}{\partial n}\Big|_{S_0} = f\cdot\bar{n} + v^k v\cdot \bar{n}_{x^k}|_{S_0},$$

(10.12) $$\frac{\partial p}{\partial n}\Big|_{S_2} = -b_t - v^k v_{x^k}\cdot\bar{n}|_{S_2} + f\cdot\bar{n}|_{S_2},$$

$$\frac{\partial p}{\partial n}\Big|_{S_1} = -\eta_t - \eta_\mu \eta_{\tau_\mu} \cdot \bar{n} + \eta_n(\eta_{\mu|\tau_\mu} + \eta_\mu \operatorname{div}\bar{\tau}_\mu + \eta_n \operatorname{div}\bar{n}) + f\cdot\bar{n}\Big|_{S_1} +$$
$$+ \eta_n \eta \cdot (\bar{n}\cdot\nabla)\bar{n} \, .$$

The expression $\frac{\partial p}{\partial n}\Big|_{S_1}$ was calculated in the same way as (3.36). To write (10.12) more conveniently we calculate

$$v^k v_{x^k}\cdot\bar{n}\Big|_{S_2} = (v_\mu \tau^k_\mu + b n^k)\, v_{x^k}\cdot\bar{n}\Big|_{S_2} = v_\mu v_{,\tau_\mu}\cdot\bar{n}\Big|_{S_2} + b\, v_{,n}\cdot\bar{n}\Big|_{S_2} =$$

$$= \left(v_\mu b_{,\tau_\mu} - v_\mu v\cdot\bar{n}_{,\tau_\mu} + b\, v_{n,n} - b\, v\cdot\bar{n}_{,n}\right)\Big|_{S_2} .$$

Calculating $v_{n,n}\Big|_{S_2}$ from the equation

$$\operatorname{div} v\Big|_{S_2} \equiv \left(v_{\mu,\tau_\mu} + v_\mu \operatorname{div}\bar{\tau}_\mu + v_{n,n} + b \operatorname{div}\bar{n}\right)\Big|_{S_2} = 0 ,$$

we obtain

$$(10.13)\quad v^k v_{x^k}\cdot\bar{n}\Big|_{S_2} = \left(v_\mu b_{,\tau_\mu} - v_\mu v\cdot\bar{n}_{,\tau_\mu} - b(v_{\mu,\tau_\mu} + v_\mu \operatorname{div}\bar{\tau}_\mu + b\operatorname{div}\bar{n} + v_{n,n})\right)\Big|_{S_2} .$$

Using (10.13) in (10.12) we obtain the following problem:

$$(\beta)\quad \begin{aligned} &\Delta p = \operatorname{div} f - v^k_{x^i} v^i_{x^k} ,\\ &\frac{\partial p}{\partial n}\Big|_{S_0} = f\cdot\bar{n}\Big|_{S_0} + v_\nu v\cdot\bar{n}_{,\tau_\nu}\Big|_{S_0} ,\\ &\frac{\partial p}{\partial n}\Big|_{S_2} = \left(f\cdot\bar{n} - b_t - v_\mu b_{,\tau_\mu} + v_\mu v\cdot\bar{n}_{,\tau_\mu} + b(v_{\mu,\tau_\mu} + v_\mu \operatorname{div}\bar{\tau}_\mu + \operatorname{div}\bar{n} + v_\mu \alpha_\mu)\right)\Big|_{S_2} ,\\ &\frac{\partial p}{\partial n}\Big|_{S_1} = -\eta_t - \eta_\mu \eta_{\tau_\mu}\cdot\bar{n} + \eta_n(\eta_{\mu,\tau_\mu} + \eta_\mu \operatorname{div}\bar{\tau}_\mu + \eta_n \operatorname{div}\bar{n}) + \eta_n \eta_\mu \alpha_\mu\Big|_{S_1} + f\cdot\bar{n}\Big|_{S_1} , \end{aligned}$$

where $\alpha_\mu = \bar{n}\cdot\nabla\bar{n}\cdot\bar{\tau}_\mu$, $\mu = 1,2$,

$$\operatorname{div}\bar{\tau}_\mu = \frac{1}{H_1 H_2 H_3}\,\frac{\partial}{\partial\tau_\mu}\left(\frac{H_1 H_2 H_3}{H_\mu}\right), \quad \operatorname{div}\bar{n} = \frac{1}{H_1 H_2 H_3}\,\frac{\partial}{\partial n}(H_1 H_2) .$$

In the above boundary expressions we see that $\mathcal{L}_\mu$ and $\operatorname{div} \bar{n}$ depend on an extension of H_1, H_2 and $\bar{n}$ in some neighbourhood of $\partial\Omega$, because they depend on the normal derivative to the boundary.

At last, as the third problem we have the evolution problem:

$$(\gamma)\qquad \begin{aligned} &\omega_t + v^k \omega_{x^k} - \omega^k v_{x^k} = F \equiv \operatorname{rot} f, \\ &\omega|_{t=0} = \omega_0 \equiv \operatorname{rot} a, \\ &\omega|_{S_1} = \chi . \end{aligned}$$

The tangent components of the vector χ to S_1 we shall calculate from Lamb equations projected on S_1 and described in curvilinear coordinates as [Ko, 3] :

$$(10.14)\qquad v_{i,t} + \frac{1}{H_i} \nabla_i\left(\frac{v^2}{2} + p\right) - (v \times \omega)_i = f_i \quad , \; i = 1,2,3,$$

where $v_i = v \cdot e_i$, $\nabla_i = e_i \cdot \nabla$, $f_i = f \cdot e_i$, and e_i are introduced in Section 3. Therefore, from (10.14) we have

$$(10.15)\qquad \begin{aligned} &\omega \cdot \bar{\tau}_1|_{S_1} = \omega_1|_{S_1} \equiv \chi_1 = \eta_n^{-1}\left[\eta_{2,t} + \frac{1}{H_2}\left(\frac{\eta^2}{2} + p\right)_{,\tau_2} + \eta_1 \omega_3|_{S_1} - f_2|_{S_1}\right], \\ &\omega \cdot \bar{\tau}_2|_{S_1} = \omega_2|_{S_1} \equiv \chi_2 = \eta_n^{-1}\left[-\eta_{1,t} - \frac{1}{H_1}\left(\frac{\eta^2}{2} + p\right)_{,\tau_1} + \eta_2 \omega_3|_{S_1} + f_1|_{S_1}\right], \end{aligned}$$

and

$$(10.16)\qquad \omega \cdot \bar{n}|_{S_1} = \omega_3|_{S_1} = \frac{1}{H_1 H_2}\left[\frac{\partial(H_2 \eta_2)}{\partial \tau_1} - \frac{\partial(H_1 \eta_1)}{\partial \tau_2}\right] \equiv \chi_3 ,$$

where

$$(10.17)\qquad \chi = \chi_1 \bar{\tau}_1 + \chi_2 \bar{\tau}_2 + \chi_3 \bar{n} .$$

To prove the existence of solutions of the problem (α,β,γ) which is equivalent to the problem (3.1) $\div$ (3.3) and (3.6), we use the method of successive approximations. For a given $\overset{k-1}{\omega}$ we calculate $\overset{k}{v}$ from the problem (α) and for a given $\overset{k}{v}$ we obtain $\overset{k}{p}$ from the problem (β). At last for given $\overset{k}{v}$ and $\overset{k}{p}$ we get $\overset{k}{\omega}$ from the problem (γ). Now we shall show that the problem of existence of solutions of the problem (α,β,γ) can not be proved in Sobolev spaces. For a given $\overset{k-1}{\omega}$ the problem (α) has a unique solution and the following estimates are valid:

$$(10.18)\quad \|\overset{k}{v}\|_{3,r} \le C(\|\overset{k-1}{\omega}\|_{2,r} + \|\bar{b}\|_{3-1/r,r,\partial\Omega}) \le C(\|\overset{k-1}{\omega}\|_{2,r} + \|\eta\|_{3-1/r,r,S_1} + \|b\|_{3-1/r,r,S_2})$$

and

$$(10.19)\quad \|\overset{k}{v}_t\|_{2,r} \le C(\|\overset{k-1}{\omega}_t\|_{1,r} + \|\eta_t\|_{2-1/r,r,S_1} + \|b_t\|_{2-1/r,r,S_2}),$$

where $r > n$, $n=2,3$. For a given $\overset{k}{v}$ the problem (β) has also a unique solution and the following estimates are valid:

$$(10.20)\quad \begin{aligned} \|\overset{k}{p}\|_{3,r} \le C(&\|\mathrm{div} f\|_{1,r} + \|\overset{k}{v}\|^2_{3,r} + \|f\cdot\bar{n}\|_{2-1/r,r,\partial\Omega} + \|b_t\|_{2-1/r,r,S_2} + \\ &+ \|b\|^2_{3-1/r,r,S_2} + \|\eta_t\|_{2-1/r,r,S_1} + \|\eta\|^2_{3-1/r,r,S_1}), \end{aligned}$$

and

$$(10.21)\quad \begin{aligned} \|\overset{k}{p}_t\|_{2,r} \le C(&\|\mathrm{div} f\|_r + \|\overset{k}{v}_t\|_{2,r}\|\overset{k}{v}\|_{3,r} + \|f_t\cdot\bar{n}\|_{1-1/r,r,\partial\Omega} + \|b_{tt}\|_{1-1/r,r,S_2} + \\ &+ \|b_t\|_{2-1/r,r,S_2}\|b\|_{3-1/r,r,S_2} + \|\eta_{tt}\|_{1-1/r,r,S_1} + \|\eta_t\|_{2-1/r,r,S_1}\|\eta\|_{3-1/r,r,S_1}), \end{aligned}$$

where $r > n$, $n=2,3$. For the problem (γ), similarly as in the proofs of theorems from sections 4,5 and 6, we obtain the differential inequality for given v and p

$$\frac{d}{dt}|\overset{k}{\omega}|^{r}_{2,1,r} \le -\int_{\partial\Omega} \overset{k}{v}\cdot\bar{n}\,(|\overset{k}{\omega}|^{r}+|\overset{k}{\omega}_t|^{r}+|\overset{k}{\omega}_x|^{r}+|\overset{k}{\omega}_{xt}|^{r}+|\overset{k}{\omega}_{xx}|^{r})ds +$$

(10.22)
$$+ C_1|\overset{k}{v}|_{3,1,r}|\overset{k}{\omega}|^{r}_{2,1,r} + C_2|F|_{2,1,r}|\overset{k}{\omega}|^{r-1}_{2,1,r} ,$$

where the surface integral I we shall estimate as the following:

(10.23) $I \le |X|^{r}_{2,1,r,S_1}(C_3 + C_4|\overset{k}{v}|^{r}_{2,1,r,S_1} + C_5|\overset{k}{v}|^{2r}_{2,1,r,S_1}) + C_6$,

where C_i, $i=1,\dots,6$, are constants. From (10.15) and (10.16) it follows that in the right-hand side of (10.23) there appear norms $\|\overset{k}{p}\|_{3,r,S_1}$ and $\|\overset{k}{p}_t\|_{2,r,S_1}$ which are not estimated in the problem (β). This is the same situation as in the case of the problem (A, B, C). However, the problem (α,β,γ) can be solved in the Hölder spaces [Kaz, 4].

The above considerations imply that problems (A, B, C) and (α,β,γ) can be solved in spaces $\lim_{r\to\infty} W^{l}_{r}(\Omega)$ or $\lim_{l\to\infty} W^{l}_{r}(\Omega$ but it is not possible, because we can not obtain an apriori estimate for solutions of these problems in these spaces. From the Calderon-Zygmunt theorem about an estimate for singular integrals [Ca, 1] it follows, that apriori estimates for problems (B), (α), (β) are not valid for $r = \infty$. Because of the nonlinearity of problems (A), (γ), it is not possible to obtain apriori estimates for these problems for $l = \infty$, also.

In a compressible fluid we have proved the existence of solutions such that $v_x, v_t, \varrho_x, \varrho_t \in C^{1/2}(\Omega)$ (see Theorem 8.1). Similarly, we can prove that v, ϱ belong to $C^{k}(\Omega^{T})$, where k is arbitrary integer. In this case we have proved the existence of solutions for vanishing normal component of velocity on the boundary only.

PART 2

A. A unique solvability of problems (A) and (D)

We shall consider the following system of equations:

$$(A.1) \qquad \frac{dy}{d\tau} = v(y,\tau), \quad y(x,t;\tau)\big|_{\tau=t} = x,$$

where $v(y,\tau) \in \Pi^{2}_{1,\infty,r}(\Omega^T)$, $\Omega \subset \mathbb{R}^3$, $r > 3$. Then from the theory of differential equations [Ma, 1] it follows that there exists a unique solution of the problem (A.1). Now we obtain some relations on solutions of (A.1). From (A.1) we have

$$(A.2) \qquad y(x,t;\tau) = x + \int_t^\tau v(y,s)\,ds,$$

so the quantities $\frac{\partial y^i}{\partial x^s}$ satisfy the following equations:

$$(A.3) \quad \frac{d}{d\tau}\frac{\partial y^i}{\partial x^s} = \frac{\partial v^i}{\partial y^k}\frac{\partial y^k}{\partial x^s}$$

We introduce the quantity $J = \det \left\| \frac{\partial y^i}{\partial x^k} \right\|$, so $J = J(x,t;\tau)$ and $J(x,t;t) = 1$. Let A^j_i be the minor of the element $\frac{\partial y^i}{\partial x^j}$ of matrix $\left(\frac{\partial y^i}{\partial x^j}\right)$, then

$$\frac{dJ}{d\tau} = A^j_i \frac{d}{d\tau}\frac{\partial y^i}{\partial x^j} = A^j_i \frac{\partial v^i}{\partial y^k}\frac{\partial y^k}{\partial x^j} = \frac{\partial v^i}{\partial y^k}\delta_{ik} J = J \operatorname{div} v$$

because $\frac{\partial y^k}{\partial x^j} A^j_i = \delta^k_i J$. Therefore, integrating this equation we obtain

$$(A.4) \quad J(x,t;\tau) = \exp \int_t^\tau \frac{\partial v^k(y,s)}{\partial y^k}\,ds,$$

and we have the estimate

$$\text{(A.5)}\qquad \exp - \left| \int_t^\tau \frac{\partial v^k}{\partial y^k}(y,s)\,ds \right| \leq J(x,t;\tau) \leq \exp \left| \int_t^\tau \frac{\partial v^k}{\partial y^k}(y,s)\,ds \right| .$$

If div v =0, then $J(x,t;\tau) = J(x,t;t) = 1$ .From (A.5) it follows that there exists the inverse transformation to (A.2) such that

$$\text{(A.6)}\qquad x(y,t;\tau) = y - \int_t^\tau v(y,s)\,ds .$$

From (A.6) we get

$$\text{(A.7)}\qquad \frac{d}{d\tau}\frac{\partial x^i}{\partial y^j} = -\frac{\partial v^i}{\partial x^k}\frac{\partial x^k}{\partial y^j} .$$

Now we shall consider the system (3.12) of ordinary differential equations

$$\text{(A.8)}\qquad \frac{d}{d\tau}\omega^k(y(x,t;\tau),\tau) = \omega^j(y(x,t;\tau),\tau)\frac{\partial v^k}{\partial y^j}(y(x,t;\tau),\tau) + F^k(y(x,t;\tau),\tau),$$

with initial conditions determined by

$$\text{(A.9)}\qquad \omega_0^j(x,t) = \begin{cases} \omega^j(y(x,t;\tau),\tau)\big|_{\tau=0} = \omega_0^j(y(x,t;0),0), \\ \omega^j(y(x,t;\tau),\tau)\big|_{\tau=t_*(x,t)} = \eta(y(x,t;t_*(x,t)),t_*(x,t)) = \end{cases}$$

$$= \eta(x',t_*(x,t)),\quad x'\in S_1 .$$

From the above considerations we have that $A_k^j = (-1)^{j+k}\dfrac{\partial J}{\partial y^k_{,x^j}}$ and $J = 1$.From the theory of ordinary linear differential equations [Ma, 1] it follows that the problem (A.8), (A.9) can be solved in the following way:

$$\text{(A.10)}\qquad \begin{aligned} \omega^j(y(x,t;\tau),\tau) &= \omega^k(y(x,t;t),t)\frac{\partial y^j}{\partial x^k}(x,t;\tau) \quad + \\ &+ \frac{\partial y^j}{\partial x^k}(x,t;\tau)\int_t^\tau A_l^k\left(\frac{\partial y}{\partial x}(x,t;s)\right)F^l(y(x,t;s),s)\,ds , \end{aligned}$$

where $\tau = t^{\prime}(x,t) = \begin{cases} 0 \\ t_*(x,t) \end{cases}$ (see Section 3). From (A.10) we obtain

$$\omega^k(x,t) = A^k_j(y_x(x,t;t^{\prime}(x,t)))\,\omega^j(y(x,t;t^{\prime}(x,t)),t^{\prime}(x,t)) +$$

$$\text{(A.11)} \qquad - \int_t^{t^{\prime}(x,t)} A^k_l(y_x(x,t;s))\,F^l(y(x,t;s),s)\,ds \; .$$

Now we shall show that if $v \in \Pi^2_{1,\infty,r}(\Omega^T)$, $r > 3$, then the solution (A.11) of the problem (A.8), (A.9) belongs to $\Pi^1_{0,\infty,r}(\Omega^T)$, $r > 3$. To show this we assume that $\omega \in C^1(\Omega^T)$, $v \in C^2(\Omega^T)$, $F \in C^1(\Omega^T)$, $\eta \in C^1(\partial\Omega^T)$ and $\omega_0 \in C^1(\Omega)$. Differentiating (A.11) with respect to x^s we obtain

$$\omega^k_{x^s}(x,t) = A^k_{j,y^l_{x^i}}\, y^l_{x^i x^s}\, \omega^j(y(x,t;t^{\prime}(x,t)),t^{\prime}(x,t)) + A^k_j\, \omega^j_{y^l}(y,t^{\prime}(x,t))\, y^l_{x^s} +$$

$$\text{(A.12)} \quad - \int_t^{t^{\prime}(x,t)} \left[A^k_{l,y^\sigma_{x^l}}\, y^\sigma_{x^l x^s} F^l + A^k_l F^l_{,y^i} y^i_{x^s}\right] ds + \left[\frac{d}{d\tau} A^k_j(y_x(x,t;\tau))\,\omega^j(y(x,t;\tau),\tau) + \right.$$

$$\left. + A^k_j\left(\omega^j_{y^i}\frac{dy^i}{d\tau} + \omega^j_{,\tau}\right) - A^k_j(y_x(x,t;\tau))\,F^j(y(x,t;\tau),\tau)\right]\Bigg|_{\tau = t^{\prime}(x,t)} t^{\prime}_{x^s} \; .$$

From (A.3) and the definition of A^i_j after some calculations we have

$$\text{(A.13)} \qquad \frac{d}{d\tau} A^k_j = - \frac{\partial v^s}{\partial x^j} A^k_s \; .$$

Using (A.13) in (A.12) we see that the coefficient of $t^{\prime}_{x^s}$ vanishes, so instead of (A.12) we obtain

$$\omega^k_{x^s}(x,t) = A^k_{j,y^l_{x^i}}\, y^l_{x^i x^s}\, \omega^j(y(x,t;t^{\prime}(x,t)),t^{\prime}(x,t)) + A^k_j\, \omega^j_{y^l}(y,t^{\prime}(x,t))\, y^l_{x^s} +$$

$$\text{(A.14)} \qquad - \int_t^{t^{\prime}(x,t)} \left[A^k_{l,y^\sigma_{x^i}}\, y^\sigma_{x^i x^s} F^l + A^k_l F^l_{,y^i} y^i_{x^s}\right] ds \; .$$

Differentiating (A.11) with respect to t we get

$$\dot{\omega}^j_t(x,t) = A^k_{j,y^s_{x^s}}\, y^s_{x^s t}\, \omega^j(y(x,t;t'(x,t)),t'(x,t)) + A^k_l(y_x(x,t;t'))F^l(y,t') +$$

(A.15)
$$+A^k_j\, \omega^j_{y^s}(y(x,t;t'(x,t)),t'(x,t))\, y^s_t + \Big[\frac{d}{d\tau} A^k_j(y_x(x,t;\tau))\, \dot{\omega}^j(y(x,t;\tau),\tau) +$$

$$+ A^k_j \frac{d\dot{\omega}^j(y(x,t;\tau),\tau)}{d\tau} - A^k_j F^j(y(x,t;\tau),\tau)\Big]\Big|_{\tau=t'(x,t)} t'_t +$$

$$+\int_{t'(x,t)}^{t} \left[A^k_{l,y^s_{x^s}}\, y^s_{x^s t} F^l + A^k_l F^l_{y^s} y^s_t \right] ds$$

Similarly as in the case of (A.12), the coefficient of t'_t vanishes because of (A.13) and the form of the equation (A.8) for $\tau = t'(x,t)$. Therefore, instead of (A.15) we have

$$\dot{\omega}^j_t(x,t) = A^k_{j,y^s_{x^s}}\, y^s_{x^s t}\, \omega^j(y(x,t;t'(x,t)),t'(x,t)) + A^k_j\, \omega^j_{y^s}(y,t'(x,t))\, y^s_t +$$

(A.16)
$$+\int_{t'(x,t)}^{t} \left[A^k_{l,y^s_{x^s}}\, y^s_{x^s t} F^l + A^k_l F^l_{,y^s} y^s_t \right] ds .$$

From (A.11), (A.14) and (A.16) we have the estimate

$$\|\omega\|_r + \|\omega_x\|_r + \|\omega_t\| \le C \|\, |y_x| \big[(|y_{xx}| + |y_{xt}| + 1)\, |\omega(y(x,t;t'(x,t)),$$

(A.17)
$$t'(x,t))| + (|y_x| + |y_t|)\, |\omega_y(y(x,t;t'(x,t)), t'(x,t))| \big] \|_r +$$

$$+\Big\| \int_t^{t'(x,t)} |y_x(s)| \big[|F(y(x,t;s),s)|(1+|y_{xx}(s)|) + |F_{,y}(y(x,t;s),s)|\,(|y_x|+|y_t|) \big] ds \Big\|_r .$$

Now we have to estimate the derivatives of y(x,t;s) appearing in (A.17). From (A.3) we have $\frac{d}{d\tau}|y_x| \le |v_y||y_x|$, so

(A.18)
$$|y_x| \le \exp\left| \int_t^{t'(x,t)} |v_y(y(x,t;s),s)|\, ds \right| .$$

Moreover, from (A.2) we have

$$y^k_{x^ix^j} = \int_t^\tau \left(v^k_{,y^py^q} \, y^p_{x^i} y^q_{x^j} + v^k_{y^p} \, y^p_{x^ix^j} \right) ds$$

from which we obtain

$$(A.19) \qquad |y_{xx}| \leq C |y_x|^2 |v_{yy}| \exp \left| \int_t^\tau |v_y| \, ds \right| .$$

At last from (A.2) we have $y_{xt} = v_x$ and $y_t = v$ .Using these estimates in (A.17) we get

$$(A.20) \qquad \begin{aligned} &\|\omega\|_{1,r} + \|\omega_t\|_r \leq C \exp\left[C \max_{s,t} \max_x |v_y|\right] \left(\|\omega_0\|_{1,r} + \|\eta\|_{1,r,S_1} \right) + \\ &+ C \exp\left[C \max_{s,t} \max_x |v_y|\right] \Big[\| v_{yy}(y(x,t;t'(x,t)),t'(x,t)) \, \omega(y(x,t;t'(x,t)),t'(x,t)) \|_r + \\ &+ \max_{s,t} \Big(\| F(y(x,t;s),s) \|_r + \| F(y(x,t;s),s) \, v_{yy}(y(x,t;s),s) \|_r + \\ &+ \| F_y(y(x,t;s),s) \|_r \Big) \Big] . \end{aligned}$$

To estimate the expressions in (A.20) we extend the domain Ω to Ω_ε and the function $v \in W^{2,1}_{r,\infty}(\Omega^T)$ to a function $\tilde{v} \in W^{2,1}_{r,\infty}(\Omega^T_\varepsilon)$ where $\Omega^T_\varepsilon = \Omega_\varepsilon \times [0,T]$, in such a way, that $\tilde{v}|_{\partial\Omega_\varepsilon} = 0$ [Lu, 1] and

$$(A.21) \qquad \|\tilde{v}\|_{W^{2,1}_{r,\infty}(\Omega^T_\varepsilon)} \leq C \|v\|_{W^{2,1}_{r,\infty}(\Omega^T)} .$$

The extended vector field $\tilde{v}(x,t)$ generate the following curves

$$(A.22) \qquad \begin{aligned} &\frac{d\tilde{y}(x,t;\tau)}{d\tau} = \tilde{v}(y(x,t;\tau),\tau), \\ &\tilde{y}(x,t;t) = x , \end{aligned}$$

which are some extension of curves $y = y(x,t;\tau)$ on the domain Ω_ε .Then the transformation $x \to \tilde{y}(x,t;\tau)$ is an automorfism of the domain Ω_ε .Moreover,we assume that the function F is extended in the following way

$$\|\tilde F\|_{W^{1,0}_{r,\infty}(\Omega^T_\varepsilon)} \leq C\|F\|_{W^{1,0}_{r,\infty}(\Omega^T)} . \tag{A.23}$$

Using the above considerations we estimate (A.20) as follow

$$\begin{aligned} &\|\omega\|_{1,r}+\|\omega_t\|_r \leq C\exp \max_{\Omega^T_\varepsilon}|\tilde v_{\tilde y}|\Big[\|\omega_0\|_{1,r}+\|\eta\|_{1,r,S_1}+ \\ &+\max_t\|\eta\|_{1,r,S_1}\max_t\|\tilde v_{\tilde y\tilde y}\|_{r,\Omega_\varepsilon}+\max_t\big(\|\tilde F(\tilde y,t)\|_{r,\Omega_\varepsilon}+ \\ &+\max_{\Omega_\varepsilon}|\tilde F(\tilde y,t)|\,\|\tilde v_{\tilde y\tilde y}(\tilde y,t)\|_{r,\Omega_\varepsilon}+\|\tilde F_{\tilde y}(\tilde y,t)\|_{r,\Omega_\varepsilon}\Big]\leq C\exp\max_t\|v\|_{1,r}\cdot \\ &\cdot\big(\|\omega_0\|_{1,r}+\|\eta\|_{1,r,S_1}+\max_t\|F\|_{1,r}\big)\big(1+\max_t\|v\|_{2,r}\big) . \end{aligned} \tag{A.24}$$

Using the fact that the space C^k is dense in W^1_r we can choose for $v\in\Pi^2_{1,\infty,r}(\Omega^T)$, $\eta\in\Pi^1_{0,\infty,r}(S^T_1)$, $\omega_0\in W^1_r(\Omega)$ and $F\in\Pi^1_{0,\infty,r}(\Omega^T)$ the sequences $\overset{n}{v}\in C^2(\Omega^T)$, $\overset{n}{\eta}\in C^1(S^T_1)$, $\overset{n}{\omega}_0\in C^1(\Omega)$ and $\overset{n}{F}\in C^1(\Omega^T)$ such that $\overset{n}{v}$, $\overset{n}{\eta}$, $\overset{n}{\omega}_0$ and $\overset{n}{F}$ converges to v , η , ω_0 and F ,respectively in norms of these spaces.For given functions $\overset{n}{v}$, $\overset{n}{\eta}$, $\overset{n}{\omega}_0$ and $\overset{n}{F}$ we seek $\overset{n}{\omega}$ in the form (A.14) (for which we have estimates (A.24)) from problems

$$\begin{aligned} &\overset{n}{\omega}_t+\overset{n}{v}{}^k\overset{n}{\omega}_{x^k}-\overset{n}{\omega}{}^k\overset{n}{v}_{x^k}=\overset{n}{F}, \\ &\overset{n}{\omega}\big|_{t=0}=\overset{n}{\omega}_0, \\ &\overset{n}{\omega}\big|_{S_1}=\overset{n}{\eta} . \end{aligned} \tag{A.25}$$

Passing with n to infinity we obtain the same estimate for the limit function ω ,so it belongs to $\Pi^1_{0,\infty,r}(\Omega^T)$ .To show that the limit function is a solution of the problem (A.25),instead of (A.25) we consider the integral identity:

$$(A.26)\quad -\int_{\Omega^T}\Big[+\overset{n}{\omega}\varphi_t+\overset{n}{v}{}^k\,\overset{n}{\omega}\,\varphi_{x^k}+\overset{n}{\omega}{}^k\,\overset{n}{v}_{x^k}\varphi+\overset{n}{F}\varphi\Big]\,dx\,dt+\int_{\Omega}\overset{n}{\omega}_0(x)\varphi(x,0)\,dx+$$
$$+\int_{S_1^T}\overset{n}{v}_n(s,t)\,\overset{n}{\eta}(s,t)\,\varphi\,ds\,dt,$$

which is valid for arbitrary smooth function φ such that $\varphi(x,T)=0$, $\varphi|_{S_0\cup S_2}=0$. From the Lebesgue theorem we can pass with n to infinity obtaining the same identity for the limit function ω, therefore ω is a solution of the problem (A.25). Hence we have proved the following result:

Lemma A.1

Let $v\in\Pi^2_{1,\infty,r}(\Omega^T)$, $\omega_0\in W^1_r(\Omega)$, $\eta\in\Pi^1_{c,\infty,r}(S_1^T)$ and $F\in\Pi^1_{0,\infty,r}(\Omega^T)$, $r>3$, then there exists a unique solution of the problem (A.8), (A.9) such that $\omega\in\Pi^1_{0,\infty,r}(\Omega^T)$.

Proof. The uniqueness follows from the following considerations. Let us assume that we have two solutions of the problem (A.8), (A.9), $\overset{1}{\omega}$ and $\overset{2}{\omega}$. Let $\eta=\overset{1}{\omega}-\overset{2}{\omega}$, then for η we have the equation

$$\frac{d}{d\tau}\eta^k(y(x,t;\tau),\tau)=\eta^j(y(x,t;\tau),\tau)\,\frac{\partial v^k}{\partial y^j}(y(x,t;\tau),\tau),$$

$$(A.27)\qquad \eta(y(x,t;\tau),\tau)\big|_{\tau=t'(x,t)}=0,$$

from which it follows that $\eta=0$. This ends the proof.

The same considerations as the above we can repeat for the problem (D) :

$$\frac{d}{d\tau}v(y(x,t;\tau),\tau)=\nabla_y p(y(x,t;\tau),\tau)+f(y(x,t;\tau),\tau),$$

$$(A.28)\qquad v(y(x,t;\tau),\tau)\big|_{\tau=t'(x,t)}=\begin{cases}a(x),\\ \eta(y(x,t;t_*(x,t)),t_*(x,t)).\end{cases}$$

The results we shall formulate in the following lemma:

Lemma A.2

Let $p \in \Pi^{3}_{2,\infty,\gamma}(\Omega^{T})$, $f \in \Pi^{1}_{0,\infty,\gamma}(\Omega^{T})$, $a \in W^{2}_{\gamma}(\Omega)$ and $\eta \in \Pi^{2}_{1,\infty,\gamma}(S^{T}_{1})$ .Then there exists a unique solution of the problem (A.28) such that $v \in \Pi^{2}_{1,\infty,\gamma}(\Omega^{T})$.

In the case of Hölder spaces the similar lemmas can be formulated.

B. Trace theorems

In this section we shall consider trace theorems for domains with edges. At first we shall formulate some results about a function given in a dihedral angle $\mathcal{D}_\vartheta \subset \mathbb{R}^n$ and its trace on the boundary. Using these results and the partition of unity, the trace theorem can be obtained for a domain with edges.

Lemma B.1

Let $u \in W_p^1(\mathcal{D}_\vartheta)$, then $u|_{\Gamma_0} \in W_p^{1-1/p}(\Gamma_0)$, $u|_{\Gamma_\vartheta} \in W_p^{1-1/p}(\Gamma_\vartheta)$ and the following estimate is valid

$$\|u|_{\Gamma_0}\|_{1-1/p,p,\Gamma_0} + \|u|_{\Gamma_\vartheta}\|_{1-1/p,p,\Gamma_\vartheta} \leq C\|u\|_{1,p,\mathcal{D}_\vartheta}. \tag{B.1}$$

Conversely, let $\alpha \in W_p^{1-1/p}(\Gamma_0)$ and $\beta \in W_p^{1-1/p}(\Gamma_\vartheta)$, $\alpha|_\Gamma = \beta|_\Gamma$ then there exists a function $u \in W_p^1(\mathcal{D}_\vartheta)$ such that

$$u|_{\Gamma_0} = \alpha, \quad u|_{\Gamma_\vartheta} = \beta, \tag{B.2}$$

and

$$\|u\|_{1,p,\mathcal{D}_\vartheta} \leq C(\|\alpha\|_{1-1/p,p,\Gamma_0} + \|\beta\|_{1-1/p,p,\Gamma_\vartheta}). \tag{B.3}$$

Moreover, we assume that ϑ is less than π.

Proof. It is sufficient to prove this lemma for a smooth function. Now we shall prove the first part of the lemma. If $u \in W_p^1(\mathcal{D}_\vartheta)$ then from the Sobolev theorems it follows that $u|_{\Gamma_0} \in L_2(\Gamma_0)$ and $u|_{\Gamma_\vartheta} \in L_2(\Gamma_\vartheta)$. We restrict ourselves to the boundary Γ_0, so $u|_{\Gamma_0} = u(x',0)$, where x' are coordinates on Γ_0. To show (B.1) we have to estimate the expression

$$\text{(B.4)}\qquad \int_{\Gamma_0} |z'|^{-n+2-p}\, dz' \int_{\Gamma_0} |u(x'+z',0)-u(x',0)|^p\, dx' .$$

To do this we use the inequality [Fa, 1]:

$$\text{(B.5)}\qquad \begin{aligned} |u(x'+z',0)-u(x',0)| &\leq |u(x'+z',|z'|)-u(x'+z',0)| + \\ &+|u(x',|z'|)-u(x',0)| + |u(x'+z',|z'|)-u(x',|z'|)| . \end{aligned}$$

With the first two terms of the right-hand side of (B.5), we estimate the integral (B.4) by

$$\int_{\Gamma_0} |z'|^{-n+2-p}\, dz' \int_{\Gamma_0} |u(x',|z'|)-u(x',0)|^p dx' = \tfrac{1}{2}\varkappa_{n-1}\int_{\Gamma_0} dx' \int_0^\infty t^{-p}|u(x',t)-u(x',0)|^p dt$$

$$\leq \tfrac{1}{2}\varkappa_{n-1}\left(\frac{p}{p-1}\right)^p \int_{\Gamma_0} dx' \int_0^\infty |u_t|^p dt \leq \tfrac{1}{2}\varkappa_{n-1}\left(\frac{p}{p-1}\right)^p \int_{\mathcal{D}_\vartheta} dx\, |u_x|^p ,$$

where we have used the Hardy inequality [Be, 1], and where $\varkappa_{n-1}$ is the surface measure of the unit sphere in $\mathbb{R}^n$. The last term of the right-hand side of the inequality [B.5] can be written as

$$u(x'+z',|z'|)-u(x',|z'|) = \sum_{i=1}^{n-1} z'_i \int_0^1 u_i(x'+tz',|z'|)\, dt ,$$

where $u_i(x,y)=\frac{\partial u}{\partial x^i}$. Therefore, we have

$$\int_{\Gamma_0} |z'|^{-n+2-p}\, dz' \int_{\Gamma_0} |u(x'+z',|z'|)-u(x',|z'|)|^p dx' \leq \int_{\Gamma_0} |z'|^{-n+2} dz' \int_0^1 dt\, |u_{x'}(x'+tz',|z'|)|^p dx' =$$

$$= \int_{\Gamma_0} |z'|^{-n+2} dz' \int_{\Gamma_0} |u_{x'}(x',|z'|)|^p dx' \leq \tfrac{1}{2}\varkappa_{n-1} \int_{\mathcal{D}_\vartheta} |u_x|^p dx ,$$

so the estimate (B.1) has been proved.

Now we shall prove the second part of the lemma. From Lemma 7.45 [Ad, 1] it follows that functions α, β defined on $\overset{\circ}{\mathbb{R}}{}^{n-1}_{+}$ can be extended on all $\mathbb{R}^{n-1}$ in such a way that the extended functions $\tilde{\mathcal{L}}$, $\tilde{\beta}$ satisfy the following relations:

(B.6)
$$\|\tilde{\mathcal{L}}\|_{1-1/p,p,\mathbb{R}^{n-1}} \leqslant C\|\mathcal{L}\|_{1-1/p,p,\mathbb{R}^{n-1}_{+}} ,$$
$$\|\tilde{\beta}\|_{1-1/p,p,\mathbb{R}^{n-1}} \leqslant C\|\beta\|_{1-1/p,p,\mathbb{R}^{n-1}_{+}} .$$

We assume that Γ_0 is a half (n-1)-plane $x_2=0, x_1 \geqslant 0, z_i \in \mathbb{R}^1$, $i=1,\dots,n-2$, and Γ_ϑ a half n-1 -plane directed under an angle ϑ to Γ_0 .At first we define a function u_1 ,which is an extension of the function $\tilde{\mathcal{L}}$ on a half space $x_2 \geqslant 0$, as

(B.7) $$u_1\big|_{x_2=0} = \tilde{\mathcal{L}} .$$

This function will be sought in the form

(B.8)
$$u_1(x_1,x_2,z_1,\dots,z_{n-2}) = \int_{\mathbb{R}^{n-1}} dt\, \tilde{\mathcal{L}}(x_1+t_1x_2, z_1+t_2x_2,\dots,z_{n-2}+t_{n-1}x_2)K(t)\cdot$$
$$\cdot\zeta(x_2) \equiv u_1'\zeta(x_2),$$

where $K(t)\in C_0^\infty(G)$, $\int_G K(t)dt=1$, G is a compact set such that $G=\{t\in\mathbb{R}^{n-1}: t_1\geqslant t_0>0, |t|\leqslant 1\}$ for $\vartheta\leqslant\frac{\pi}{2}$ and $G=\{t\in\mathbb{R}^{n-1}: t_1\leqslant -t_0<0, |t|\leqslant 1\}$ for $\vartheta\geqslant\pi/2$ .Moreover, $\zeta(x)\in C^\infty(\mathbb{R}^1_+)$ such that $\zeta(x)=1$ for $|x|\leqslant 1/2$ and $\zeta(x)=0$ for $|x|\geqslant 1$ .Now we introduce coordinates connected with the direction γ_ϑ .We denote the derivative tangent to γ_ϑ by $\frac{\partial}{\partial s}=\cos\vartheta\frac{\partial}{\partial x_1}+\sin\vartheta\frac{\partial}{\partial x_2}$,and the derivative normal to γ_ϑ by $\frac{\partial}{\partial\tau}=\sin\vartheta\frac{\partial}{\partial x_1}-\cos\vartheta\frac{\partial}{\partial x_2}$,so $s=\cos\vartheta x_1+\sin\vartheta x_2$ and $\tau=\sin\vartheta x_1-\cos\vartheta x_2$. We see that Γ_ϑ is determined by $\tau=0$.

Now we shall seek a function u_2 satisfying the relation

(B.9) $$u_2|_{\Gamma_\vartheta} = \tilde{\beta} - u_1|_{\Gamma_\vartheta} \equiv \tilde{\hat{\beta}}(s_1, z_1, \dots, z_{n-2}),$$

and

(B.10) $$u_2|_{\Gamma_0} = 0 .$$

From the compatibility condition, written in the assumptions of the lemma, we have

(B.11) $$\tilde{\hat{\beta}}|_{x'=0} = 0.$$

From (B.9) and (B.10) we construct the function u_2 in the form

(B.12) $$u_2(x_1, x_2, z_1, \dots, z_{n-2}) = \int_{\mathbb{R}^{n-1}} dt\ \tilde{\hat{\beta}}(s+t_1\tau, z_1+t_2\tau, \dots, z_{n-2}+t_{n-1}\tau)\mathcal{K}_1(t)\chi(x) \equiv \equiv u_2' \Phi(\varphi),$$

where $\mathcal{K}_1(t) \in C_0^\infty(G)$ with compact set G, $\chi(x) = \Phi(\varphi)\zeta(\tau)$, $\varphi = \operatorname{arctg}\frac{x_2}{x_1}$, $\Phi(\varphi) = 2\frac{\varphi - \vartheta/2}{\vartheta}$ for $\vartheta/2 \leq \varphi \leq \vartheta$ and $\Phi(\varphi) = 0$ for $0 \leq \varphi \leq \vartheta/2$. From (B.9) we have

(B.13) $$\tilde{\hat{\beta}}(s, z_1, \dots, z_{n-2}) = \tilde{\beta}(s, z_1, \dots, z_{n-2}) - \int_{\mathbb{R}^{n-1}} dt'\ \tilde{\alpha}(s(\cos\vartheta + t_1'\sin\vartheta), z_1 + t_2'\sin\vartheta s, \dots, z_{n-2} + t_{n-1}'\sin\vartheta s)\, K(t')\,\zeta(s\sin\vartheta).$$

Therefore, using (B.13) we obtain

(B.14) $$\tilde{\hat{\beta}}(s+t_1\tau, z_1+t_2\tau, \dots, z_{n-2}+t_{n-1}\tau) = \tilde{\beta}(s+t_1\tau, z_1+t_2\tau, \dots, z_{n-2}+t_{n-1}\tau) + \int_{\mathbb{R}^{n-1}} dt'\ \tilde{\alpha}((\cos\vartheta + t_1'\sin\vartheta)(s+t_1\tau), z_1+t_2\tau+t_2'\sin\vartheta(s+t_1\tau), \dots$$

$$\ldots, z_{n-2}+t_{n-1}\tau + t'_{n-1}\sin\vartheta(s+t_1\tau)) \, K(t') \, \Upsilon((s+t_1\tau)\sin\vartheta) \, .$$

From the above relations we see that the function

$$(B.15) \qquad u(x',z) = u_1(x',z) + u_2(x',z),$$

where $x'=(x_1,x_2)$, $z=(z_1,\ldots,z_{n-2})$, is a solution of the problem (B.2). Let us take

$$(B.16) \qquad u_2 = v_1 + v_2 \equiv (v_1'+v_2')\,\Phi(\gamma),$$

where

$$(B.17) \qquad v_1' = \int_{\mathbb{R}^{n-1}} dt \;\tilde{\beta}(s+t_1\tau, z_1+t_2\tau,\ldots,z_{n-2}+t_{n-1}\tau)\, K_1(t)$$

and v_2' is the rest part of u_2'. Similarly as it was done in [Fa, 1], we have

$$(B.18) \qquad \|D_x u_1\|_{p,\mathcal{D}_\vartheta} + \|D_x v_1'\|_{p,\mathcal{D}_\vartheta} \leq C\left(\|\alpha\|_{1-1/p,p,\Gamma_0} + \|\beta\|_{1-1/p,p,\Gamma_\vartheta}\right).$$

Now we shall consider v_2'. Knowing that $\tilde{\alpha}$ depends on arguments expressed in (B.14), we have

$$D_s\tilde{\alpha} = D_1\tilde{\alpha}\,(\cos\vartheta + t_1'\sin\vartheta) + \sum_{i=2}^{n-1} D_i\tilde{\alpha}\, t_i'\sin\vartheta = \frac{1}{\tau}D_{t_1}\tilde{\alpha},$$

$$(B.19) \qquad \begin{aligned} D_\tau\tilde{\alpha} &= D_1\tilde{\alpha}\, t_1(\cos\vartheta + t_1'\sin\vartheta) + \sum_{i=2}^{n-1} D_i\tilde{\alpha}(t_2 + t_1t_i'\sin\vartheta) = \\ &= \frac{1}{\tau}D_{t_1}\tilde{\alpha}\, t_1 + \frac{1}{\tau}\sum_{i=2}^{n-1} D_{t_i}\tilde{\alpha}\, t_i, \end{aligned}$$

$$D_{z_i}\tilde{\alpha} = D_i\tilde{\alpha} = \frac{1}{\tau}D_{t_{i+1}}\tilde{\alpha}, \quad i=1,\ldots,n-2,$$

where D_i denotes a derivative with respect to i-th argument of $\tilde{\mathcal{I}}$ .Using (B.19) in v_2' and after integration by parts we get

$$D_\sigma v_2' = \int\limits_{\mathbb{R}^{n-1}} dt \int\limits_{\mathbb{R}^{n-1}} dt' \; \frac{1}{\tau}\Big[\tilde{\mathcal{I}}((\cos\vartheta + t_1'\sin\vartheta)(s+t_1\tau), z_1+t_2\tau+t_2'\sin\vartheta(s+t_1\tau),$$

(B.20)
$$\ldots, z_{n-2}+t_{n-1}\tau+t_{n-1}'\sin\vartheta(s+t_1\tau)) - \tilde{\mathcal{I}}((\cos\vartheta+t_1'\sin\vartheta)s, z_1+t_2'\sin\vartheta s,\ldots$$

$$\ldots, z_{n-2}+t_{n-1}'\sin\vartheta s)\,\Psi_\sigma(t,t',\tau) + \int\limits_{\mathbb{R}^{n-1}} dt \int\limits_{\mathbb{R}^{n-1}} dt' \; \tilde{\mathcal{I}}(\quad) K_1(t) K(t') \cdot$$

$$\cdot D_\sigma(\zeta((s+t_1\tau)\sin\vartheta)\,\zeta(\tau)) \,,$$

where $\sigma = \{s,\tau,z_1,\ldots,z_{n-2}\}$,and

$$\Psi_s(t,t',\tau) = D_{t_1}\big[K_1(t)\zeta((s+t_1\tau)\sin\vartheta)\big] K(t')\,\zeta(\tau),$$

(B.21)
$$\Psi_\tau(t,t',\tau) = \Big\{D_{t_1}\big[t_1 K_1(t)\zeta((s+t_1\tau)\sin\vartheta)\big] + \\ + \sum_{i=2}^{n-1} D_{t_i} K_1(t)\zeta((s+t_1\tau)\sin\vartheta)\Big\} K(t')\,\zeta(\tau),$$

$$\Psi_{z_i}(t,t',\tau) = D_{t_i} K_1(t)\zeta((s+t_1\tau)\sin\vartheta)\, K(t')\,\zeta(\tau),$$

$$\operatorname{supp}\Psi_\sigma \subseteq \{(t,t',\tau)\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}^1_+ : |t|\le C, |t'|\le C, \tau\le C\} .$$

Using Lemma B.2 we have the estimate for the first term of (B.20), so we have to estimate the second one only. Introducing new variables $z_i' = z_i + t_{i+1}\tau + t_{i+1}'\sin\vartheta(s+t_1\tau)$, $i=1,\ldots,n-2$, the second term of (B.20) is estimated by

(B.22)
$$\Big(\int\limits_{\mathbb{R}^{n-1}} dt \int\limits_{\mathbb{R}^{n-1}} dt' \int\limits_{\mathbb{R}^n_+} ds\, d\tau\, dz' \; |\tilde{\mathcal{I}}((\cos\vartheta+t_1'\sin\vartheta)(s+t_1\tau), z_1',\ldots,z_{n-2}')|^p \cdot$$

$$\cdot |D_\tau(\zeta((s+t_1\tau)\sin\vartheta)\zeta(\tau))\, K_1(t)\, K(t')|^p\Big)^{1/p} .$$

Introducing new variables $s'=(\cos\vartheta+t_1'\sin\vartheta)s$, $\tau'=(\cos\vartheta+t_1'\sin\vartheta)t_1\tau$, (B.22) is estimated by $C\|\alpha\|_{p,\Gamma_0}$ if $\left|\frac{K_1^p(t)}{t_1}\right|\le C$, $|\cos\vartheta+t_1'\sin\vartheta|\ge k_0>0$, $t'\in\operatorname{supp}K$ and $\sup_{s,t,\tau,t'}|D_\tau(\zeta((s+t_1\tau)\sin\vartheta)\zeta(\tau))K_1(t)K(t')|\le C$.
Therefore, we have obtained

$$(B.23)\quad \|D_x v_2'\|_{p,\mathcal{D}_\vartheta}\le C\|\alpha\|_{1-1/p,p,\Gamma_0}\ .$$

At last we consider

$$\|D_x u_2\|_{p,\mathcal{D}_\vartheta}\le\|D_x u_2'\Phi(\varphi)\|_{p,\mathcal{D}_\vartheta}+\|u_2'D_x\Phi\|_{p,\mathcal{D}_\vartheta}\le\|D_x u_2'\Phi(\varphi)\|_{p,\mathcal{D}_\vartheta}+$$

$$(B.24)\quad +C\|u_2'\varrho^{-1}\dot\Phi\|_{p,\mathcal{D}_\vartheta}\le C\|D_x u_2'\|_{p,\mathcal{D}_\vartheta}\le C(\|\alpha\|_{1-1/p,p,\Gamma_0}+\|\beta\|_{1-1/p,p,\Gamma_\vartheta}),$$

where $\dot\Phi(\varphi)=\frac{2}{\vartheta}$ for $\frac{\vartheta}{2}\le\varphi\le\vartheta$, $\dot\Phi(\varphi)=0$ for $0\le\varphi\le\frac{\vartheta}{2}$, and $\varrho=\sqrt{x_1^2+x_2^2}$.
For $p\ne2$ the estimate for the term $\|u_2'\dot\Phi\varrho^{-1}\|_{p,\mathcal{D}_\vartheta}$ follows from (B.10) and the Hardy inequality [Be, 1]. For $p=2$ it follows from the inequality [Kon, 2]:

$$(B.25)\quad \int_{\mathcal{D}_\vartheta}\frac{u^2}{\varrho^2}\,dx\le C\int_{\mathcal{D}_\vartheta}|\nabla u|^2dx\ ,$$

which is valid for a function from $H^1(\mathcal{D}_\vartheta)$ and vanishing on Γ_0. In our case this condition is satisfied, because $u_2'\dot\Phi|_{\Gamma_0}=0$.
Similarly as (B.22) and under the same assumptions the norm $\|u_2\|_{p,\mathcal{D}_\vartheta}$ is estimated by $C(\|\alpha\|_{p,\Gamma_0}+\|\beta\|_{p,\Gamma_\vartheta})$. This ends the proof.

Lemma B.2

Let α be such that the term

$$(B.26)\quad \langle\alpha\rangle_{1-1/p,p,\Gamma_0}:=\left(\int_{\Gamma_0}dx\int_{\Gamma_0}dy\ \frac{|\alpha(x)-\alpha(y)|^p}{|x-y|^p}\right)^{1/p}$$

is bounded, then the following estimate holds:

$$I \equiv \Big(\int\limits_{\mathbb{R}^n_+} dx \Big| \int\limits_{\mathbb{R}^{n-1}} dt \int\limits_{\mathbb{R}^{n-1}} dt' \; |\tau|^{-1} \big(\tilde{\mathcal{L}}((\cos\vartheta + t_1' \sin\vartheta)(s+t_1\tau), z_1 + t_2\tau + t_2' \sin\vartheta (s+t_1\tau),$$

$$(B.27) \qquad \ldots, z_{n-2} + t_{n-1}\tau + t_{n-1}' \sin\vartheta (s+t_1\tau)) - \tilde{\mathcal{L}}((\cos\vartheta + t_1'\sin\vartheta)s, z_1 + t_2' \sin\vartheta s,$$

$$\ldots, z_{n-2} + t_{n-1}' \sin\vartheta s)\big) \Psi(s,\tau,t,t') \Big|^p \Big)^{1/p} \leq C \langle \mathcal{L} \rangle_{1-\gamma_p, p, \Gamma_0},$$

where $\tilde{\mathcal{L}}$ is the extended function of $\mathcal{L}$ and $\Psi \in C_0^\infty$ is such that $t' \in \operatorname{supp} \Psi$ if $|\cos\vartheta + t_1' \sin\vartheta| \geqslant k_0 > 0$, $|t| \leq 1$, $|t'| \leq 1$.

Proof. Introducing new variables $t_i'' = t_i + t_i' \sin\vartheta\, t_1$, $i=2,\ldots, n-1$, $t_1'' = t_1$, $|t''| \leq 2$, we obtain

$$I \leq \Big(\int\limits_{\mathbb{R}^n_+} dx \int\limits_{\mathbb{R}^{n-1}} dt \int\limits_{\mathbb{R}^{n-1}} dt' \; |\tau|^{-p} \big| \tilde{\mathcal{L}}((\cos\vartheta + t_1' \sin\vartheta)(s+t_1\tau), z_1 + t_2''\tau + t_2' \sin\vartheta s, \ldots$$

$$(B.28) \qquad \ldots, z_{n-2} + t_{n-1}''\tau + t_{n-1}' \sin\vartheta s) - \tilde{\mathcal{L}}((\cos\vartheta + t_1'\sin\vartheta)s, z_1 + t_2'\sin\vartheta s, \ldots,$$

$$\ldots, z_{n-2} + t_{n-1}' \sin\vartheta s)\big|^p \, |\Psi(t_1', \tau, t_1'', t_i'' - t_i' \sin\vartheta t_1)|^p \Big)^{1/p} \equiv I_1 .$$

Introducing new variables $|t''| = r$, $t_i'' = \omega_i r$, $\tilde{y}_i' = \omega_i \tau$, $|\tilde{y}| = \tau$, $|\omega| = 1$, $i = 1, \ldots, n-1$, and $\tilde{x}_j' = z_{j-1} + t_j' \sin\vartheta s$, $\tilde{x}_1' = s$, $j = 2, \ldots, n-1$ (B.28) implies

$$I_1 \leq \Big(\int\limits_0^2 r^{n-2} dr \int\limits_{\mathbb{R}^{n-1}} dt' \int\limits_{|\omega|=1} d\omega \int\limits_0^\infty \frac{d|\tilde{y}'|}{|\tilde{y}'|^p} \int\limits_{\mathbb{R}^{n-1}} d\tilde{x}' \, |\tilde{\mathcal{L}}((\cos\vartheta + t_1' \sin\vartheta)(\tilde{x}_1' + r\tilde{y}_1'), \tilde{x}_i' + r\tilde{y}_i') +$$

$$- \mathcal{L}((\cos\vartheta + t_1' \sin\vartheta)\tilde{x}_1', \tilde{x}_i')|^p \, |\Psi(t_1'', t_i'' - t_i' \sin\vartheta t_1, t', \tau)|^p \Big)^{1/p} \equiv I_2 ,$$

where index i denotes that variables with i=2,...,n-2 there appear. Introducing next variables: $\tilde{x}_1 = (\cos\vartheta + t_1' \sin\vartheta)\tilde{x}_1'$, $\tilde{x}_i = \tilde{x}_i'$, $\tilde{y}_1 = (\cos\vartheta + t_1' \sin\vartheta)\tilde{y}_1'$, $\tilde{y}_i = \tilde{y}_i'$, $i = 2, \ldots, n-1$ we see that

$$|\tilde{y}'| = \sqrt{\frac{\tilde{y}_1^2}{(\cos\vartheta + t_1'\sin\vartheta)^2} + \tilde{y}_i^2} \geqslant \sqrt{\frac{1}{4}\tilde{y}_1^2 + \tilde{y}_i^2} \geqslant \frac{1}{2}|\tilde{y}|,$$

so we have

$$I_2 \leq \left(\int_0^2 r^{n-2} dr \int_{\mathbb{R}^{n-1}} dt' \int_{\mathbb{R}^{n-1}} \frac{2^{p+n-2}}{(\cos\vartheta + t_1'\sin\vartheta)^2} \frac{d\tilde{y}}{|\tilde{y}|^{n-2+p}} \int_{\mathbb{R}^{n-1}} d\tilde{x} \, |\tilde{\alpha}(\tilde{x}+r\tilde{y}) - \tilde{\alpha}(\tilde{x})|^p |\Psi|^p \right)^{1/p} \equiv$$

$$\equiv I_3 .$$

Using the properties of the support of Ψ and introducing new variables $\tilde{z} = r\tilde{y}$ we get

$$I_3 \leq \left(\frac{2^{p+n-2}}{k_0} \right)^{2/p} \sup_{\mathrm{supp}\,\Psi} |\Psi| \left(\int_{\mathbb{R}^{n-1}} d\tilde{z} \int_{\mathbb{R}^{n-1}} d\tilde{x} \, \frac{|\tilde{\alpha}(\tilde{x}+\tilde{z}) - \tilde{\alpha}(\tilde{x})|^p}{|\tilde{z}|^p} \right)^{1/p} .$$

This ends the proof.

Now we shall prove the main result of this section:

Theorem B.1

Let $u \in W_p^l(\mathcal{D}_\vartheta)$, then $\frac{\partial^k u}{\partial n^k}\Big|_{\Gamma_0} \in W_p^{l-k-1/p}(\Gamma_0)$, $\frac{\partial^{k'} u}{\partial n^{k'}}\Big|_{\Gamma_\vartheta} \in W_p^{l-k'-1/p}(\Gamma_\vartheta)$, and we have the estimate

$$\text{(B.29)} \quad \left\| \frac{\partial^k u}{\partial n^k}\Big|_{\Gamma_0} \right\|_{l-k-1/p,p,\Gamma_0} + \left\| \frac{\partial^{k'} u}{\partial n^{k'}}\Big|_{\Gamma_\vartheta} \right\|_{l-k'-1/p,p,\Gamma_\vartheta} \leq C \|u\|_{l,p,\mathcal{D}_\vartheta} .$$

Now we shall formulate the inverse result. Let us assume that $\alpha \in W_p^{l-k-1/p}(\Gamma_0)$ and $\beta \in W_p^{l-k'-1/p}(\Gamma_\vartheta)$, then there exists a function $u \in W_p^l(\mathcal{D}_\vartheta)$ such that

$$\text{(B.30)} \quad \frac{\partial^k u}{\partial n^k}\Big|_{\Gamma_0} = \alpha, \quad \frac{\partial^{k'} u}{\partial n^{k'}}\Big|_{\Gamma_\vartheta} = \beta ,$$

and

$$\text{(B.31)} \quad \|u\|_{l,p,\mathcal{D}_\vartheta} \leq C(\|\alpha\|_{l-k-1/p,p,\Gamma_0} + \|\beta\|_{l-k'-1/p,p,\Gamma_\vartheta}) .$$

Proof. It is sufficient to prove this theorem for a smooth function. We shall consider the case $k=k'=1$ only. In the other cases the proof is similar. We use notations introduced in the proof of Lemma B.1. The first part of the theorem can be done similarly as in Lemma B.1 for the function $w(x',0)$ equal $D_{x'}^{l-2}\frac{\partial u}{\partial n}\Big|_{\Gamma_0}$ or $D_{x'}^{l-2}\frac{\partial u}{\partial n}\Big|_{\Gamma_\vartheta}$, where x' belong to Γ_0 or Γ_ϑ, respectively, because from the Sobolev theorems it follows that $\frac{\partial u}{\partial n}\Big|_{\Gamma_0}\in W_p^{l-2}(\Gamma_0)$ and $\frac{\partial u}{\partial n}\Big|_{\Gamma_\vartheta}\in W_p^{l-2}(\Gamma_\vartheta)$.

To prove the second part of the theorem we recall [Ad, 1], [Be, 1], that functions α, β can be extended on $\mathbb{R}^{n-1}$ in such a way that the extended functions $\tilde{\alpha}$, $\tilde{\beta}$ satisfy the relations:

$$\text{(B.32)}\qquad \begin{aligned} \|\tilde{\alpha}\|_{l-1/p,p,\mathbb{R}^{n-1}} &\leq C\|\alpha\|_{l-1/p,p,\mathbb{R}^{n-1}_+}, \\ \|\tilde{\beta}\|_{l-1/p,p,\mathbb{R}^{n-1}} &\leq C\|\beta\|_{l-1/p,p,\mathbb{R}^{n-1}_+}. \end{aligned}$$

At first we define a function u_1, which is an extension of the function $\tilde{\alpha}$ on the half-space $x_2\geq 0$, as

$$\text{(B.33)}\qquad \frac{\partial u_1}{\partial x_2}\Big|_{x_2=0} = \tilde{\alpha}.$$

This function will be constructed in the form

$$\text{(B.34)}\qquad \begin{aligned} u_1(x_1,x_2,z_1,\dots,z_{n-2}) &= x_2\int_{\mathbb{R}^{n-1}} dt\; \tilde{\alpha}(x_1+t_1x_2, z_1+t_2x_2,\dots \\ &\dots, z_{n-2}+t_{n-1}x_2)K(t)\zeta(x_2) \equiv u_1'\zeta(x_2). \end{aligned}$$

Changing variables in (B.34): $y_1=x_1+t_1x_2$, $y_2=z_1+t_2x_2,\dots,y_{n-1}=z_{n-2}+t_{n-1}x_2$, where $(y_1-x_1)^2+(y_2-z_1)^2+\dots+(y_{n-1}-z_{n-2})^2=x_2^2$, we obtain

$$(B.35)\quad u_1(x_1,x_2,z_1,\dots,z_{n-2}) = x_2^{-n+2}\int_{\mathbb{R}^{n-1}} dy\, \mathcal{I}(y) K\left(\frac{y_1-x_1}{x_2}, \frac{y_2-z_1}{x_2}, \dots, \frac{y_{n-1}-z_{n-2}}{x_2}\right) \zeta(x_2),$$

where $K(t) \in C_0^\infty(G)$, G is a compact set and $\int_G K(t)\,dt = 1$. Moreover, $\zeta(x) \in C^\infty(\mathbb{R}^1_+)$ such that $\zeta(x) = 1$ for $x \leq 1/2$ and $\zeta(x) = 0$ for $x \geq 1$ .Then

$$\frac{\partial u_1}{\partial \tau} = x_2^{-n+2}\int_{\mathbb{R}^{n-1}} dy \Big[(n-2)x_2^{-1}\cos\vartheta\, K(\;) + x_2^{-2}K_1'(\;)((y_1-x_1)\cos\vartheta - x_2\sin\vartheta) +$$

$$(B.36)\quad + x_2^{-2}\sum_{i=1}^{n-2} K_{i+1}'(\;)(y_{i+1}-z_i)\cos\vartheta\Big]\mathcal{I}(y)\zeta(x_2) - x_2^{-n+2}\int_{\mathbb{R}^{n-1}} dy\, \mathcal{I}(y) K(\;)\zeta'(x_2)\cos\vartheta,$$

where $K_i'(\;)$ denotes the derivative of K with respect to i-th argument.Coming back to the variables t we get

$$\frac{\partial u_1}{\partial \tau} = \int_{\mathbb{R}^{n-1}} dt\; \mathcal{I}(x_1+t_1x_2, z_1+t_2x_2, \dots, z_{n-2}+t_{n-1}x_2)\Big[(n-2)\cos\vartheta\, K(t) +$$

$$(B.37)\quad + K_1'(t)(t_1\cos\vartheta - \sin\vartheta) + \sum_{i=1}^{n-2} K_{i+1}'(t)\,t_{i+1}\cos\vartheta\Big]\zeta(x_2) +$$

$$- x_2\int_{\mathbb{R}^{n-1}} dt\; \mathcal{I}(\;)\, K(t)\,\zeta'(x_2)\cos\vartheta.$$

We demand that

$$\left.\frac{\partial u_1}{\partial \tau}\right|_{x'=0} = \mathcal{L}(0,z_1,\dots,z_{n-2})\Big[(n-2)\cos\vartheta + \int_{\mathbb{R}^{n-1}} dt\,\Big[K_1'(t)(t_1\cos\vartheta - \sin\vartheta) +$$

$$(B.38)\quad + \sum_{i=1}^{n-2} K_{i+1}'(t)\,t_{i+1}\cos\vartheta\Big]\Big] = \beta(0,z_1,\dots,z_{n-2}),$$

where $x' = (x_1, x_2)$ .We introduce the notation $\frac{\partial}{\partial s^i}\zeta(x_2) = (\sin\vartheta)^i D^i \zeta(x_2)$ where $D\zeta$ denotes a derivative with respect to its argument. Then

$$\frac{\partial^i}{\partial s^i}\frac{\partial u_1}{\partial \tau} = \int_{\mathbb{R}^{n-1}} dt\Big[(n-2)\cos\vartheta\, K(t) + K_1'(t)(t_1\cos\vartheta - \sin\vartheta) + \sum_{i=1}^{n-2} K_{i+1}'(t)t_{i+1}\cos\vartheta\Big]\cdot \tag{B.39}$$
$$\cdot \sum_{j=0}^{i} \frac{\partial^j}{\partial s^j}\tilde{\alpha}\,\frac{\partial^{i-j}}{\partial s^{i-j}}\zeta(x_2) - \int_{\mathbb{R}^{n-1}} dt\, K(t)\sum_{j=0}^{i-1}\frac{\partial^j}{\partial s^j}\tilde{\alpha}\,\frac{\partial^{i-j-1}}{\partial s^{i-j-1}}\zeta(x_2),$$

where we have omited terms with x_2, and

$$\frac{\partial^i}{\partial s^i}\tilde{\alpha} - \sum_{\sigma=0}^{i}\cos^\sigma\vartheta\,\sin^{i-\sigma}\vartheta \sum_{|\varkappa|=i-\sigma} D_1^\sigma D_\varkappa^{i-\sigma}\alpha t^\varkappa,$$

where $\varkappa$ is multiindex, $\varkappa=(\varkappa_1,\dots,\varkappa_{n-1})$. Moreover, we demand the following condition

$$\frac{\partial^i}{\partial s^i}\frac{\partial u_1}{\partial \tau}\Big|_{x'=0} = \int_{\mathbb{R}^{n-1}} dt\Big[(n-2)\cos\vartheta\, K(t) + K_1'(t)(t_1\cos\vartheta - \sin\vartheta) + \tag{B.40}$$
$$+\sum_{i=1}^{n-2} K_{i+1}'(t)t_{i+1}\cos\vartheta\Big]\frac{\partial^i}{\partial s^i}\alpha\Big|_{x'=0} - \int_{\mathbb{R}^{n-1}} dt\, K(t)\frac{\partial^{i-1}}{\partial s^{i-1}}\Big|_{x'=0} = D_s^i\beta\big|_{x'=0},$$

where $i=1,\dots,l-1$. Now we shall seek the function u_2 satisfying the relation

$$\frac{\partial u_2}{\partial n}\Big|_{\Gamma_\vartheta} = \beta - \frac{\partial u_1}{\partial n}\Big|_{\Gamma_\vartheta} \equiv \hat{\beta}, \tag{B.41}$$

where $\frac{\partial}{\partial n}$ denotes the derivative normal to Γ_ϑ, and

$$\frac{\partial u_2}{\partial n}\Big|_{\Gamma_0} = 0. \tag{B.42}$$

From (B.38) and (B.40) we know that

$$\frac{\partial^i}{\partial s^i}\hat{\beta}\Big|_{x'=0} = 0, \quad i \le l-1. \tag{B.43}$$

From (B.41) and (B.42) we construct the function u_2 in the following form:

$$u_2(x_1,x_2,z_1,\dots,z_{n-2}) = \tau\int_{\mathbb{R}^{n-1}} dt\ \tilde{\hat{\beta}}(s+t_1\tau, z_1+t_2\tau,\dots,z_{n-2}+t_{n-1}\tau)\, K_1(t)\,\chi(x) \equiv \tag{B.44}$$

$$\equiv u_2^{1}\,\Phi(\varphi)$$

where $\tilde{\hat{\beta}}$ is an extension of $\hat{\beta}$ on all $\mathbb{R}^{n-1}$ and all other symbols are described in (B.8) and (B.12). From (B.41) and (B.44) we get

$$\begin{aligned}
&\tilde{\hat{\beta}}(s+t_1\tau, z_1+t_2\tau,\dots,z_{n-2}+t_{n-1}\tau) = \tilde{\beta}(s+t_1\tau, z_1+t_2\tau,\dots,z_{n-2}+t_{n-1}\tau) + \\
&- \int_{\mathbb{R}^{n-1}} dt'\left[(n-2)\cos\vartheta\, K(t') + K_1'(t')(t'\cos\vartheta-\sin\vartheta) + \sum_{i=1}^{n-2} K'_{i+1}t'_{i+1}\cos\vartheta\right]\cdot \\
&\cdot \mathcal{L}((\cos\vartheta+t_1'\sin\vartheta)(s+t_1\tau), z_1+t_2\tau+t_2'\sin\vartheta(s+t_1\tau),\dots,z_{n-2}+t_{n-1}\tau+t_{n-1}'\sin\vartheta(s+t_1\tau))\cdot \\
&\cdot \zeta(\sin\vartheta(s+t_1\tau)) - \sin\vartheta(s+t_1\tau)\int_{\mathbb{R}^{n-1}} dt'\, K(t')\,\tilde{\mathcal{L}}(\quad)\,\zeta'(\sin\vartheta(s+t_1\tau))\cos\vartheta\ .
\end{aligned} \tag{B.45}$$

From the above relations we see that the function

$$u(x_1,x_2,z_1,\dots,z_{n-2}) = u_1(\quad) + u_2(\quad) \tag{B.46}$$

is a solution of the problem (B.30). From (B.44) and (B.45) we see that u_2 can be divided into three parts

$$u_2 = u_{2,1} + u_{2,2} + u_{2,3}\ . \tag{B.47}$$

From [Fa,1] we have the estimate

$$\|u_1\|_{l,p,\mathcal{D}_\vartheta} + \|u'_{2,1}\|_{l,p,\mathcal{D}_\vartheta} \le C(\|\mathcal{L}\|_{l-1-1/p,p,\Gamma_0} + \|\beta\|_{l-1/p-1,p,\Gamma_\vartheta})\ . \tag{B.48}$$

From (B.44) and (B.45) we have

$$D^k_{x'}(u'_{2,2}+u'_{2,3}) = \sum_{j=0}^{k} \int_{\mathbb{R}^{n-1}} dt \int_{\mathbb{R}^{n-1}} dt' \Big[\Phi(t')\mathcal{K}_1(t) D^j_{x'}\tilde{\alpha}\, D^{k-j}_{x'}(\zeta(\sin\vartheta(s+t_1\tau))\zeta(\tau)\tau) + \tag{B.49}$$

$$-K(t')\mathcal{K}_1(t)\, D^j_{x'}\tilde{\alpha}\, D^{k-j}_{x'}(\tau\sin\vartheta(s+t_1\tau)\zeta'(\sin\vartheta(s+t_1\tau))\zeta(\tau))\cos\vartheta\Big],$$

where $k=0,\dots,l$, $x'=(s,\tau)$, and

$$D^k_{z_i}(u'_{2,2}+u'_{2,3}) = \sum_{j=0}^{k} \int_{\mathbb{R}^{n-1}} dt \int_{\mathbb{R}^{n-1}} dt'\, D^k_{z_i}\tilde{\alpha}\,\Big[\Phi(t')\zeta(\sin\vartheta(s+t_1\tau)) + \tag{B.50}$$

$$-K(t')\sin\vartheta(s+t_1\tau)\zeta'(\sin\vartheta(s+t_1\tau))\cos\vartheta\Big]\mathcal{K}_1(t)\,\tau\,\zeta(\tau),$$

where $i=1,\dots,n-2$, $k=0,\dots,l$, and

$$\Phi(t') = (n-2)\cos\vartheta\, K(t') + K'_1(t')(t'\cos\vartheta-\sin\vartheta) + \sum_{i=1}^{n-2} K'_{i+1}(t')\, t'_{i+1}\cos\vartheta .$$

Using (B.19) we integrate twice times by parts in the l-th derivative of (B.49), (B.50) and once in the $l-1$-th derivative. Then using Lemmas B.1, B.2 we obtain the estimate

$$\|u'_{2,2}+u'_{2,3}\|_{l,p,\mathcal{D}_\vartheta} \le c\,(\|\alpha\|_{l-1-1/p,p,\Gamma_0} + \|\beta\|_{l-1-1/p,p,\Gamma_\vartheta}). \tag{B.51}$$

At last we consider the function u_2

$$\|D^k_x u_2\|_{p,\mathcal{D}_\vartheta} \le \sum_{j=0}^{k} \|D^j_x u'_2\, D^{k-j}_x \Phi(\varphi)\|_{p,\mathcal{D}_\vartheta} \le c\sum_{j=0}^{k} \|D^j_x u'_2\, \varrho^{-(k-j)}\dot{\Phi}\|_{p,\mathcal{D}_\vartheta}, \tag{B.52}$$

where $k=0,\dots,l$, and the same notations as in (B.24) are used. To estimate (B.52), for all cases apart from $p=2$ and $k-j=1$ we use the Hardy inequality, because the condition (B.43) is satisfied. For $p=2$ and $k-j=1$ we use the inequality (B.25). This concludes the proof.

C. Problems (B) and (E) in domain with dihedral angles π/n

We start this section with the following lemma:

Lemma C.1

The solutions of the problem (B) can be expressed as the sum

(C.1) $$v = v_1 + v_2,$$

where $v_1 = \nabla u$, u is a solution of the Neumann problem

(C.2) $$\Delta u = 0, \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = b,$$

and $v_2 = \operatorname{rot} e$, e is a solution of the third problem

(C.3) $$-\Delta e = \omega, \qquad e_\tau\big|_{\partial\Omega} = 0, \quad \operatorname{div} e\big|_{\partial\Omega} = 0$$

where $e_\tau = e\cdot\bar\tau$ and $\bar\tau$ is a vector tangent to $\partial\Omega$.

Proof. From (C.1) v_2 is a solution of the problem

(C.4) $$\operatorname{rot} v_2 = \omega, \quad \operatorname{div} v_2 = 0, \qquad v_2\cdot\bar n\big|_{\partial\Omega} = 0.$$

From Lemma 1 in [By, 1] it follows that for the problem $\operatorname{div} v_2 = 0$, $v_2\cdot\bar n|_{\partial\Omega} = 0$, there exists a unique vector e such that $v_2 = \operatorname{rot} e$, $\operatorname{div} e = 0$, $e_\mu|_{\partial\Omega} = 0$, where $e_\mu = e\cdot\bar\tau_\mu$, $\mu = 1, 2$, and $\bar\tau_\mu$ are vectors tangent to the boundary. Therefore, instead of (C.4) we have

(C.5) $$-\Delta e = \omega, \qquad \operatorname{div} e = 0, \qquad e_\mu\big|_{\partial\Omega} = 0.$$

Using that $\operatorname{div}\omega = 0$, the second equation in (C.5) we replace by the boundary condition $\operatorname{div} e|_{\partial\Omega} = 0$, because the problem $\Delta \operatorname{div} e = 0$, $\operatorname{div} e|_{\partial\Omega} = 0$ has the unique solution $\operatorname{div} e = 0$. Therefore, instead of (C.5) we have (C.3). This ends the proof.

Now we shall consider the Neumann problem in ,

$$\text{(C.6)} \qquad \Delta u = f \;, \quad \frac{\partial u}{\partial n}\Big|_{\Gamma_0 \cup \Gamma_\vartheta} = 0 .$$

We assume that the function f has a compact support. In the case of the Neumann problem, using the method of reflection, the function f can be extended on $\mathbb{R}^3$ in the following way:

$$\text{(C.7)} \qquad \tilde{f}(r, k\tfrac{\pi}{n} + \varphi, x_3) = \tilde{f}(r, k\tfrac{\pi}{n} - \varphi, x_3) ,$$

where $\varphi \in [0, \frac{\pi}{n}]$, $k = 1, \dots, 2n$, and r , φ are the polar coordinates. Using the extended function (C.7) we can write a solution of (C.6) in the form

$$\text{(C.8)} \qquad u(x) = \int_{\mathbb{R}^3} \frac{\tilde{f}(x')}{|x - x'|}\, dx' .$$

Therefore, from the Calderon-Zygmunt integrals theorem [Ca, 1] and the Riesz potential [St, 1] we have the estimate

$$\text{(C.9)} \qquad \|u\|_{l+2,p,\mathcal{D}_\vartheta \cap K_R(0)} \leq c\, \|\tilde{f}\|_{l,p,\mathbb{R}^3} \leq c\, \|f\|_{l,p,\mathcal{D}_\vartheta} ,$$

where $K_R(0)$ is a ball with the center in the origin and the radius $R < \infty$, such that $\operatorname{supp} f \subset \mathcal{D}_\vartheta \cap K_R(0)$.

For the Dirichlet problem, instead of (C.7) we have

$$\text{(C.10)} \qquad \tilde{f}(r, k\tfrac{\pi}{n} + \varphi, x_3) = -\tilde{f}(r, k\tfrac{\pi}{n} - \varphi, x_3) .$$

For the Dirichlet-Neumann problem we have

$$\tilde{f}(r, k\tfrac{\pi}{n}+\varphi, x_3) = -\tilde{f}(r, k\tfrac{\pi}{n}-\varphi, x_3) \qquad \text{for } k=2,4,\dots,2n, \tag{C.11}$$

$$\tilde{f}(r, k\tfrac{\pi}{n}+\varphi, x_3) = \tilde{f}(r, k\tfrac{\pi}{n}-\varphi, x_3) \qquad \text{for } k=1,3,\dots,2n-1.$$

For (C.10) or (C.11) a solution of the Dirichlet or the Dirichlet-Neumann problem can be written in the form (C.8) and the estimate (C.9) is valid also.Therefore,we have proved the following lemma:

Lemma C.2

Let $f \in W_p^l(\mathcal{D}_\vartheta)$ and has a compact support,then there exists a solution of the Neumann,the Dirichlet and the Dirichlet-Neumann problems described by (C.8),for which the estimate (C.9) is valid.

The problem (C.3) in dihedral angle $\mathcal{D}_\vartheta$ separates into two problems:

$$\Delta e_3 = \omega_3, \quad e_3|_{\Gamma_0} = e_3|_{\Gamma_\vartheta} = 0, \tag{C.12}$$

and

$$\Delta e_i = \omega_i, \quad i=1,2,$$

$$e_1|_{\Gamma_0} = 0, \quad e_1\cos\vartheta + e_2\sin\vartheta|_{\Gamma_\vartheta} = 0, \tag{C.13}$$

$$\frac{\partial e_2}{\partial n}\Big|_{\Gamma_0} = 0, \quad \frac{\partial e_1}{\partial n}\sin\vartheta - \frac{\partial e_2}{\partial n}\cos\vartheta\Big|_{\Gamma_\vartheta} = 0.$$

For the problem (C.13) the extended functions $\tilde{\omega}_i$,i=1,2,over R^3 we can calculate from the following system of equations:

$$\tilde{\omega}_1(r,k\tfrac{\pi}{n}+\varphi,x_3)\cos\tfrac{\pi}{n}+\tilde{\omega}_2(r,k\tfrac{\pi}{n}+\varphi,x_3)\sin\tfrac{\pi}{n}=$$

(C.14)

$$=-\tilde{\omega}_1(r,k\tfrac{\pi}{n}-\varphi,x_3)\cos\tfrac{\pi}{n}-\tilde{\omega}_2(r,k\tfrac{\pi}{n}-\varphi,x_3)\sin\tfrac{\pi}{n},$$

$$\tilde{\omega}_1(r,k\tfrac{\pi}{n}+\varphi,x_3)\sin\tfrac{\pi}{n}-\tilde{\omega}_2(r,k\tfrac{\pi}{n}+\varphi,x_3)\cos\tfrac{\pi}{n}=$$

(C.15)

$$=\tilde{\omega}_1(r,k\tfrac{\pi}{n}-\varphi,x_3)\sin\tfrac{\pi}{n}-\tilde{\omega}_2(r,k\tfrac{\pi}{n}-\varphi,x_3)\cos\tfrac{\pi}{n},$$

for $k=1,3,\dots,2n-1$, and

$$\tilde{\omega}_1(r,k\tfrac{\pi}{n}+\varphi,x_3)=-\tilde{\omega}_1(r,k\tfrac{\pi}{n}-\varphi,x_3),$$

(C.16)

$$\tilde{\omega}_2(r,k\tfrac{\pi}{n}+\varphi,x_3)=\tilde{\omega}_2(r,k\tfrac{\pi}{n}-\varphi,x_3),$$

for $k=2,4,\dots,2n$. Therefore, using the extended functios $\tilde{\omega}_i$, $i=1,2$, we can write a solution of (C.13) in the form

$$e_i(x)=\int_{\mathbb{R}^3}\frac{\tilde{\omega}_i(x')}{|x-x'|}\,dx', \tag{C.17}$$

and the following estimate is valid

$$\|e_i\|_{l+2,p,\mathcal{D}_\vartheta\cap K_R(0)}\le c\|\tilde{\omega}_i\|_{l,p,\mathcal{D}_\vartheta}\le c\left(\|\omega_1\|_{l,p,\mathcal{D}_\vartheta}+\|\omega_2\|_{l,p,\mathcal{D}_\vartheta}\right). \tag{C.18}$$

Hence, we have proved:

Lemma C.3

Let $\omega_i\in W_p^l(\mathcal{D}_\vartheta)$ and have compact supports. Then there exist solutions of the problem (C.12), (C.13) described by (C.8) or (C.17), and the estimates (C.9) or (C.18) are valid, respectively.

Now we shall consider the above problems in the domain Ω

with dihedral angles π/n .At first we shall consider the Neumann problem

(C.19) $$-\Delta u = f \quad , \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = h ,$$

when the following compatibility condition is satisfied

(C.20) $$\int_{\Omega} f\,dx = \int_{\partial\Omega} h\,ds \ .$$

We assume that $f \in W_p^l(\Omega)$, $h \in W_p^{l+1-1/p}(\partial\Omega)$ .As the weak solution of the problem (C.19) we denote a solution of the following equation

(C.21) $$(\nabla u, \nabla \eta)_{\Omega} + (h, \eta)_{\partial\Omega} = (f, \eta)_{\Omega} \quad , \forall \eta \in H^1(\Omega),$$

where $(\alpha, \beta)_{\sigma} = \int_{\sigma} \alpha(x)\beta(x)\,dx$.
From the Fredholm theorem it follows that there exists a unique weak solution $u \in H^1(\Omega)$ of the problem (C.19) such that

(C.22) $$\|u\|_{1,2,\Omega} \leq c\,(\|f\|_{2,\Omega} + \|h\|_{2,\partial\Omega})$$

and $\int_{\Omega} u\,dx = 0$.
Now we shall formulate the following theorem:

Theorem C.1

Let $f \in W_p^l(\Omega)$, $h \in W_p^{l+1-1/p}(\partial\Omega)$, Ω is a domain with angles between two surfaces only, equal π/n ,n=2,3,....,.Then there exists a solution of the problem (C.19) such that

and

(C.23) $$\|u\|_{l+2,p,\Omega} \leq c\,(\|f\|_{l,p,\Omega} + \|h\|_{l+1-1/p,\partial\Omega}) \ .$$

Proof.We shall show that the weak solution $u \in H^1(\Omega)$ belongs to $W_p^{l+2}(\Omega)$ .From the theory of elliptic equations it follows that in a neighbourhood of any interior point as well as in a neighbourhood of any point near a smooth part of the boundary $u \in W_p^{l+2}(\Omega')$,and we have the estimate

$$\text{(C.24)} \qquad \|u\|_{l+2,p,\Omega'} \leqslant C(\|f\|_{l,p,\Omega''} + \|h\|_{l+1-1/p,\,p,\,\Omega''\cap\partial\Omega}),$$

where $\bar{\Omega}' \subset \Omega''$ denotes any of these two neighbourhoods.

Finally we have to find a similar estimate in a neighbourhood of the edges.Let $\xi \in L$,where L is any of edges of the domain Ω ,and $\Omega_\lambda(\xi) = \Omega \cap K_\lambda(\xi)$,where $\lambda \leqslant \frac{d}{2}$ .Let $\Gamma_0(\xi)$, $\Gamma_\vartheta(\xi)$ be spaces tangent to the boundary $\partial\Omega$ at the point $\xi \in L$,where the angle between two surfaces is $\vartheta = \frac{\pi}{n}$,n=2,3, ...,.Now we define d as a value ,for which there exists a map - ping $T \in C^{l+2}(\Omega_d(\xi))$ transforming $\Omega_d(\xi)$ into the neighbourhood of the point $T(\xi) \in TL$ in the dihedral angle $\mathcal{D}_\vartheta$,where $T(L)$ is the intersection of its planes $\Gamma_0 = T\Gamma_0(\xi)$ and $\Gamma_\vartheta = T\Gamma_\vartheta(\xi)$ .We introduce a function $\zeta_\lambda = \zeta(\frac{|x-\xi|}{d})$,such that $\zeta(\tau) \in C^\infty(0,\infty)$, $\zeta(\tau) = 1$ for $\tau \leqslant 1/2$ and $\zeta(\tau) = 0$ for $\tau \geqslant 1$. Using the transformation $z = Tx$ instead of ∇ we obtain the following operator $\hat{\nabla} = \frac{\partial z_m}{\partial x}\Big|_{x=T^{-1}(z)} \frac{\partial}{\partial z_m} \equiv \mathcal{A}(z)\cdot\nabla$ .To calculate the boundary conditions in new coordinates we introduce a vector $\bar{n}_0 = \bar{n}(\xi)$ such that

$$\bar{n}(x)\Big|_{x=T^{-1}(z)} = \frac{\mathcal{A}(z)\cdot\bar{n}_0}{|\mathcal{A}(z)\cdot\bar{n}_0|} \equiv B(z)\bar{n}_0\,, \text{ where } B(0) = \mathcal{A}(0) = I\ .$$

Introducing the notation $\hat{f}(z) = f(x)|_{x=T^{-1}(z)}$ we obtain the following equation for the function $\hat{v}_\lambda = \hat{\zeta}_\lambda \hat{u}$ in the dihedral angle $\mathcal{D}_\vartheta$:

(C.25) $$\nabla^2 \hat{v}_\lambda = (\nabla^2 - \hat{\nabla}^2)\hat{v}_\lambda + \hat{g},$$

where $\hat{\nabla}^2 = A^k A^l \nabla_k \nabla_l$, and

$$\hat{g} = \hat{\zeta}_\lambda \hat{f} + 2\hat{\nabla}\hat{\zeta}_\lambda \hat{\nabla}\hat{u} + \hat{\nabla}^2\hat{\zeta}_\lambda \hat{u} + \hat{\nabla}(A)\hat{\nabla}(\hat{\zeta}_\lambda)\hat{u} + \hat{\nabla}(A)\hat{\nabla}(\hat{v}_\lambda).$$

Now the boundary conditions have the form

(C.26) $$\bar{n}_0 \cdot \nabla \hat{v}_\lambda \big|_{\Gamma_0 \cup \Gamma_\vartheta} = \hat{\zeta}_\lambda \hat{h} + \bar{n}_0 \cdot (\nabla - \hat{\nabla})\hat{v}_\lambda + (\bar{n}_0 - \bar{n}) \cdot \hat{\nabla}\hat{v}_\lambda + \bar{n} \cdot \hat{\nabla}(\hat{\zeta}_\lambda)\hat{u} \equiv \equiv \hat{d} + \bar{n}_0 \cdot (\nabla - \hat{\nabla})\hat{v}_\lambda + (\bar{n}_0 - \bar{n}) \cdot \hat{\nabla}\hat{v}_\lambda .$$

Functions on the right-hand side of equations (C.25) and (C.26) are different from zero in the domain $\hat{\Omega}_\lambda = T\Omega_\lambda$ .Thus we can extend them on $\mathcal{D}_\vartheta \setminus \hat{\Omega}_\lambda$ as null functions.Therefore,we consider the problem (C.25),(C.26) as the Neumann problem with respect to $\hat{v}$ in the domain $\mathcal{D}_\vartheta$ .Knowing that $A, B \in C^{l+2}(\hat{\Omega}_\lambda)$ and using (C.9) and the Theorem B.1,we obtain

(C.27) $$\|\hat{v}_\lambda\|_{2,p,\mathcal{D}_\vartheta} \le C_1 \|\hat{g}\|_{p,\mathcal{D}_\vartheta} + C_2 \|\hat{d}_1\|_{1-1/p,p,\partial\mathcal{D}_\vartheta} + C_3 \lambda \|\hat{v}_\lambda\|_{2,p,\mathcal{D}_\vartheta} .$$

For sufficiently small λ we get

(C.28) $$\|u\|_{2,p,\Omega_{\lambda/2}} \le C_4 \|f\|_{p,\Omega_\lambda} + C_5 \|u\|_{1,p,\Omega_\lambda} + C_6 \|h\|_{1-1/p,p,\partial\Omega_\lambda} ,$$

where $C_1, \ldots, C_6$ are constants.Using the interpolation inequality we obtain

(C.29) $$\|u\|_{2,p,\Omega_{\lambda/2}} \le C_4 \|f\|_{p,\Omega_\lambda} + \varepsilon \|u\|_{2,p,\Omega_\lambda} + C_7 \|u\|_{1,2,\Omega_\lambda} + C_6 \|h\|_{1-1/p,p,\partial\Omega_\lambda} .$$

We sum (C.29) over all neighbourhoods of the edges, the inequality (C.24) over all remaining neighbourhoods, and we use the estimate (C.22) for the weak solution. Then we obtain the required estimate

$$(C.30)\qquad \|u\|_{2,p,\Omega} \leq c\,(\|f\|_{p,\Omega} + \|h\|_{1-1/p,p,\partial\Omega}).$$

Now we prove the existence of the solution u in the space $W_p^2(\Omega)$. In order to do that we have to estimate functions $\hat{g}$ and $\hat{d}$. Using the interpolation inequality we have

$$(C.31)\qquad \|\hat{g}\|_{p,\mathcal{D}_\vartheta} \leq c_8 \|\hat{f}\|_{p,\mathcal{D}_\vartheta} + c_9 \lambda \|\hat{v}_\lambda\|_{2,p,\mathcal{D}_\vartheta} + c_{10} \|\hat{v}_\lambda\|_{1,2,\mathcal{D}_\vartheta},$$

and

$$(C.32)\qquad \|\hat{d}\|_{1-1/p,p,\partial\mathcal{D}_\vartheta} \leq c_{11} \|\hat{h}\|_{1-1/p,p,\partial\mathcal{D}_\vartheta} + c_{12}\lambda \|\hat{v}_\lambda\|_{2,p,\mathcal{D}_\vartheta} + c_{13} \|\hat{v}_\lambda\|_{1,2,\mathcal{D}_\vartheta}.$$

Therefore, using the method of successive approximations for small λ we obtain the existence $\hat{v}_\lambda \in W_p^2(\mathcal{D}_\vartheta)$, hence $u \in W_p^2(\Omega)$.

Now we apply the method of induction to show that $u \in W_p^{l+2}(\Omega)$. From (C.25), (C.26) using (C.9) we obtain

$$\|\hat{v}_\lambda\|_{s+2,p,\mathcal{D}_\vartheta} \leq c_1' \|\hat{g}\|_{s,p,\mathcal{D}_\vartheta} + c_2' \|\hat{d}\|_{s+1-1/p,p,\partial\mathcal{D}_\vartheta} + c_3' \lambda \|\hat{v}_\lambda\|_{s+2,\lambda,\mathcal{D}_\vartheta},$$

where s=0, 1, ..., l, hence we have

$$\|u\|_{s+2,p,\Omega_{\lambda/2}} \leq c_4' \|f\|_{s,p,\Omega_\lambda} + c_5' \|u\|_{s+1,p,\Omega_\lambda} + c_6' \|h\|_{s+1-1/p,p,\partial\Omega_\lambda} + c_7' \lambda \|u\|_{s+2,p,\Omega_\lambda}.$$

Thus for sufficiently small λ and $u \in W_p^{s+1}(\Omega)$, s=0, 1, ..., l, we obtain the required estimate, and using the method of successive approximations we prove the existence. This ends the proof.

As the weak solution of the problem (C.3) we denote a solution of the following equation

$$\text{(C.33)}\quad \int_\Omega \nabla e\cdot\nabla\eta\, dx + \int_{\partial\Omega} e_n\eta_n \operatorname{div}\bar{n}\, ds = \int_\Omega \omega\cdot\eta\, dx\ , \forall\eta\in H^1(\Omega),\quad \eta_\tau|_{\partial\Omega}=0,$$

where $e_n=e\cdot\bar{n}$, $e_\tau=e\cdot\bar{\tau}$. From the Fredholm theorem it follows that there exists a unique solution $e\in H^1(\Omega)$ of (C.33) such that

$$\text{(C.34)}\quad \|e\|_{1,2,\Omega} \leq c\|\omega\|_{2,\Omega}.$$

Now we formulate the following theorem:

Theorem C.2

Let $\omega\in W_p^l(\Omega)$, Ω is a bounded domain with angles equal $\frac{\pi}{n}$, n=2,3,..., between two surfaces only. Then there exists a unique solution of the problem (C.3) such that $e\in W_p^{l+2}(\Omega)$ and

$$\text{(C.35)}\quad \|e\|_{l+2,p,\Omega} \leq c\,\|\omega\|_{l,p,\Omega}.$$

Proof. The proof is the same as the proof of Theorem C.1, with the only difference such that instead of equations (C.25), (C.26) for $\hat{e}'_\lambda=\hat{\zeta}_\lambda\hat{e}$ we have

$$\text{(C.36)}\quad \nabla^2\hat{e}'_\lambda = (\nabla^2-\hat{\nabla}^2)\,\hat{e}'_\lambda + \hat{g}_1$$

where $\hat{g}_1 = \hat{\zeta}_\lambda\hat{\omega} + 2\hat{\nabla}\hat{\zeta}_\lambda\hat{\nabla}\hat{e} + AA\cdot\nabla\nabla(\hat{\zeta}_\lambda)\hat{e} + \hat{\nabla}(A)\hat{\nabla}\hat{e}'_\lambda$,

and

$$\hat{e}'_\tau \bar{\tau}_{o\mu}\big|_{\partial\Omega} = (1 - B(z))\,\bar{\tau}_{o\mu}\hat{e}'_\lambda \,, \tag{C.37}$$

$$\operatorname{div} \hat{e}'_\lambda\big|_{\partial\Omega} = (\operatorname{div} - \hat{\operatorname{div}})\,\hat{e}'_\lambda + \hat{\nabla}(\hat{\zeta}_\lambda)\,\hat{e}_\lambda \,,$$

where $\bar{\tau}_\mu(x)\big|_{x = T^{-1}(z)} = B(z)\,\bar{\tau}_{o\mu}$, $\mu = 1, 2$.

Moreover, we use the existence of the weak solution determined by (C.33).

D. Problems (B) and (E) in domain with edges

We start this section with Lemmas 4.17 and 4.19, from [Kon, 1], which we adopt them to our case.

Lemma D.1

Let the functions $f_{k_1k_2}(x) \in H^1(G)$, $k_1+k_2=k$, $k_1 \geqslant 0$, $k_2 \geqslant 0$, are integers, $x=(x_1,x_2)$ are the cartesian coordinates in $\mathbb{R}^2$, be given in a domain $G \subset \mathbb{R}^2$ such, that $0 \in G$.

Then there exists a function $u \in H^{k+1}(G)$ such, that

$$(D.1)\qquad \frac{\partial^{k_1+k_2}}{\partial x_1^{k_1}\partial x_2^{k_2}}u - f_{k_1k_2} = K(|x|)\, f_{k_1k_2} \in H^1_0(G;0),$$

$$(D.2)\qquad \left.\frac{\partial^{|\alpha|}u}{\partial x^{\alpha}}\right|_{x=0} = 0 \quad ,$$

$|\alpha| \leqslant k-1$, $k \geqslant 1$ and for $k=0$ we have not any condition similar to (D.2), where $\alpha=(\alpha_1,\alpha_2)$ is a multiindex and $K(|x|)$ is a smooth function with compact support such, that

$$(D.3)\qquad \lim_{r\to 0} K(r)\, r^{-1} = c_0 < \infty \quad , \quad r=|x| .$$

Moreover, the following estimate is valid

$$(D.4)\qquad \left\| \frac{\partial^{k_1+k_2}}{\partial x_1^{k_1}\partial x_2^{k_2}}u - f_{k_1k_2} \right\|_{H^1_0(G;0)} \leqslant C \| f_{k_1k_2} \|_{H^1(G)} .$$

Proof. Without loss of generality we can assume that a segment I of $x_1=0$ belongs to G. We assume that $0 \in I$. We shall consider the following functions determined on

$$(D.5)\qquad u_{k-j,j}(x_2) = \frac{1}{(j-1)!}\int_0^{x_2}(x_2-\tau)^{j-1} f_{k-j,j}(0,\tau)\left(1+K(\tau)\right)d\tau \ ,$$

for $k \geqslant j \geqslant 1$, and

(D.6) $$u_{k,0}(x_2) = f_{k0}(0,x_2)(1+K(x_2)),$$

where the behaviour of the function $K(\tau)$ in neighbourhood of zero is described by (D.3). From the definition (D.5) the function $u_{k-j,j}(x_2)$ with derivatives of all orders up to $j-1$ is equal zero for $x_2=0$ and its j-th derivative is equal $f_{k-j,j}(0,x_2) \in H^{1/2}(I)$. For $k=0$ we have only the expression (D.6). Therefore, $u_{k-j,j}(x_2) \in H^{j+1/2}(I)$, $j=1,\dots,k$, and $u_{k,0}(x_2) \in H^{1/2}(I)$, so from the Slobodetzki theorem about traces there exists a function $u \in H^{k+1}(G)$, such that

(D.7)
$$\frac{\partial^{k-j}}{\partial x_1^{k-j}} u \Big|_{x_1=0} = u_{k-j,j}(x_2), \quad j=1,\dots,k, \text{ and}$$
$$\frac{\partial^{k}}{\partial x_1^{k}} u \Big|_{x_1=0} = u_{k,0}(x_2).$$

The function $u(x_1,x_2)$ is the seeking function, because on the segment I we have

(D.8) $$\frac{\partial^{j}}{\partial x_2^{j}} \frac{\partial^{k-j}}{\partial x_1^{k-j}} u \Big|_{x_1=0} = f_{k-j,j}(0,x_2)(1+K(x_2)),$$

where $j=0,\dots,k$. From (D.5), (D.6) and (D.7) we can construct the function u in the form

(D.9) $$u(x_1,x_2) = \sum_{j=0}^{k} \frac{1}{(k-j)!} x_1^{k-j} \int_{\mathbb{R}^1} K_j(t)\, u_{k-j,j}(x_2+tx_1)\,dt,$$

where $K_j(t)$, $j=0,\dots,k$, are smooth functions with compact support such, that $\int_{\mathbb{R}^1} K_j(t)\,dt = 1$. Now from the property (D.3) of the function $K(x_2)$ we obtain the estimate (D.4) for the function u

$$\left\| \frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}} u - f_{k_1k_2} \right\|_{H_0^1(G;0)} \leq C \left\| \frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}} u - f_{k_1k_2} \right\|_{H_0^{1/2}(I;0)} =$$

(D.10)

$$= C \| K(x_2) f_{k_1k_2} \|_{H_0^{1/2}(I;0)} \leq C \| K(|x|) f_{k_1k_2}(x_1,x_2) \|_{H_0^1(G;0)} \leq C \| f_{k_1k_2} \|_{H^1(G)}.$$

This ends the proof.

Now we shall consider the Neumann problem with homogeneous boundary conditions in the domain d_ϑ :

(D.11) $$-\Delta' u = f, \quad \frac{\partial u}{\partial n}\Big|_{\partial d_\vartheta} = 0,$$

where $\Delta' = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ in coordinates introduced in Section 2.

Lemma D.2

Let $f \in H^k(d_\vartheta)$, $k \geqslant 1$, then there exists a function $v \in H^{k+2}(d_\vartheta)$ such that

(D.12) $$\Delta' v + f_{k-2} \in H_0^k(d_\vartheta;0), \quad \frac{\partial v}{\partial n}\Big|_{\partial d_\vartheta} = 0, \quad \frac{\partial^i v}{\partial x'^i}\Big|_{x'=0} = 0, \quad i < k+1,$$

and the following estimates are valid

(D.13) $$\| \Delta' v + f_{k-2} \|_{H_0^k(d_\vartheta;0)} \leq C \| f \|_{H^k(d_\vartheta)},$$

(D.14) $$\| v \|_{H^{k+2}(d_\vartheta)} \leq C \| f \|_{H^k(d_\vartheta)},$$

where $f_{k-2} = f - \sum_{j=0}^{k-2} P_f^j(x')$ for $k \geqslant 2$ and $f_{-1} = f$ for $k=1$,

$P_f^j(x') = \frac{1}{j!} \sum_{|\alpha|=j} \frac{\partial^j f}{\partial x'^\alpha}\Big|_{x'=0} x'^\alpha$, $\alpha = (\alpha_1, \alpha_2)$ is a multiindex.

Proof. Let $\frac{\partial}{\partial l_1} = \frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial l_2} = \cos\vartheta \frac{\partial}{\partial x_1} + \sin\vartheta \frac{\partial}{\partial x_2}$ be two directions along γ_0 and γ_ϑ , respectively, which constitute the boundary of d_ϑ .Let us consider the equations

$$\frac{\partial^k}{\partial l_s^k}\frac{\partial \omega}{\partial n_s}=0,\quad s=1,2,$$

(D.15)
$$\frac{\partial^{k-1}}{\partial x_1^{i_1}\partial x_2^{i_2}}(\overset{\prime}{\Delta}\omega+f)=0,\quad i_1+i_2=k-1,\ k\geq 1,$$

where $\frac{\partial}{\partial n_1}=\frac{\partial}{\partial x_2}$, $\frac{\partial}{\partial n_2}=-\sin\vartheta\frac{\partial}{\partial x_1}+\cos\vartheta\frac{\partial}{\partial x_2}$. The equations (D.15) constitute an algebraic system of k+2 equations on k+ 2 functions $D^{k+1}_{j_1j_2}\omega\equiv\frac{\partial^{k+1}}{\partial x_1^{j_1}\partial x_2^{j_2}}\omega$, $j_1+j_2=k+1$.The equations (D.15) are linearly independent so we can calculate $D^{k+1}_{j_1j_2}\omega$ as a linear combination of $D^{k-1}_{i_1i_2}f$ .We shall denote them as $\omega^{(k+1)}_{j_1j_2}$, $j_1+j_2=k+1$.It is evident that $\omega^{(k+1)}_{j_1j_2}\in H^1(d_\vartheta)$ .Lemma D.1 implies that there exists a function $v\in H^{k+2}(d_\vartheta)$ such that

(D.16)
$$\left.\frac{\partial^i v}{\partial x^\alpha}\right|_{x'=0}=0,\quad |\alpha|=i,\ i<k+1,$$

(D.17)
$$\frac{\partial^{k+1}}{\partial x_1^{j_1}\partial x_2^{j_2}}v-\omega^{(k+1)}_{j_1j_2}=-K(|x'|)\,\omega^{(k+1)}_{j_1j_2}\in H^1_0(d_\vartheta;0),\quad j_1+j_2=k+1,$$

where $K(\tau)$ is determined in Lemma D.1.From the explicit expression of functions $\omega^{(k+1)}_{j_1j_2}$ we obtain (D.14) ,and from (D.17) we also get

(D.18)
$$\frac{\partial^{k-1}}{\partial x_1^{i_1}\partial x_2^{i_2}}(\overset{\prime}{\Delta}v+f)=K(|x'|)\frac{\partial^{k-1}}{\partial x_1^{i_1}\partial x_2^{i_2}}f\in H^1_0(d_\vartheta;0),\ i_1+i_2=k-1,\ k\geq 1.$$

From (D.18),using Lemma D.1 and the Hardy inequality (2.21) ,we obtain (D.13).Now we shall show that the boundary condition $\left.\frac{\partial v}{\partial n}\right|_{\partial d_\vartheta}=0$ is satisfied.From the first equation of (D.15) and from (D.17) we have

(D.19)
$$\frac{\partial^k}{\partial l_s^k}\frac{\partial v}{\partial n_s}=0,\quad s=1,2,$$

and from (D.16) we get

$$\text{(D.20)}\quad \frac{\partial^{\sigma}}{\partial x_i^{\sigma}}\,\frac{\partial v}{\partial n_s}\bigg|_{x'=0} = 0\,,\quad s=1,2,\ \sigma\le k-1,\ i=1,2.$$

From (D.19) and (D.20) after simple calculations we have

$$\text{(D.21)}\quad \frac{\partial v}{\partial n}\bigg|_{\partial d_{\vartheta}} = 0\,.$$

This ends the proof.

From [So,2] we rewrite the following lemma:

Lemma D.3

Let $m \geqslant 0$ be an integer number such that $m+2 < \frac{\pi}{\vartheta}$, and let $Q_m(x')$ be a set of all homogeneous polynomials of degree m. Then the problem

$$\text{(D.22)}\quad \begin{aligned} &\Delta' u = Q_m(x')\,, \quad x'\in d_{\vartheta}\,,\\ &\frac{\partial u}{\partial n}\bigg|_{\gamma_0\cup\gamma_{\vartheta}} = 0, \end{aligned}$$

has a solution $u \in Q_{m+2}(x')$.

Proof. In polar coordinates any homogeneous polynomial of degree m has the form

$$Q_m(x') = r^m \sum_{j=0}^{[\frac{m}{2}]} \left[a_j \cos(m-2j)\varphi + b_j \sin(m-2j)\varphi\right]$$

and conversely. Let $u = r^{m+2}P(\varphi)$, then $P(\varphi)$ is a solution of the problem

$$\frac{d^2P}{d\varphi^2} + (m+2)^2 P = \sum_{j=0}^{[\frac{m}{2}]} \left[a_j \cos(m-2j)\varphi + b_j \sin(m-2j)\varphi\right],$$

$$\frac{dP}{d\varphi}\bigg|_{\varphi=0} = \frac{dP}{d\varphi}\bigg|_{\varphi=\vartheta} = 0\,.$$

Because $m+2 \neq \frac{\pi}{\vartheta}$,this problem has a unique solution of the for

$$\sum_{j=0}^{[\frac{m}{2}]} \left[a_j' \cos(m-2j)\varphi + b_j' \sin(m-2j)\varphi\right] + c\cos(m+2)\varphi + d\sin(m+2)\varphi ,$$

which it is a homogeneous polynomial.This ends the proof.

Now we shall consider the Neumann problem in $\mathcal{D}_\vartheta$:

$$\text{(D.23)} \quad -\Delta u = f, \quad \frac{\partial u}{\partial n}\Big|_{\Gamma_0 \cup \Gamma_\vartheta} = 0 .$$

After the Fourier transformation with respect to x_3 we obtain

$$\text{(D.24)} \quad -\Delta' \tilde{u} + \xi^2 \tilde{u} = \tilde{f}, \quad \frac{\partial \tilde{u}}{\partial n}\Big|_{\partial d_\vartheta} = 0,$$

where $\tilde{u}$, $\tilde{f}$ are the Fourier transformates of u and f, respectively.As the weak solution of the problem (D.24) we mean a solution of the following integral identity

$$\text{(D.25)} \quad \int_{d_\vartheta} \nabla'\tilde{u}\cdot\nabla'\eta \, dx' + \xi^2 \int_{d_\vartheta} \tilde{u}\,\eta \, dx' = \int_{d_\vartheta} \tilde{f}\,\eta \, dx' , \quad \forall \eta \in H^1(d_\vartheta),$$

where $\nabla' = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ .Let $\tilde{f} \in L_2(d_\vartheta)$,then from the Riesz theorem there exists a weak solution of (D.24) in $H^1(d_\vartheta)$ such that

$$\text{(D.26)} \quad |\xi|^2 \int_{d_\vartheta} \left(|\nabla'\tilde{u}|^2 + |\xi|^2 |\tilde{u}|^2\right) dx' \leq c \int_{d_\vartheta} |\tilde{f}|^2 dx' .$$

Moreover,we formulate the following result

Lemma D.4

Let $\mu \in [0,1)$ and $\int_{d_\vartheta} |\tilde{f}|^2 |x'|^{2\mu} dx' < \infty$ .Then there exists a weak solution of the problem (D.24) such that

$$\text{(D.27)} \; |\xi|^2 \int_{d_\vartheta} \left(|\nabla'\tilde{u}|^2 + |\xi|^2|\tilde{u}|^2\right)|x'|^{2\mu} dx' + |\xi|^{2-2\mu} \int_{d_\vartheta} \left(|\nabla'\tilde{u}|^2 + |\xi|^2|\tilde{u}|^2\right) dx' \leq C(\mu,\vartheta) \int_{d_\vartheta} |\tilde{f}|^2 |x'|^{2\mu} dx' .$$

Proof. We consider

$$\Big|\int_{d_\vartheta} \tilde{f}\eta \, dx'\Big|^2 \leq \int_{d_\vartheta} |\tilde{f}|^2 |x'|^{2\mu} dx' \int_{d_\vartheta} |\eta| |x'|^{-2\mu} dx' .$$

Using the interpolation inequality

$$(D.28) \quad \int_{d_\vartheta} |\tilde{u}|^2 |x'|^{-2\mu} dx' \leq h^{2-2\mu} \int_{d_\vartheta} |\nabla' \tilde{u}|^2 dx' + C_1 h^{-2\mu} \int_{d_\vartheta} |\tilde{u}|^2 dx' ,$$

which is valid for an arbitrary $\tilde{u} \in H^1(d_\vartheta)$, a parameter $h>0$ and $\mu<1$, in the particular case of $h=|\xi|^{-1}$ we have

$$(D.29) \quad |\xi|^{2-2\mu} \int_{d_\vartheta} |\eta|^2 |x'|^{-2\mu} dx' \leq \int_{d_\vartheta} (|\nabla'\eta|^2 + C_1 |\xi|^2 |\eta|^2) dx' .$$

Therefore, the Riesz theorem implies the existence of a weak solution of (D.24). Moreover, putting $\eta = \tilde{u}$ in (D.25) and using (D.29) we obtain

$$(D.30) \quad |\xi|^{2-2\mu} \int_{d_\vartheta} (|\nabla'\tilde{u}|^2 + |\xi|^2 |\tilde{u}|^2) dx' \leq C^2 \int_{d_\vartheta} |\tilde{f}|^2 |x'|^{2\mu} dx' .$$

Now we shall show that the integral $|\xi|^2 \int_{d_\vartheta} (|\nabla'\tilde{u}|^2 + |\xi|^2 |\tilde{u}|^2) |x'|^{2\mu} dx'$ converges and we shall estimate it. To do this we assume $\eta = \tilde{u}\, \Psi_\mu(x', \xi, R)$ in (D.25), where R is sufficiently large value $(R > |\xi|^{-1})$ and

$$\Psi_\mu(x',\xi,R) = \begin{cases} |\xi|^{-2\mu} & \text{for } |x'| \leq |\xi|^{-1}, \\ |x'|^{2\mu} & \text{for } |\xi|^{-1} \leq |x'| \leq R' = \min(R^{2\mu}, \max(|x'|^{2\mu}, |\xi|^{-2\mu})), \\ R^{2\mu} & \text{for } |x'| \geq R \end{cases}$$

Hence, we obtain

$$\text{(D.31)}\quad \int_{d_\vartheta} (|\nabla'\bar{u}|^2 + |\gamma|^2 |\bar{u}|^2)\varphi_\mu \, dx' = -\int_{d_\vartheta} \nabla'\bar{u}\nabla'\varphi_\mu \bar{\bar{u}} dx' + \int_{d_\vartheta} \tilde{f}\varphi_\mu \bar{\bar{u}}\, dx' .$$

To estimate the integrals in the right-hand side of (D.31) we use the following inequalities

$$\Big|\int_{d_\vartheta} \tilde{f}\bar{\bar{u}}\varphi_\mu dx'\Big| \le \Big(\int_{d_\vartheta} |\tilde{f}|^2\varphi_\mu dx'\Big)^{1/2}\Big(\int_{d_\vartheta} |\bar{u}|^2\varphi_\mu dx'\Big)^{1/2},$$

$$\Big|\int_{d_\vartheta} \nabla'\bar{u}\cdot\nabla'\varphi_\mu \bar{\bar{u}} dx'\Big| \le 2\mu \Big(\int_{|\gamma|^{-1}\le|x'|\le R} |\nabla'\bar{u}|^2|x'|^{2\mu} dx'\Big)^{1/2} \Big(\int_{|\gamma|^{-1}\le|x'|\le R} |\bar{u}|^2|x'|^{2\mu-2} dx'\Big)^{1/2} \le$$

$$\le 2\mu \Big(\int_{d_\vartheta} |\nabla\bar{u}|^2\varphi_\mu dx'\Big)^{1/2} \Big(|\gamma|^{2-2\mu}\int_{d_\vartheta} |\bar{u}|^2 dx'\Big)^{1/2} .$$

Hence, using (D.30) we get

$$|\gamma|^2\int_{d_\vartheta} (|\nabla'\bar{u}|^2 + |\gamma|^2|\bar{u}|^2)\varphi_\mu dx' \le 2\int_{d_\vartheta} |\tilde{f}|^2|x'|^{2\mu}dx' + 8\mu^2|\gamma|^{4-2\mu}\int_{d_\vartheta} |\bar{u}|^2 dx' \le c\int_{d_\vartheta} |\tilde{f}|^2|x'|^{2\mu}dx',$$

where the left-hand side is bounded for all R. Passing with R to infinity we conclude the proof.

Let us assume that

$$\chi_\mu = \begin{cases} |x'|^\mu & \text{for } |x'|\le R, \\ R^\mu & \text{for } |x'|\ge R, \end{cases} \quad \text{so } \chi_\mu \in H^1(d_\vartheta).$$

Putting in (D.25) $\eta = \bar{\bar{u}}\chi_{2(\mu+s}$, $\mu<1$, we obtain

$$\text{(D.32)}\quad \begin{aligned} \int_{d_\vartheta} (|\nabla'\bar{u}|^2 + |\gamma|^2|\bar{u}|^2)\chi_{2(\mu+s)} dx' &\le c\Big[|\gamma|^{-2}\int_{d_\vartheta} |\tilde{f}|^2\chi_{2(\mu+s)}dx' + \\ &+ \int_{d_\vartheta} |\bar{u}|^2\chi_{2(\mu+s-1)}dx'\Big], \end{aligned}$$

where s is an arbitrary natural number. Therefore, after the inductive considerations and passing with R to infinity we get

$$(D.33)\quad |\gamma|^2 \int_{d_\vartheta} (|\nabla'\tilde{u}|^2+|\gamma|^2|\tilde{u}|^2)|x'|^{2(\mu+s)}dx' \leq c\sum_{\sigma=0}^{s} |\gamma|^{2(\sigma-s)} \int_{d_\vartheta} |\tilde{f}|^2|x'|^{2(\mu+\sigma)}dx'$$

for $\gamma \neq 0$ .Now we shall formulate the following theorem

Theorem D.1

Let $\tilde{f} \in \mathcal{E}^k(d_\vartheta)$ has a compact support and

$$(D.34)\quad \frac{\pi}{\vartheta} > 1+k ,$$

then there exists a unique solution of (D.24) such, that

$$(D.35)\quad \sum_{s=0}^{k} |\gamma|^{2(k-s)} \|\tilde{u}-\tilde{v}_s-\sum_{j=0}^{s}\tilde{P}_j\|^2_{H_0^{s+2}(d_\vartheta;0)} \leq c\sum_{s=0}^{k} |\gamma|^{2s} \|\tilde{f}\|^2_{H^s(d_\vartheta)} ,$$

where $\tilde{v}_0 = \tilde{u}-\tilde{u}(0)$, $\tilde{v}_s \in H^{s+2}(d_\vartheta)$, $s \geq 1$. For $s = 1$, $\tilde{v}_1$ is a solution of the problem

$$\Delta'\tilde{v}_1+\tilde{f}-|\gamma|^2\tilde{u} = K(|x'|)(\tilde{f}-|\gamma|^2\tilde{u}) \in H_0^1(d_\vartheta;0),\quad \frac{\partial\tilde{v}_1}{\partial n}\Big|_{\partial d_\vartheta}=0,$$

$$(D.36)\quad \frac{\partial^i \tilde{v}_1}{\partial x'^\alpha}\Big|_{x'=0} = 0,\quad |\alpha| = i,\ i<2,$$

where $\tilde{u} \in H^1(d_\vartheta)$ is a weak solution of (D.24), such that (see Lemma D.2)

$$(D.37)\quad \|\Delta'\tilde{v}_1+\tilde{f}-|\gamma|^2\tilde{u}\|_{H_0^1(d_\vartheta;0)} \leq c\|\tilde{f}\|_{\mathcal{E}^1(d_\vartheta)},\quad \|\tilde{v}_1\|_{H^3(d_\vartheta)} \leq c\|\tilde{f}\|_{\mathcal{E}^1(d_\vartheta)}.$$

For $s \geq 2$, $\tilde{v}_s$ is a solution of the problem

$$(D.38)\quad D_{x'}^{s-1}(\Delta'\tilde{v}_s-|\gamma|^2\tilde{v}_{s-2}+\tilde{f}) = K(|x'|)\,D_{x'}^{s-1}(\tilde{f}-|\gamma|^2\tilde{v}_{s-2}) \in H_c^1(d_\vartheta;0),$$

$$\frac{\partial^i \tilde{v}_s}{\partial x'^\alpha}\Big|_{x'=0} = 0, \ |\alpha| = i < s+1, \ \frac{\partial \tilde{v}_s}{\partial n}\Big|_{\partial d_\vartheta} = 0.$$

From Lemma D.2 we know that $\tilde{v}_s$ satsfies the estimates

$$\|\Delta' \tilde{v}_s - |\xi|^2 \tilde{v}_{s-2} + \tilde{f}_{s-2}\|_{H_0^s(d_\vartheta;0)} \leq C\|\tilde{f}\|_{\mathcal{E}^s(d_\vartheta)}, \quad s \geq 2,$$

(D.39)

$$\|\tilde{v}_s\|_{H^{s+2}(d_\vartheta)} \leq C\|\tilde{f}\|_{\mathcal{E}^s(d_\vartheta)},$$

where $\tilde{f}_{s-2} = \tilde{f} - \sum_{j=0}^{s-2} \tilde{P}_f^j(x')$, $\tilde{P}_f^j(x') = \sum_{|\alpha|=j} \frac{1}{\alpha!} D^j_{x'^\alpha} f\big|_{x'=0} x'^\alpha$

and $\alpha = (\alpha_1, \alpha_2)$ is multiindex.The constants in (D.35), (D.37) and (D.39) do not depend on ξ .

Moreover, $\tilde{P}_0 = \tilde{u}(0)$, $\tilde{P}_1 = 0$ and $\tilde{P}_s$, $s \geq 2$, is a solution of the problem

(D.40) $$-\Delta' \tilde{P}_s = \tilde{P}_f^{s-2}(x') - |\xi|^2 \tilde{P}_{s-2}, \quad \frac{\partial \tilde{P}_s}{\partial n}\Big|_{\partial d_\vartheta} = 0.$$

From the estimate (D.35) and from properties of $H_0^s(d_\vartheta;0)$ we see that

(D.41) $$\tilde{P}_j = \frac{\partial^j \tilde{u}}{\partial x'^\alpha}\Big|_{x'=0} x'^\alpha$$ where $|\alpha| = j$, α is multiindex.

Proof.Using the existence of weak solutions we estimate the second derivatives of function $\tilde{u}$ .To do this we consider the following problem:

(D.42) $$-\Delta' \tilde{v} = \tilde{f} - |\xi|^2 \tilde{u} \equiv \tilde{g}, \quad \frac{\partial \tilde{v}}{\partial n}\Big|_{\partial d_\vartheta} = 0.$$

From (D.26) we see that $\tilde{g} \in L_2(d_\vartheta)$ .Therefore,from the Kondratiev theorem [Kon, 1] we have that $\tilde{v} \in H_0^2(d_\vartheta;0)$ and

$$(D.43)\qquad \|\tilde{v}\|_{H_0^2(d_\vartheta;0)} \leq c\|\tilde{g}\|_{L_2(d_\vartheta)} \leq c\|\tilde{f}\|_{L_2(d_\vartheta)} .$$

From the fact that $\tilde{f}$ has a compact support we have $\tilde{f} \in L_{2,\mu+1}(d_\vartheta;0)$, $0<\mu<1$. Therefore, from (D.32) we have $\tilde{g} \in L_{2,\mu+1}(d_\vartheta;0)$, and from the Kondratiev theorem it follows that $\tilde{u} \in H^2_{1+\mu}(d_\vartheta;0)$, where $\tilde{u}$ is a solution of the problem (D.42) for $\tilde{g} \in L_{2,\mu+1}(d_\vartheta;0)$. From the second Kondratiev theorem we have $\tilde{u} = a(\xi)+b(\xi)\ln r+\tilde{v}$ but $\nabla'\tilde{u} \in L_2(d_\vartheta)$ so $b=0$. Moreover, from the theorems of imbeddings and Lemma 2.3 we see that $\tilde{v}$ is continuous and $\tilde{v}(0)=0$, so $a=\tilde{u}(0)$ and from (D.43) we get

$$(D.44)\qquad \|\tilde{u}-\tilde{u}(0)\|_{H_0^2(d_\vartheta;0)} \leq c\|\tilde{f}\|_{L_2(d_\vartheta)} .$$

Let $\tilde{f} \in \Xi^1(d_\vartheta)$, then from (D.26) it follows that $|\xi|^2\tilde{u} \in H^1(d_\vartheta)$ and we have the estimate

$$(D.45)\qquad \|\xi^2\tilde{u}\|_{H^1(d_\vartheta)} \leq c\|\tilde{f}\|_{\Xi^1(d_\vartheta)} .$$

Therefore, $\tilde{g} \in H^1(d_\vartheta)$ and $\|\tilde{g}\|_{H^1(d_\vartheta)} \leq c\|\tilde{f}\|_{\Xi^1(d_\vartheta)}$. Hence, from Lemma D.2 there exists a function $\tilde{v}_1 \in H^3(d_\vartheta)$ such that

$$\tilde{h}_1 = \Delta'\tilde{v}_1 + \tilde{g} \in H^1_0(d_\vartheta;0) , \quad \frac{\partial^i \tilde{v}}{\partial x^i}\Big|_{x=0} = 0 , \quad i<2 , \quad \frac{\partial \tilde{v}}{\partial n}\Big|_{\partial d_\vartheta} = 0 ,$$

and we have the estimates

$$(D.46)\qquad \|\tilde{h}_1\|_{H_0^1(d_\vartheta;0)} \leq c\|\tilde{f}\|_{\Xi^1(d_\vartheta)} , \quad \|\tilde{v}_1\|_{H^3(d_\vartheta)} \leq c\|\tilde{f}\|_{\Xi^1(d_\vartheta)} .$$

Therefore, we shall consider the problem

$$(D.47)\qquad -\Delta'(\tilde{u}-\breve{v}_1)=\check{h}_1\,,\quad \frac{\partial(\tilde{u}-\breve{v}_1)}{\partial n}\Big|_{\partial d_\vartheta}=0\,.$$

Similarly as above, using the second Kondrat'ev theorem and knowing that $\frac{\partial\tilde{u}}{\partial x'}\big|_{x'=0}=0$, we obtain the existence of the function $\tilde{\omega}_1=\tilde{u}-\breve{v}_1-\tilde{u}(0)\in H^3_0(d_\vartheta;0)$ such that

$$(D.48)\qquad \|\tilde{\omega}_1\|_{H^3_0(d_\vartheta;0)}\leq c\,\|\tilde{f}\|_{\Xi^1(d_\vartheta)}\,.$$

Now we shall use the method of induction. Let $\tilde{f}\in\Xi^s(d_\vartheta)$, where $2\leq s\leq k$. The function $\tilde{u}$ on the s-step we shall denote by $\tilde{u}_s$. We shall seek it in the form

$$(D.49)\qquad \tilde{u}_s=\tilde{\omega}_s+\breve{v}_s+\sum_{j=2}^{s}\tilde{P}_j\,.$$

On the inductive assumption we have $\tilde{u}_{s-2}=\tilde{\omega}_{s-2}+\breve{v}_{s-2}+\sum_{j=2}^{s-2}\tilde{P}_j$, where

$$\|\tilde{\omega}_{s-2}\|_{H^{s-2}_0(d_\vartheta;0)}\leq c\,\|\tilde{f}\|_{\Xi^{s-2}(d_\vartheta)}\,,$$

$$\|\breve{v}_{s-2}\|_{H^{s-2}(d_\vartheta)}\leq c\,\|\tilde{f}\|_{\Xi^{s-2}(d_\vartheta)}\,.$$

Hence, we have

$$(D.50)\qquad \|\Upsilon^2\tilde{\omega}_{s-2}\|_{H^{s-2}_0(d_\vartheta;0)}\leq c\,\|\tilde{f}\|_{\Xi^s(d_\vartheta)}\,,$$

$$(D.51)\qquad \|\Upsilon^2\breve{v}_{s-2}\|_{H^{s-2}(d_\vartheta)}\leq c\,\|\tilde{f}\|_{\Xi^s(d_\vartheta)}\,,$$

and we assume that $\tilde{u}_s$ is a solution of the problem

$$(D.52)\quad -\Delta'\tilde{u}_s + \xi^2(\tilde{u}_{s-2} + \sum_{j=2}^{s-2} \tilde{p}_j) = \tilde{f}, \quad \frac{\partial \tilde{u}_s}{\partial n}\Big|_{\partial d_\vartheta} = 0 .$$

Using (D.49), instead of (D.52) we consider the following two problems

$$-\Delta'\tilde{\omega}_s = -\xi^2\tilde{\omega}_{s-2} - \xi^2\tilde{v}_{s-2} + \Delta'\tilde{v}_s + \tilde{f}_{s-2},$$

$$(D.53)\quad \frac{\partial \tilde{\omega}_s}{\partial n}\Big|_{\partial d_\vartheta} = 0,$$

and

$$(D.54)\quad -\Delta\tilde{P}_s = \tilde{P}_f^{s-2}(x') - \xi^2\tilde{P}_{s-2}, \quad \frac{\partial \tilde{P}_s}{\partial n}\Big|_{\partial d_\vartheta} = 0 .$$

From Lemma D.2 we know that there exists a function $\tilde{v}_s \in H^{s+2}(d_\vartheta)$ such that $\Delta'\tilde{v}_s - \xi^2\tilde{v}_{s-2} + \tilde{f}_{s-2} \in H_0^s(d_\vartheta;0)$ and

$$(D.55)\quad \|\Delta'\tilde{v}_s - \xi^2\tilde{v}_{s-2} + \tilde{f}_{s-2}\|_{H_0^s(d_\vartheta;0)} \le c\|\tilde{f}\|_{\Xi^s(d_\vartheta)}, \quad \|\tilde{v}_s\|_{H^{s-2}(d_\vartheta)} \le c\|\tilde{f}\|_{\Xi^s(d_\vartheta)}.$$

Moreover, from (D.50) and (D.51) we have

$$\tilde{h}_s = \Delta'\tilde{v}_s - \xi^2\tilde{v}_{s-2} + \tilde{f}_{s-2} - \xi^2\tilde{\omega}_{s-2} \in H_0^s(d_\vartheta;0)$$

and

$$(D.56)\quad \|\tilde{h}_s\|_{H_0^s(d_\vartheta;0)} \le c\|\tilde{f}\|_{\Xi^s(d_\vartheta)} .$$

Therefore, from the Kondratiev theorem [Kon, 1] it follows that there exists a solution of the problem (D.53) such that $\tilde{\omega}_s \in H_0^{s+2}(d_\vartheta;0)$ if the condition (D.34) is satisfied and the following estimate is valid

$$\text{(D.57)} \quad \|\breve{\omega}_s\|_{H_0^{s+2}(d_\vartheta;0)} \leq C\|\breve{h}_s\|_{H_0^s(d_\vartheta;0)} \leq C\|\tilde{f}\|_{\mathcal{E}^s(d_\vartheta)} .$$

Using Lemma D.3 we see that the problem (D.54) has a solution in the form of homogeneous polynomial of degree s. From the inductive considerations, for every s we have

$$\text{(D.58)} \quad u = \breve{\omega}_s + \breve{v}_s + \sum_{j=2}^{s} \tilde{P}_j$$

and

$$\text{(D.59)} \quad u = \breve{\omega}_{s-1} + \breve{v}_{s-1} + \sum_{j=2}^{s-1} \tilde{P}_j$$

Hence, we have the relation

$$\text{(D.60)} \quad \breve{\omega}_s + \breve{v}_s - \breve{v}_{s-1} + \breve{P}_s = \breve{\omega}_{s-1} .$$

We shall show that (D.60) is valid. We introduce the functions

$$\text{(D.61)} \quad \begin{aligned} \breve{\omega}_s' &= (1 - \zeta(|x'|))(\breve{v}_s - \breve{v}_{s-1} + \breve{P}_s) + \breve{\omega}_s , \\ \breve{\omega}_{s-1}' &= \breve{\omega}_{s-1} - \zeta(|x'|)(\breve{v}_s - \breve{v}_{s-1} + \breve{P}_s) , \end{aligned}$$

where $\zeta(\varrho) = 1$ for $\varrho < 1$, $\zeta(\varrho) = 0$ for $\varrho > 2$ and ζ is a smooth function. We shall show that functions (D.61) are solutions of the same problem, hence they are equal themselves, so (D.60) is valid. Functions (D.61) are solutions of the following problems:

$$-\Delta' \breve{\omega}_s' = -\Delta' \breve{\omega}_s - \Delta'(\breve{v}_s - \breve{v}_{s-1} + \breve{P}_s)(1 - \zeta(|x'|)) +$$

$$+2\nabla'\zeta\nabla'(\check{v}_s-\check{v}_{s-1}+\check{P}_s)+(\check{v}_s-\check{v}_{s-1}+\check{P}_s)\Delta'\zeta(|x'|)\,, \tag{D.62}$$

$$\frac{\partial\check{\omega}'_s}{\partial n}\Big|_{\partial d_\vartheta}=0\,,$$

and

$$-\Delta'\check{\omega}'_{s-1}=-\Delta'\check{\omega}_{s-1}+\Delta'(\check{v}_s-\check{v}_{s-1}+\check{P}_s)\zeta(|x'|)+$$

$$+2\nabla'\zeta\cdot\nabla'(\check{v}_s-\check{v}_{s-1}+\check{P}_s)+(\check{v}_s-\check{v}_{s-1}+\check{P}_s)\Delta'\zeta(|x'|)\,, \tag{D.63}$$

$$\frac{\partial\check{\omega}'_{s-1}}{\partial n}\Big|_{\partial d_\vartheta}=0\,.$$

The right-hand sides of (D.62) and (D.63) are equal one to each other if $\Delta'\check{\omega}_s+\Delta'(\check{v}_s-\check{v}_{s-1}+\check{P}_s)=\Delta'\check{\omega}_{s-1}$,
what is true if

$$\check{f}_{s-2}+\check{P}_f^{\,s-2}=\check{f}_{s-3} \tag{D.64}$$

and

$$\check{\omega}_{s-2}+\check{v}_{s-2}=\check{\omega}_{s-3}+\check{v}_{s-3}-\check{P}_{s-2}\,. \tag{D.65}$$

The equality (D.64) follows from the definition of $\check{f}_s$,but (D.65) from the inductive assumptions.From (D.64) and (D.65) the right-hand sides of (D.62) and (D.63) belong to $H_0^s\cap H_0^{s-1}$ so functions $\check{\omega}'_s$, $\check{\omega}'_{s-1}$ are solutions of the same problem. This ends the proof.

Lemma D.5

Let $\frac{I}{\vartheta}>k+1$, $f\in H^k(\mathcal{D}_\vartheta)$ and has a compact support. Then there exists a solution of (D.23) such that

$$\sum_{s=0}^{k}\int_{-\infty}^{\infty}\|\nabla_2^{k-s}(u-v_s-\sum_{j=0}^{s}P_j)\|^2_{H_0^{s+2}(d_\vartheta;0)}+\|\nabla_2^{k+2}u\|^2_{L_2(\mathcal{D}_\vartheta)}+ \tag{D.66}$$

$$+\int_{\infty}^{\infty} dz \, \|\nabla_z^{k+1} u\|^2_{H^1(d_\vartheta)} \le c \|f\|^2_{H^k(\mathcal{D}_\vartheta)} .$$

Proof. The proof follows from Theorem D.1, (D.26) and the Parseval identity.

Lemma D.6

Let $f \in H^k(\mathcal{D}_\vartheta)$ and has a compact support. Then for an arbitrary solution of the problem (D.22) we have the estimate

$$\text{(D.67)} \quad \|u\|_{H^{k+2}(\mathcal{D}_\vartheta \cap K_R(\Gamma))} \le C(R) \|f\|_{H^k(\mathcal{D}_\vartheta \cap K_R(\Gamma))} + C_1(R) \|u\|_{L_2(\mathcal{D}_\vartheta) \cap K_R(\Gamma))},$$

where $K_R(\Gamma) = \{x \in \mathbb{R}^3;\ \varrho(x,\Gamma) \le R\}$, $C(R)$, $C_1(R)$ are functions of R such that $C(R) \to$ const for $R \to 0$ and $C_1(R) \to \infty$ for $R \to 0$.

Proof. Let us consider the following expression

$$\|u\|^2_{H^{k+2}(\mathcal{D}_\vartheta \cap K_R(\Gamma))} = \|\nabla_z^{k+2} u\|^2_{L_2(\mathcal{D}_\vartheta \cap K_R(\Gamma))} + \int_{-\infty}^{\infty} dz \, \|\nabla_z^{k+1} u\|^2_{H^1(d_\vartheta \cap K_R(0))} +$$

$$+ \sum_{i=0}^{k} \int_{-\infty}^{\infty} dz \, \|\nabla_z^{k-i} u\|^2_{H^{i+2}(d_\vartheta \cap K_R(0))} ,$$

where the last term we estimate as:

$$\sum_{i=0}^{k} \int_{-\infty}^{\infty} dz \, \|\nabla_z^{k-i}(u - v_i - \sum_{j=0}^{i} p_j) + \nabla_z^{k-i}(v_i + \sum_{j=0}^{i} p_j)\|^2_{H^{i+2}(d_\vartheta \cap K_R(0))}$$

$$\le \sum_{i=0}^{k} \int_{-\infty}^{\infty} dz \Big(\|\nabla_z^{k-i}(u - v_i - \sum_{j=0}^{i} p_j)\|^2_{H^{i+2}(d_\vartheta \cap K_R(0))} + \|\nabla_z^{k-i}(v_i + \sum_{j=0}^{i} p_j)\|_{H^{i+2}(d_\vartheta \cap K_R(0))} \Big)$$

$$\le C_1'(R) \|f\|^2_{H^{k+2}(\mathcal{D}_\vartheta)} + C_2' \sum_{i=0}^{k} \sum_{j=0}^{i} \int_{-\infty}^{\infty} dz \, \|\nabla_z^{k-i} p_j\|^2_{H^{i+2}(d_\vartheta \cap K_R(0))}$$

$$\le C_3' \|f\|^2_{H^{k+2}(\mathcal{D}_\vartheta)} + C_4' \sum_{i=0}^{k} \sum_{j=0}^{i} \operatorname*{ess\,sup}_{x' \in d_\vartheta \cap K_R(0)} \int_{-\infty}^{\infty} dz \, |\nabla_z^{k-i} \nabla_{x'}^{j} u|^2 .$$

From the theorems of imbeddings, the last term in the above expression we estimate as:

$$\Big(\sum_{i=0}^{k}\sum_{j=0}^{i}\operatorname*{ess\,sup}_{x'\in d_\vartheta\cap K_R(0)}\int_{-\infty}^{\infty}dz\,|\nabla_z^{k-i}\nabla_{x'}^{j}u|^2\Big)^{1/2}\leq \varepsilon\|u\|_{H^{k+2}(\mathcal{D}_\vartheta\cap K_R(\Gamma))}+$$

$$+C_1(R)\,\|u\|_{L_2(\mathcal{D}_\vartheta\cap K_R(\Gamma))}\,.$$

This ends the proof.

Using the above results we shall prove the existence of solutions of the Neumann problem (C.19), (C.20) in domain Ω with edges.

Theorem D.2

Let $f\in H^k(\Omega)$, $h\in H^{k-1/2}(\partial\Omega)$, Ω is a domain with edges with smooth parts of the boundary of class C^{k+2} and

$$\frac{\pi}{\alpha_0}>k+1\,, \tag{D.68}$$

where α_0 is the maximal angle between two surfaces.

Then there exists a solution $u\in H^{k+2}(\Omega)$ of the Neumann problem (C.19), (C.20) such that

$$\|u\|_{H^{k+2}(\Omega)}\leq C(\|f\|_{H^k(\Omega)}+\|h\|_{H^{k+1/2}(\partial\Omega)})\,. \tag{D.69}$$

Proof. We shall use the existence of weak solution (C.21) and the estimate (C.22). In the neighbourhood of edges we obtain the problem (C.25), (C.26). From results of Lemma D.6 and Theorem B.1 we obtain

$$\|\hat v_\lambda\|_{2,2,\mathcal{D}_{\vartheta(\xi)}}\leq C_1\|\hat g\|_{2,\mathcal{D}_{\vartheta(\xi)}}+C_2\|\hat d\|_{1/2,2,\partial\mathcal{D}_{\vartheta(\xi)}}+\lambda C_3\|\hat v_\lambda\|_{2,2,\mathcal{D}_{\vartheta(\xi)}}. \tag{D.70}$$

Assuming that λ is sufficiently small and using the result of Theorem B.1,we get

$$\|u\|_{2,2,\Omega_{\lambda/2}} \leq C_4\|f\|_{2,\Omega_\lambda} + C_5\|u\|_{1,2,\Omega_\lambda} + C_6\|h\|_{1/2,2,\partial\Omega_\lambda} . \tag{D.71}$$

We sum (D.71) over all neighbourhood of the edges,the inequality (C.24) over all remaining neighbourhoods,and we use the estimate (C.21) for the weak solution.Then we obtain the required estimate

$$\|u\|_{2,2,\Omega} \leq C(\|f\|_{2,\Omega} + \|h\|_{1/2,2,\partial\Omega}) \tag{D.72}$$

Now we shall prove that the weak solution belongs to $H^2(\Omega)$. In order to do this we have to estimate functions $\hat{g}$ and $\hat{d}$:

$$\|\hat{g}\|_{2,\mathcal{D}_{\vartheta(\gamma)}} \leq C_7\|\hat{f}\|_{2,\mathcal{D}_{\vartheta(\gamma)}} + C_8\|\hat{v}_\lambda\|_{1,2,\mathcal{D}_{\vartheta(\gamma)}} , \tag{D.73}$$

and

$$\|\hat{d}\|_{1/2,2,\partial\mathcal{D}_{\vartheta(\gamma)}} \leq C_9\|\hat{h}\|_{1/2,2,\partial\mathcal{D}_{\vartheta(\gamma)}} + C_{10}\|\hat{v}_\lambda\|_{1,2,\mathcal{D}_{\vartheta(\gamma)}} . \tag{D.74}$$

Therefore,using in (C.25), (C.26) the method of successive approximations for small λ ,we prove the existence $\hat{v}_\lambda \in H^2(\mathcal{D}_{\vartheta(\gamma)})$, hence $u \in H^2(\Omega)$.

Now we apply the method of induction to show that $u \in H^{k+2}(\Omega)$. From (C.25),(C.26) ,using the Lemma D.6 and Theorem B.1,we obtain

$$\|\hat{v}_\lambda\|_{s+2,2,\mathcal{D}_{\vartheta(\gamma)}} \leq C_1'\|\hat{g}\|_{s,2,\mathcal{D}_{\vartheta(\gamma)}} + C_2'\|\hat{d}\|_{s+1/2,2,\partial\mathcal{D}_{\vartheta(\gamma)}} + C_3'\lambda\|\hat{v}_\lambda\|_{s+2,2,\mathcal{D}_{\vartheta(\gamma)}} ,$$

where s = 0,...,k, hence, we have

$$\|u\|_{s+2,2,\Omega_{\lambda/2}} \le C_4'\|f\|_{s,2,\Omega_\lambda} + C_5'\|h\|_{s+1/2,2,\partial\Omega_\lambda} + C_6'\|u\|_{s+1,2,\Omega_\lambda}.$$

For $u \in H^{s+1}(\Omega)$, s=0,1,...,k, we obtain the required estimate, and using the method of successive approximations we prove the existence of solutions. This ends the proof.

After the Fourier transformation with respect to x_3, instead of (C.12) we have

$$\text{(D.75)} \qquad -\Delta'\breve{e}_3 + \xi^2\breve{e}_3 = \tilde{\omega}_3 \;, \; \breve{e}_3|_{\gamma_0} = \breve{e}_3|_{\gamma_\vartheta} = 0 \;,$$

and instead of (C.13) we get

$$-\Delta'\breve{e}_i + \xi^2\breve{e}_i = \tilde{\omega}_i \;, \; i = 1,2,$$

$$\text{(D.76)} \qquad \breve{e}_1|_{\gamma_0} = 0, \; \breve{e}_1\cos\vartheta + \breve{e}_2\sin\vartheta|_{\gamma_\vartheta} = 0,$$

$$\frac{\partial\breve{e}_2}{\partial\varphi}\Big|_{\gamma_0} = 0; \; \frac{\partial\breve{e}_1}{\partial\varphi}\sin\vartheta - \frac{\partial\breve{e}_2}{\partial\varphi}\cos\vartheta\Big|_{\gamma_\vartheta} = 0 \;.$$

Similarly as in the case of the Neumann problem we can prove the following theorem:

Theorem D.3

Let $\tilde{\omega}_i \in \xi^k(d_\vartheta)$, i=1,2,3, have compact supports, and

(D.77) $\frac{\pi}{\vartheta} > 1+k$ for the problem (D.75),

(D.78) $\frac{\pi}{\vartheta} > 2+k$ for the problem (D.76)

(see Lemma 2.5) .

Then there exists a unique solution of the problem (D.75) and a unique solution of the problem (D.76) such that

$$\sum_{s=0}^{k} \sum_{\sigma} |\xi|^{2(k-s)} \| \tilde{e}_\sigma - \tilde{e}'_{\sigma,s} - \sum_{j=0}^{s} \tilde{Q}_j^\sigma \|^2_{H_0^{s+2}(d_\vartheta;0)} \tag{D.79}$$
$$\leq c \sum_{s=0}^{k} \sum_{\sigma} |\xi|^{2s} \| \tilde{\omega}_\sigma \|^2_{H^s(d_\vartheta)} ,$$

where $\sigma = \{1,2\}$ or $\{3\}$,and $\tilde{e}'_{\sigma,s} \in H^{s+2}(d_\vartheta)$ is calculated in such a way that

$$D_{x'}^{s-1}(\Delta' \tilde{e}'_{\sigma,s} - \xi^2 \tilde{e}'_{\sigma,s-2} + \tilde{\omega}_{\sigma,s-2}) \in H_0^1(d_\vartheta;0),$$

$$\frac{\partial^i \tilde{e}_{\sigma,s}}{\partial x'^i}\Big|_{x'=0} = 0 \quad , \quad i < s+1,$$

and it satisfies the boundary conditions (D.75) and (D.76), respectively.Moreover,we have

$$\sum_{\sigma} \| \Delta' \tilde{e}'_{\sigma,s} - \xi^2 \tilde{e}'_{\sigma,s-2} + \tilde{\omega}_{\sigma,s-2} \|^2_{H_0^s(d_\vartheta;0)} \leq c \sum_{\sigma} \| \tilde{\omega}_\sigma \|^2_{\Sigma^s(d_\vartheta)} , \tag{D.80}$$

where $\tilde{\omega}_{\sigma,s} = \tilde{\omega}_\sigma - \sum_{j=0}^{s} \tilde{P}^j_{\omega_\sigma}(x')$ .A solution of the problem

$$-\Delta' \tilde{Q}_s^3 = \tilde{P}^{s-2}_{\omega_3}(x') - \xi^2 \tilde{Q}^3_{s-2} , \quad \tilde{Q}^3_s|_{\partial d_\vartheta} = 0, \tag{D.81}$$

we denote by $\tilde{Q}_s^3$,and a solution of the problem

$$-\Delta' \tilde{Q}_s^i = \tilde{P}^{s-2}_{\omega_i}(x') - \xi^2 \tilde{Q}^i_{s-2} , \quad i = 1,2,$$

$$\tilde{Q}^1_s|_{\gamma_0} = 0, \quad \tilde{Q}^1_s \cos\vartheta + \tilde{Q}^2_s \sin\vartheta|_{\gamma_\vartheta} = 0, \tag{D.82}$$

$$\frac{\partial \tilde{Q}^2_s}{\partial \varphi}\Big|_{\gamma_0} = 0, \quad \frac{\partial \tilde{Q}^1_s}{\partial \varphi}\sin\vartheta - \frac{\partial \tilde{Q}^2_s}{\partial \varphi}\cos\vartheta\Big|_{\gamma_\vartheta} = 0,$$

we denote by $\tilde{Q}_s^{\{1,2\}} := (\tilde{Q}_s^1, \tilde{Q}_s^2)$, where $\tilde{Q}_0^\sigma = \tilde{Q}_1^\sigma = 0$.
From the properties of the space $H_0^s(d_\vartheta;0)$ we have

$$\text{(D.83)}\qquad \tilde{Q}_j^\sigma = \frac{\partial^j \tilde{e}^\sigma}{\partial x'^\alpha}\bigg|_{x'=0} x'^\alpha, \ |\alpha| = j, \ \alpha = (\alpha_1, \alpha_2).$$

Similarly, as in Lemma D.6 we can prove the following lemma:

Lemma D.7

Let $\omega_i \in H^k(\mathcal{D}_\vartheta)$, $i = 1,2,3$, have compact supports. Then for arbitrary solutions of the problems (C.12), (C.13) we have the estimate

$$\text{(D.84)}\qquad \begin{aligned} \sum_\sigma \|e_\sigma\|_{H^{k+2}(\mathcal{D}_\vartheta \cap K_R(\Gamma))} &\le C(R) \sum_\sigma \|\omega_\sigma\|_{H^k(\mathcal{D}_\vartheta \cap K_R(\Gamma))} + \\ &+ C_1(R) \sum_\sigma \|e_\sigma\|_{L_2(\mathcal{D}_\vartheta \cap K_R(\Gamma))}, \end{aligned}$$

where the same notations as in Lemma D.6 are used.
In the same way as it was done for Theorem C.2, we can prove the following theorem:

Theorem D.4

Let $\omega \in H^k(\Omega)$, Ω is a bounded domain with edges and smooth parts of the boundary of class C^{k+2}. Then for

$$\text{(D.85)}\qquad \frac{\pi}{\alpha_0} > k+2,$$

there exists a solution $e \in H^{k+1}(\Omega)$ of the problem (C.3) such that

$$\text{(D.86)}\qquad \|e\|_{k+1,2,\Omega} \le C\|\omega\|_{k,2,\Omega},$$

where C is a constant.

E. Solvability of the Dirichlet and Neumann problem in twodimensional domain with corners

At the begining we shall consider the Dirichlet and Neumann problems in the angle d_ϑ

$$(E.1)\qquad \Delta u = f\,,\quad u|_{\partial d_\vartheta} = h \quad \text{or} \quad \frac{\partial u}{\partial n}\Big|_{\partial d_\vartheta} = g\,,$$

where $\partial d_\vartheta = \gamma_c \cup \gamma_\vartheta$, $f \in W_p^l(d_\vartheta)$, $h \in W_p^{l+2-1/p}(\partial d_\vartheta)$, $g \in W_p^{l+1-1/p}(\partial d_\vartheta)$.
We assume that f , g , h have compact supports. Using Theorem B.1 we see that there exists functions $\tilde{h} \in W_p^{l+2}(d_\vartheta)$ and $\breve{g} \in W_p^{l+2}(d_\vartheta)$ such that $\tilde{h}|_{\partial d_\vartheta} = h$, $\frac{\partial \breve{g}}{\partial n}\big|_{\partial d_\vartheta} = g$ and

$$(E.2)\qquad \|\tilde{h}\|_{l+2,p,d_\vartheta} \le c\,\|h\|_{l+2-1/p,p,\partial d_\vartheta}\,,$$

$$(E.3)\qquad \|\breve{g}\|_{l+2,p,d_\vartheta} \le c\,\|g\|_{l+1-1/p,p,\partial d_\vartheta}\,.$$

Therefore, instead of the problem (E.1) we have

$$(E.4)\qquad \Delta v = f - \Delta\tilde{h} \equiv \tilde{f}\,,\quad v|_{\partial d_\vartheta} = 0\,,\quad \text{where} \quad v = u - \tilde{h}\,,$$

and

$$(E.5)\qquad \Delta v = f - \Delta\breve{g} \equiv \breve{f}\,,\quad \frac{\partial v}{\partial n}\Big|_{\partial d_\vartheta} = 0\,,\quad \text{where} \quad v = u - \breve{g}\,.$$

Now instead of (E.4) we shall consider

$$(E.6)\qquad \Delta(v - v_0) = \tilde{f} - P_{\tilde{f}}^{l-1}(x)\,,\quad (v - v_0)|_{\partial d_\vartheta} = 0\,,$$

where according to Lemma 2.3 $\tilde{f} - P_{\tilde{f}}^{l-1}(x) \in V_{p,0}^l(d_\vartheta, 0)$, and

$$(E.7) \qquad \Delta v_0 = P_f^{l-1} \quad , \quad v_0\big|_{\partial d_\vartheta} = 0 .$$

Let us explain why we have to reduce the nonhomogeneous problem (E.1), using Theorem E.1, to the homogeneous problem (E.4), (E.5). Let (E.4) be considered as the nonhomogeneous problem, then from Lemma 2.3 we calculate $P_h^l(x)$ because $h \in W_p^{l+2-1/p}(\mathbb{R}^1)$. Therefore, we can not construct the function v_0 as a solution of (E.7) because v_0 is a polynomial of degree l+1 but the boundary condition gives a polynomial of degree l.

Using the results of [M, 1], the problem (E.6) has a unique solution $v - v_0 \in V_{p,0}^{l+2}(d_\vartheta; 0)$, p> 2, and the estimate

$$(E.8) \qquad \|v - v_0\|_{V_{p,0}^{l+2}(d_\vartheta;0)} \leq c\, \|f - P_f^{l-1}\|_{V_{p,0}^{l}(d_\vartheta;0)} ,$$

is valid if the following condition is satisfied

$$(E.9) \qquad \frac{\pi}{\vartheta} > l+2-\frac{2}{p} .$$

From the definition of the polynomial P_f^{l-1} we see that in the polar coordinates r , φ it has the form

$$(E.10) \qquad P_f^{l-1}(x) = \sum_{k=0}^{l-1} a_k(\varphi)\, r^k .$$

Therefore, we seek solutions of (E.7) in the form

$$(E.11) \qquad v_0 = \sum_{s=0}^{l+1} d_s(\varphi)\, r^s .$$

Using (E.11) and (E.10) in (E.7) we obtain the following system of differential equations

$$\ddot{\alpha}_{s+2} + (s+2)^2 \alpha_{s+2} = a_s \,, \quad \alpha_0 = \alpha_1 = 0 \,, \tag{E.12}$$

$$\alpha_{s+2}\big|_{\varphi=0} = \alpha_{s+2}\big|_{\varphi=\vartheta} = 0 \,, \tag{E.13}$$

where s=0,1,...,l-1.In the case of the Neumann problem,instead of (E.13) we have

$$\dot{\alpha}_{s+2}\big|_{\varphi=0} = \dot{\alpha}_{s+2}\big|_{\varphi=\vartheta} = 0 . \tag{E.14}$$

Solving (E.12) we obtain a solution in the form

$$\alpha_k = C_1^k \sin k\varphi + C_2^k \cos k\varphi - \frac{\cos k\varphi}{k}\int_0^\varphi a_{k-2} \sin kx\, dx + \frac{\sin k\varphi}{k}\int_0^\varphi a_{k-2}\cos kx\, dx . \tag{E.15}$$

From the boundary condition (E.13) we get $C_2^k = 0$, and C_1^k we calculate from

$$C_1^k \sin k\vartheta - \frac{\cos k\vartheta}{k}\int_0^\vartheta a_{k-2}(x)\sin kx\, dx + \frac{\sin k\vartheta}{k}\int_0^\vartheta a_{k-2}(x)\cos kx\, dx = 0 . \tag{E.16}$$

From (E.9) we see that $\frac{\pi}{\vartheta} > l+1$,but k=2,...,l+1,so the constants C_1^k can be always calculated from (E.16).For the Neumann problem $C_1^k = 0$, and C_2^k we calculate from

$$-C_2^k k \sin k\vartheta + \sin k\vartheta \int_0^\vartheta a_{k-2}(x)\sin kx\, dx + \cos k\vartheta \int_0^\vartheta a_{k-2}(x)\cos kx\, dx = 0 \tag{E.17}$$

From (E.11) and (E.15) for p> 2 we have

$$\|v_c\|_{l+2,p,d_\vartheta\cap K_R(0)} \le C \sum_{i=0}^{l-1} \max_{x\in d_\vartheta\cap K_R(0)} |D^i \tilde{f}| \le C \|\tilde{f}\|_{l,p,d_\vartheta\cap K_R(0)} \tag{E.18}$$

$$\leq C(\|f\|_{l,p,d_\vartheta\cap K_R(0)}+\|\Delta\tilde{h}\|_{l,p,d_\vartheta\cap K_R(0)})\leq C(\|f\|_{l,p,d_\vartheta\cap K_R(0)}+\|h\|_{l+2-1/p,p,\partial d_\vartheta\cap K_R(0)}),$$

where $K_R(0)$ is a ball with radius $R<\infty$ with center in the origin. In the case of the Neumann problem, instead of (E.18) we have

$$\|v_0\|_{l+2,p,d_\vartheta\cap K_R(0)}\leq C(\|f\|_{l,p,d_\vartheta\cap K_R(0)}+\|g\|_{l+1-1/p,p,\partial d_\vartheta\cap K_R(0)}). \tag{E.19}$$

Summarizing all above results we obtain the estimate

$$\begin{aligned}\|u\|_{l+2,p,d_\vartheta\cap K_R(0)}&=\|v-v_0+v_0+\tilde{h}\|_{l+2,p,d_\vartheta\cap K_R(0)}\leq\|v-v_0\|_{l+2,p,d_\vartheta\cap K_R(0)}+\\&+\|v_0\|_{l+2,p,d_\vartheta\cap K_R(0)}+\|\tilde{h}\|_{l+2,p,d_\vartheta\cap K_R(0)}\leq C(\|v-v_0\|_{V^{l+2}_{p,0}(d_\vartheta\cap K_R(0),0)}+\\&+\|f\|_{l,p,d_\vartheta\cap K_R(0)}+\|h\|_{l+2-1/p,p,\partial d_\vartheta\cap K_R(0)})\leq C(\|\tilde{f}-P^{l-1}_{\tilde{f}}\|_{V^{l}_{p,0}(d_\vartheta\cap K_R(0),0)}+\\&+\|f\|_{l,p,d_\vartheta\cap K_R(0)}+\|h\|_{l+2-1/p,p,\partial d_\vartheta\cap K_R(0)})\leq C(\|f\|_{l,p,d_\vartheta\cap K_R(0)}+\|h\|_{l+2-1/p,p,\partial d_\vartheta\cap K_R(0)}).\end{aligned} \tag{E.20}$$

Similarly we can obtain estimate for the Neumann problem. From the above results we can formulate the following theorem:

Theorem E.1

Let $f\in W^l_p(d_\vartheta)$, $h\in W^{l+2-1/p}_p(\partial d_\vartheta)$, $g\in W^{l+1-1/p}_p(\partial d_\vartheta)$ have compact supports, $p>2$ and $\pi/\vartheta>l+2-2/p$. Then there exist solutions of the Dirichlet and Neumann problems (E.1) and the following estimates are valid

$$\|u\|_{l+2,p,d_\vartheta\cap K_R(0)}\leq C(\|f\|_{l,p,d_\vartheta}+\|h\|_{l+2-1/p,p,\partial d_\vartheta}), \tag{E.21}$$

and

$$\|u\|_{l+2,p,d_\vartheta\cap K_R(0)}\leq C(\|f\|_{l,p,d_\vartheta}+\|g\|_{l+1-1/p,p,\partial d_\vartheta}), \tag{E.22}$$

where supports of f , g , h belong to $d_\vartheta \cap K_R(0)$.

Now we shall consider the problem (E.1) in a bounded domain $\Omega \subset \mathbb{R}^2$ with corners, hence we formulate the result:

Theorem E.2

Let $f \in W_p^l(\Omega)$, $h \in W_p^{l+2-1/p}(\partial\Omega)$, $g \in W_p^{l+1-1/p}(\partial\Omega)$, Ω $\mathbb{R}^2$ is a domain with corners and

$$\frac{\pi}{\alpha_0} > l+2-2/p , \tag{E.23}$$

where α_0 is the maximal angle. Let the smooth parts of the boundary be of class C^{l+2}. In the case of the Neumann problem let the following compatibility condition be satisfied

$$\int_\Omega f\,dx = \int_{\partial\Omega} g(s)\,ds . \tag{E.24}$$

Then there exist solutions of the problems (E.1) such that $u \in W_p^{l+2}(\Omega)$ and the following estimates are valid

$$\|u\|_{l+2,p,\Omega} \leqslant c(\|f\|_{l,p,\Omega} + \|h\|_{l+2-1/p,p,\partial\Omega}) , \tag{E.25}$$

$$\|u\|_{l+2,p,\Omega} \leqslant c(\|f\|_{l,p,\Omega} + \|g\|_{l+1-1/p,p,\partial\Omega}) . \tag{E.26}$$

Proof. The proof follows from Theorem E.1 and the proof of Theorem C.1.

F. Solvability of the mixed problem for a hyperbolic equation

In this section we shall consider the problem of the existence and uniqueness of solutions of the problem (F):

(F.1) $$Q^2 g - \operatorname{div}(h(g)\nabla g) = F,$$

(F)

(F.2) $$g|_{t=0} = g_0, \; g_t|_{t=0} = g_1, \; \frac{\partial g}{\partial n}\Big|_{\partial\Omega} = G,$$

where $Q = \partial_t + v\cdot\nabla$ and $v\cdot\bar{n}|_{\partial\Omega} = 0$. From (7.9) and the form of the function $h(g)$ we have

(F.3) $$h(g) \geqslant m^2 > 0,$$

where m is a constant. Moreover, the compatibility condition (7.7) is satisfied. The equation (F.1) is a hyperbolic one because its characteristic polynomial has the form

(F.4) $$\lambda_0^2 + 2v(x,t)\cdot\lambda\,\lambda_0 - \left[h(g(x,t))\lambda^2 - (v(x,t)\cdot\lambda)^2\right] = 0,$$

where $\lambda_0 = \varphi_t, \lambda_k = \varphi_{x^k}$, $k=1,\dots,n$, and $\varphi\,(x,t) = \mathrm{const}$ is a characteristic surface. The equation (F.1) is strictly hyperbolic one because from (F.4) we have two different real roots

(F.5) $$\lambda_0^{\mp} = -v\cdot\lambda \mp \sqrt{h\lambda^2},$$

and from (F.3) we have

(F.6) $$\inf_{x,t\in\Omega^T, |\lambda|=1} |\lambda_0^+ - \lambda_0^-| = \inf_{x,t\in\Omega^T} 2\sqrt{h} > 2m.$$

Now we shall consider the linearized problem (F), where $h=h(x)$. The following lemma is proved in [Ve, 1]:

Lemma F.1

Let us assume that

(1) $\partial\Omega \in C^5$, $v\cdot\bar{n}|_{\partial\Omega}=0$,

(2) $v \in \Pi^3_{1,\infty}(\Omega^T)$, $h \in \Pi^3_{1,\infty}(\Omega^T)$, $F \in \Pi^2_{1,2}(\Omega^T)$,

(3) $g_0 \in H^3(\Omega)$, $g_1 \in H^2(\Omega)$,

(4) $G \in \Pi^{5/2}_{0,2}(\partial\Omega^T)$.

Then there exist a unique solution of the linearized problem (F) such, that $g \in \Pi^3_{0,\infty}(\Omega^T)$ and the following estimate is valid

(F.7) $$|g|^2_{3,0,\infty,\Omega^T} \le C\alpha e^{C\alpha^2 T}(\|g_0\|^2_{3,2,\Omega}+\|g_1\|^2_{2,2,\Omega}+|F|^2_{2,1,\Omega^T}+|G|^2_{5/2,0,\partial\Omega^T}),$$

where

(F.8) $$\alpha = |v|^4_{3,1,\infty,\Omega^T}+|h|^2_{3,1,\infty,\Omega^T}+1.$$

Similarly as Lemma F.1 the following result can be proved:

Lemma F.2

Let $\partial\Omega \in C^5$, $v\cdot\bar{n}|_{\partial\Omega}=0$, $v \in \Pi^3_{1,\infty}(\Omega^T)$, $h \in \Pi^3_{1,\infty}(\Omega^T)$ $F \in \Pi^1_{1,2}(\Omega^T)$, $g_0 \in H^2(\Omega)$, $g_1 \in H^1(\Omega)$, $G \in \Pi^{3/2}_{0,2}(\partial\Omega^T)$, then for an arbitrary soluti- on of the problem (F) the following estimate is valid

(F.9) $$|g|^2_{2,0,\infty,\Omega^T} \le C\alpha e^{C\alpha^2 T}(\|g_0\|^2_{2,2,\Omega}+\|g_1\|^2_{1,2,\Omega}+|F|^2_{1,1,\Omega^T}+|G|^2_{3/2,0,\partial\Omega^T}),$$

where α is determined by (F.8).

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