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# ESSAYS ON STABILITY ANALYSIS AND MODEL REDUCTION

Umberto Viaro

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# Chapter 8

# Components of the forced response

In most control textbooks, as well as in many textbooks of electrical and electronics engineering, the notions of steady–state and transient response are introduced at an intuitive level without formal definitions. Often, the terms “steady state” and “transient” are not even listed in the index. This is probably due to the fact that there appears to be no need for precise definitions in the special cases of dc or ac excitations since the corresponding outputs of a stable linear time–invariant continuous–time system tend either to a constant or to a sinusoidal function of the same frequency as the input with different amplitude and phase. In the control literature, the limit response to unbounded inputs, such as polynomials, is also referred to as “steady–state” response (see, e.g., [1]) even if the term “asymptotic” seems to be more appropriate for unbounded inputs.

This chapter explores the concepts of steady–state, asymptotic and transient response for linear time–invariant systems, and suggests a precise definitions for these notions along the lines of [2] $\div$ [5]. To this purpose, the forced response is decomposed into three components, named “input”, “system” and “interaction” component [6] because they retain some characteristics that are proper, respectively, to the input, to the system, and to both. The definitions refer essentially to SISO systems described by a strictly–proper rational transfer function and to inputs whose Laplace transform is a proper rational function: it follows that such inputs are formed by linear combinations of exponentials, possibly multiplied by polynomials in $t$. Some extensions to the cases where one of the two functions is nonrational are briefly discussed in section 8.4. The linear system is assumed to be initially inert so that its output coincides with the forced response (the free response is identically zero).

## 8.1 Decomposition of the forced response

Consider a system represented by a strictly–proper rational transfer function

$$W(s) = \frac{N_w(s)}{D_w(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i} = K_w \frac{\prod_{i=1}^{m'} (s - z_i)^{\mu_i}}{\prod_{i=1}^{n'} (s - p_i)^{\nu_i}}, \tag{8.1}$$

where the polynomials $N_w(s)$ and $D_w(s)$ are coprime, and $\mu_i$ and $\nu_i$ are the multiplicities of the zeros $z_i$ and poles $p_i$, respectively. Assume that the input $u(t)$ is Laplace transformable and its Laplace transform $U(s)$ is rational and proper, that is,

$$U(s) = \frac{N_u(s)}{D_u(s)} = \frac{\sum_{i=0}^{q} d_i s^i}{\sum_{i=0}^{r} c_i s^i} = K_u \frac{\prod_{i=1}^{q'} (s - t_i)^{\pi_i}}{\prod_{i=1}^{r'} (s - v_i)^{\rho_i}}, \tag{8.2}$$

where the polynomials $N_u(s)$ and $D_u(s)$ are also coprime, and $\pi_i$ and $\rho_i$ are the multiplicities of the corresponding zeros $t_i$ and poles $v_i$.

The Laplace transform $Y_f(s)$ of the forced response $y_f(t)$ to $u(t)$, given by

$$y_f(t) = \int_0^t w(t - \tau) u(\tau) d\tau, \tag{8.3}$$

where $w(t)$ denotes the impulse response, is

$$Y_f(s) = W(s)U(s) = \frac{N_w(s)N_u(s)}{D_w(s)D_u(s)}. \tag{8.4}$$

It is also assumed that the polynomials $N_w(s)$ and $D_u(s)$ are coprime as well as the polynomials $N_u(s)$ and $D_w(s)$, so that no cancellations of factors common to the numerator and denominator of (8.4) are possible. Instead, $D_w(s)$ and $D_u(s)$ may have some roots in common. In this case, $D_w(s)$ can be written as

$$D_w(s) = \bar{D}_w(s) C_w(s), \tag{8.5}$$

whose factor $C_w(s)$ contains all of the roots of $D_w(s)$, with their multiplicities $\nu_i$, that are roots of $D_u(s)$. Similarly, $D_u(s)$ can be written as

$$D_u(s) = \bar{D}_u(s)C_u(s), \tag{8.6}$$

whose factor $C_u(s)$ contains all of the roots of $D_u(s)$, with their multiplicities $\rho_i$, that are roots of $D_w(s)$. Clearly, if all of the roots of $D_w(s)$ and $D_u(s)$ are simple, then $C_w(s) = C_u(s)$.

Define now the polynomial:

$$C(s) := C_w(s)C_u(s). \tag{8.7}$$

By construction, the three polynomials $\bar{D}_w(s)$, $\bar{D}_u(s)$ and $C(s)$ are pairwise coprime. Therefore, the following result holds.

**Proposition 8.1.1** *The Laplace transform $Y_f(s)$ of the forced response can be expressed as the sum of three strictly–proper rational functions according to:*

$$Y_f(s) = \frac{N_w(s)N_u(s)}{\bar{D}_w(s)\bar{D}_u(s)C(s)} = \frac{\bar{N}_w(s)}{\bar{D}_w(s)} + \frac{\bar{N}_u(s)}{\bar{D}_u(s)} + \frac{N_c(s)}{C(s)}. \tag{8.8}$$

**Proof** A classical result of polynomial algebra states that, given three polynomials $\alpha(s)$, $\beta(s)$ and $\gamma(s)$, with $\alpha(s)$ and $\beta(s)$ coprime, there exist pairs of polynomials $x(s)$ and $y(s)$ satisfying the diophantine equation (see, e.g., [7]):

$$\alpha(s)x(s) + \beta(s)y(s) = \gamma(s). \tag{8.9}$$

Moreover, if $\deg[\gamma(s)] < \deg[\alpha(s)] + \deg[\beta(s)]$, there exists a unique pair of polynomials $x(s)$ and $y(s)$ satisfying (8.9) such that $\deg[x(s)] < \deg[\beta(s)]$ and $\deg[y(s)] < \deg[\alpha(s)]$.
If $x(s)$ and $y(s)$ denote the unique solution of (8.9) corresponding to $\alpha(s) = \bar{D}_w(s)\bar{D}_u(s)$, $\beta(s) = C(s)$ and $\gamma(s) = N_w(s)N_u(s)$, then $Y_f(s)$ in (8.8) can be decomposed as:

$$Y_f(s) = \frac{N_w(s)N_u(s)}{\bar{D}_w(s)\bar{D}_u(s)C(s)} = \frac{y(s)}{\bar{D}_w(s)\bar{D}_u(s)} + \frac{x(s)}{C(s)}. \tag{8.10}$$

By identifying $x(s)$ in (8.10) with $N_c(s)$ in (8.8), and applying the same procedure to the first addendum at the right–hand side of (8.10), the following decomposition is obtained:

$$\frac{y(s)}{\bar{D}_w(s)\bar{D}_u(s)} = \frac{\bar{N}_w(s)}{\bar{D}_w(s)} + \frac{\bar{N}_u(s)}{\bar{D}_u(s)}, \tag{8.11}$$

which proves (8.8). □

According to (8.8), the transformed response consists of three strictly–proper components:

$$Y_f(s) = Y_w(s) + Y_u(s) + Y_c(s), \tag{8.12}$$

with

$$Y_w(s) := \frac{\bar{N}_w(s)}{\bar{D}_w(s)} = \sum_{i=1}^{n''} \sum_{j=0}^{\nu_i - 1} \frac{R_{w,ij}}{(s - q_{w,i})^{j+1}}, \tag{8.13}$$

$$Y_u(s) := \frac{\bar{N}_u(s)}{\bar{D}_u(s)} = \sum_{i=1}^{r''} \sum_{j=0}^{\rho_i - 1} \frac{R_{u,ij}}{(s - q_{u,i})^{j+1}}, \tag{8.14}$$

$$Y_c(s) := \frac{N_c(s)}{C(s)} = \sum_{i=1}^{v''} \sum_{j=0}^{\phi_i - 1} \frac{R_{c,ij}}{(s - q_{c,i})^{j+1}}, \tag{8.15}$$

where: $n''$ is the number of distinct poles $q_{w,i}$ of $W(s)$ that are not in common with $U(s)$, $r''$ is the number of distinct poles $q_{u,i}$ of $U(s)$ that are not in common with $W(s)$, $v''$ is the number of distinct common poles $q_{c,i}$, and $\phi_i$ is the sum of the multiplicities of the same pole in $W(s)$ and $U(s)$.

Correspondingly, in the time domain:

$$y_f(t) = y_w(t) + y_u(t) + y_c(t), \tag{8.16}$$

where

$$y_w(t) = LT^{-1}[Y_w(s)], \;\; y_u(t) = LT^{-1}[Y_u(s)], \;\; y_c(t) = LT^{-1}[Y_c(s)]. \tag{8.17}$$

Therefore, $y_w(t)$ is formed from modes present in $w(t)$ but not in $u(t)$, even if some coefficients $R_{w,ij}$ may well be zero, as is the case when zeros of $U(s)$ cancel poles of $W(s)$). For this reason, it is reasonable to call the $y_w(t)$ component the *system component*.

The component $y_u(t)$ is formed from modes present in $u(t)$ but not in $w(t)$, even if in this case too some $R_{u,ij}$ may be zero so that the corresponding mode is filtered out. It is reasonable to call the $y_u(t)$ component the *input component*, since its form resembles the one of $u(t)$.

The component $y_c(t)$ corresponds to poles that are common to $y_w(t)$ and $y_u(t)$, and hence it can be referred to as the *interaction* or *resonant component.*

**Example** A system with transfer function

$$W(s) = 10\frac{s+4}{(s+3)(s+5)} \tag{8.18}$$

is subjected to an input whose Laplace transform is

$$U(s) = 5\frac{3s+2.5}{(s-1)(s+3)}. \tag{8.19}$$

By partial fraction expansion and inverse transformation, the forced response turns out to be

$$y_f(t) = \frac{6750}{512}e^{-5t} + \frac{1250}{192}e^{t} - \frac{34250}{1536}e^{-3t} + \frac{61300}{1536}te^{-3t}. \tag{8.20}$$

With the previous definitions,

$$y_w(t) = \frac{6750}{512}e^{-5t}, \;\; y_u(t) = \frac{1250}{192}e^{t}, \;\; y_c(t) = -\frac{34250}{1536}e^{-3t} + \frac{61300}{1536}te^{-3t}. \tag{8.21}$$

As this example shows, $y_c(t)$ contains one or more modes that are not present in $w(t)$ and $u(t)$.

## 8.2 Steady–state and asymptotic responses

From an etymological point of view, the commonly used terms of "transient, asymptotic and steady–state response" seem to be proper only when:
(i) the system is bounded–input bounded–output (BIBO) stable, i.e., $\mathbb{R}\mathrm{e}[p_i] < 0$, $i = 1, \ldots, n'$, and
(ii) at least one input mode does not tend to zero, i.e., $\mathbb{R}\mathrm{e}[v_i] \geq 0$, for at least one $i$.

In this case, both $y_w(t)$ and $y_c(t)$ tend to zero as $t \to \infty$ since the common poles between $W(s)$ and $U(s)$, if any, are in the open LHP. Therefore, $y_c(t)$ and $y_w(t)$ can rightfully be called the *transient* terms because they (practically) "continue for only a short time" [8].

If it is further assumed that $U(s)$ has no pole in the open RHP and its purely imaginary poles $v_i$ are simple, then $u(t)$ is bounded and $y_u(t)$ may be called *steady–state* response, corresponding to the most common interpretation of the term "steady state" and consistent with the dictionary definition of "steady" as "regular" [8].

If, instead, the input is unbounded because some of the poles of $U(s)$ are in the open RHP or some of its purely imaginary zeros are not simple, then it appears more reasonable to refer to the input component as the *asymptotic* response, rather than "steady state". Actually, this asymptotic response is dominated by the mode in $y_u(t)$ with the most positive value of $\mathbb{R}\text{e}[v_i]$, or, if the maximum of $\mathbb{R}\text{e}[v_i] = 0$, with the highest power of $t$.

**Example** Let

$$W(s) = \frac{1}{s+1} \quad \text{and} \quad U(s) = \frac{1}{s^2}. \tag{8.22}$$

In this case:

$$Y_u(s) = \frac{1}{s^2} - \frac{1}{s} \tag{8.23}$$

so that the input component

$$y_u(t) = t - 1, \;\; t > 0, \tag{8.24}$$

is the asymptotic response dominated by the mode $t$.

Note that, in the control literature, the polynomial part of the forced response to a canonical input $u(t) = \frac{1}{(i-1)!}t^i$, $t > 0$, is often called steady–state response; for example, in [9, p. 78] the steady–state response to a ramp–function input is identified with a term of the form $k_0 + k_1 t$.

If the assumptions (i) and (ii) at the beginning of this section do not hold, the suggested terminology seems to be no longer meaningful. For instance, if the system is unstable, then $y_w(t)$ cannot be called transient; also, the fastest growing term of the forced response could belong to $y_w(t)$ itself rather than to $y_u(t)$. If, on the contrary, the system is stable and $u(t)$ is composed of terms that tend to zero, the overall response will be "transient"; the expression "asymptotic" could then refer to the slowest decaying term which, however, could belong to $y_w(t)$ (for a definition of dominant modes, see [10]).

In conclusion, it appears appropriate to use the terms "transient", "steady–state" or "asymptotic" only when $\mathbb{Re}[p_i] < 0, \forall i$, and $\mathbb{Re}[v_i] \geq 0$ for at least one $i$. In these cases, $y_w(t)$ is indeed transient, and the steady–state or asymptotic response corresponds to the input component $y_u(t)$ (or to its fastest growing term). Of course, the decomposition of the overall forced response into a system component, an input component and, possibly, a resonant component is always valid.

## 8.3 Properties of the input component

The steady–state response is sometimes computed by assuming that the input is applied at $t = -\infty$, using the convolution integral

$$y(t) = \int_{-\infty}^{t} u(\tau)w(t-\tau)d\tau. \tag{8.25}$$

However, care must be taken, when computing the steady–state response in this way, that the integral in question is well defined. For instance, if $\mathbb{Re}[v_i - p_j] < 0$, $\forall i$ and $\forall j$, then (8.25) diverges for all finite $t$. To see this, consider the simple case where

$$u(t) = \sum_{i=1}^{r} Q_{u,i}e^{v_i t}, \quad w(t) = \sum_{i=1}^{n} Q_{w,i}e^{p_i t} \tag{8.26}$$

so that the integral (8.25) becomes

$$\int_{-\infty}^{t} u(\tau)w(t-\tau)d\tau = \sum_{i=1}^{r}\sum_{j=1}^{n} \frac{Q_{u,i}Q_{w,i}}{v_i - p_j} e^{p_j t}\left[e^{(v_i-p_j)\tau}\right]\Big|_{-\infty}^{t} = \infty. \tag{8.27}$$

However, the integral is well defined and does define the steady state when $\mathbb{Re}[p_i] < 0$ and $\mathbb{Re}[v_i] \geq 0$, since then

$$\int_{-\infty}^{t} u(\tau)w(t-\tau)d\tau = \sum_{i=1}^{r}\left(\sum_{j=1}^{n} \frac{Q_{u,i}Q_{w,i}}{v_i - p_j}\right)e^{v_i t} \tag{8.28}$$

is finite for all finite $t$ and contains only the modes of $u(t)$.
The considered property does not hold, instead, if $u(t)$ includes exponential terms with a negative real part of $v_i$ and/or the system is unstable because the integral (8.25) does not converge in those cases.

Another interesting property of the input component $y_u(t)$ is expressed by the following proposition.

**Proposition 8.3.1** *The input component $y_u(t)$ is a linear combination of $u(t)$ and its first $r-1$ successive derivatives:*

$$y_u(t) = \sum_{h=0}^{r-1} f_h \frac{d^h u}{dt^h}, \quad t > 0. \tag{8.29}$$

**Proof** To avoid complex notation, reference is made to the case of a $U(s)$ whose poles $v_i$ are all simple. Therefore, $u(t)$ is as in (8.26) and $y_u(t)$ particularizes to

$$y_u(t) = \sum_{i=1}^{r''} R_{u,i} e^{v_i t}, \tag{8.30}$$

where, again, $r''$ denotes the number of distinct poles of $U(s)$ that are not in common with $W(s)$. It follows that the coefficients $f_h$ in (8.29) can uniquely be determined by solving the matrix equation:

$$\begin{bmatrix} 1 & v_1 & v_1^2 & \dots & v_1^{r-1} \\ 1 & v_2 & v_2^2 & \dots & v_2^{r-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & v_{r''} & v_{r''}^2 & \dots & v_{r''}^{r-1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & v_r & v_r^2 & \dots & v_r^{r-1} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \dots \\ f_{r''-1} \\ \dots \\ \dots \\ f_{r-1} \end{bmatrix} = \begin{bmatrix} R_{u,1}/Q_{u,1} \\ R_{u,2}/Q_{u,2} \\ \dots \\ R_{u,r''}/Q_{u,r''} \\ 0 \\ \dots \\ 0 \end{bmatrix} \tag{8.31}$$

whose coefficient matrix is nonsingular. □

A definition of steady–state response based on (8.29) was proposed in [11]. Precisely, the steady–state response was defined as a solution (for $t > 0$) of the system inhomogeneous equation having the form (8.29). The input component $y_u(t)$ defined in this chapter is certainly a linear combination of $u(t)$ and its derivatives but it is also a solution of the system inhomogeneous equation only if $y_c(t) \equiv 0$. This can easily be realized by considering that the system component $y_w(t)$ is a solution of the system homogeneous equation and, therefore, $y_u(t) + y_c(t)$ is a particular solution of the inhomogeneous equation; since $y_c(t)$ does not satisfy the homogeneous equation, $y_u(t)$ alone cannot be a solution of the inhomogeneous one if $y_c(t)$ is not identically zero.

## 8.4 Extensions

In the previous sections both $U(s)$ and $W(s)$ were rational functions In this section, instead, the case in which either of these functions is nonrational is briefly examined. Assume first that the nonrational term, taken to be the input transform $U(s)$ without loss of generality, may be expressed as

$$U(s) = \sum_{i=1}^{\infty} \sum_{j=0}^{\psi_i} \frac{R_{u,ij}}{(s - v_i)^{j+1}}, \tag{8.32}$$

which typically occurs when $u(t)$ is periodic with an infinite number of harmonics. For instance, the square wave of amplitude 1 and period 4:

$$u(t) = \delta_{-1}(t) + 2\sum_{i=1}^{\infty} (-1)^i \delta_{-1}(t - 2i), \tag{8.33}$$

where $\delta_{-1}(t)$ denotes the unit step function, has the following nonrational transform [12]:

$$U(s) = \frac{1}{s}\left[1 + 2\sum_{i=1}^{\infty} (-1)^i e^{-2is}\right] = \frac{1}{s}\tanh(s) \tag{8.34}$$

which can be rewritten as

$$U(s) = \sum_{i=1}^{\infty} \frac{2}{s^2 + \omega_i^2} \quad \text{with} \quad \omega_i = (2i - 1)\frac{\pi}{2}. \tag{8.35}$$

Taking the inverse Laplace transform of every term in (8.35), the following Fourier expansion is obtained:

$$u(t) = \delta_{-1}(t)\left[\sum_{i=1}^{\infty} \frac{2}{\omega_i} \sin(\omega_i t)\right]. \tag{8.36}$$

If $W(s)$ or $U(s)$ has a finite number of poles, $Y_f(s)$ can still be decomposed as in (8.12), even if, in this case, either $Y_w(s)$ in (8.13) or, respectively, $Y_u(s)$ in (8.14) will have an infinite number of terms. For instance, if $U(s)$ is given by (8.35) and

$$W(s) = \frac{1}{s + 1}, \tag{8.37}$$

then

$$Y_f(s) = \sum_{i=1}^{\infty} \frac{1}{s+1} \, \frac{2}{s^2+\omega_i^2} = \left[ \sum_{i=1}^{\infty} \frac{2}{1+\omega_i^2} \right] \frac{1}{s+1} + \sum_{i=1}^{\infty} \frac{2}{1+\omega_i^2} \frac{1+s}{s^2+\omega_i^2}. \tag{8.38}$$

The first term at the right–hand side of (8.38) is $Y_w(s)$ whereas the second one is $Y_u(s)$; no interaction term is present.

Concerning the coincidence of $y_u(t)$ with the response that the system would have produced if the input had been applied at $t = -\infty$, this indeed happens if *each addendum* of $u(t)$, as obtained from (8.32) by inverse Laplace transformation, satisfies the integrability conditions for (8.25), which is true in the above example. Specifically, the inverse Laplace transform of $Y_w(s)$ is

$$y_w(t) = U(-1)\, e^{-t} = 0.762\, e^{-t} \tag{8.39}$$

and the inverse Laplace transform of $Y_u(s)$ is

$$y_u(t) = \sum_{i=1}^{\infty} \frac{2}{1+\omega_i^2} \left[ \frac{1}{\omega_i} \sin(\omega_i t) - \cos(\omega_i t) \right] \tag{8.40}$$

which coincides with

$$y(t) = \int_{-\infty}^{t} \left[ 2 \sum_{i=1}^{\infty} \frac{1}{\omega_i} \sin(\omega_i \tau) \right] e^{-(t-\tau)} d\tau. \tag{8.41}$$

Instead, the property expressed by Proposition 8.3.1 is no longer true if $U(s)$ is nonrational, as clearly shown by the above example in which the input is a square wave: the steady–state response $y_u(t)$ is given by (8.40) and this cannot be obtained by combining the considered piecewice constant $u(t)$ and its derivatives, which are everywhere zero except for the discontinuity points.

Consider now a case in which the nonrational function has no finite poles. Precisely, assume that $u(t)$ is given by

$$u(t) = e^{-t^2}\, \delta_{-1}(t) \tag{8.42}$$

whose Laplace transform is [12]

$$U(s) = \frac{\sqrt{\pi}}{2}\, e^{\frac{s^2}{4}} \operatorname{erfc}\left[ \frac{s}{2} \right]. \tag{8.43}$$

If $W(s)$ has $n$ simple poles $p_i$, the forced response transform can be written as

$$Y_f(s) = W(s)U(s) = \sum_{i=1}^{n} \frac{\hat{R}_i}{s - p_i} + \hat{Y}_f(s), \tag{8.44}$$

where

$$\hat{R}_i = \lim_{s \to p_i} (s - p_i)W(s)U(s) \tag{8.45}$$

and $\hat{Y}_f(s)$ has no poles.
Clearly, under the above assumptions, no resonance component is present, that is, $Y_c(s) = 0$. The summation at the right–hand side of (8.44) has the form of the transform of a free evolution and, therefore, can be considered as the transform $Y_w(s)$ of the system component. For this reason, one might regard $\hat{Y}_f(s)$ as the input component because it depends on $u(t)$ and does not contain any mode of the free evolution. It is easy to see that, in this case, $\hat{y}_f(t)$ is not given by a linear combination of $u(t)$ and its derivatives. In fact, $\hat{y}_f(t)$ is a particular solution of the system inhomogeneous equation, being the difference between the particular solution $y_f(t)$ of the inhomogeneous equation and the solution $y_w(t)$ of the homogeneous one, that is,

$$\sum_{i=0}^{n} a_i \frac{d^i \hat{y}_f}{dt^i} = \sum_{i=0}^{m} b_i \frac{d^i u}{dt^i}. \tag{8.46}$$

Now, if

$$\hat{y}_f(t) = \sum_{k=0}^{K} f_k \frac{d^k u}{dt^k} \tag{8.47}$$

were true, then $u(t)$ would satisfy the homogeneous equation

$$\sum_{i=0}^{n+K} \left[ b_i - \sum_{j=0}^{i} a_{i-j} f_j \right] \frac{d^i u}{dt^i} = 0 \tag{8.48}$$

and U(s) would be rational, thus contradicting the assumption.
Furthermore, $\hat{y}_f(t)$ may not coincide with the system response to an input acting from $t = -\infty$. For example, if $U(s)$ is given by (8.43) and $W(s)$ by (8.37), the overall forced response is

$$y_f(t) = \int_0^t e^{-(t-\tau)} e^{-\tau^2} d\tau = e^{-t} \left\{ e^{\frac{1}{4}} \frac{\sqrt{\pi}}{2} \left[ \text{erfc}\left( -\frac{1}{2} \right) - \text{erfc}\left( t - \frac{1}{2} \right) \right] \right\} =$$

$$= e^{-t}\left[1.730 - 1.138\,\mathrm{erfc}\left(t - \frac{1}{2}\right)\right], \quad t > 0. \tag{8.49}$$

According to (8.44) and (8.45), the system component is

$$y_w(t) = U(-1)e^{-t} = 1.730\,e^{-t}, \quad t > 0, \tag{8.50}$$

so that

$$\hat{y}_f(t) = y_f(t) - y_w(t) = -1.138\,\mathrm{erfc}\left(t - \frac{1}{2}\right)e^{-t}, \tag{8.51}$$

whereas the response $y(t)$ for $t > 0$ to the input $u(t) = e^{-t^2}$ acting from $t = -\infty$ is

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}e^{-\tau^2}\,d\tau = e^{-t}\left\{e^{\frac{1}{4}}\frac{\sqrt{\pi}}{2}\left[2 - \mathrm{erfc}\left(t - \frac{1}{2}\right)\right]\right\} =$$
$$= e^{-t}\left\{2.276 - 1.138\,\mathrm{erfc}\left(t - \frac{1}{2}\right)\right]\right\}. \tag{8.52}$$

In the previous example it was possible to determine both $\hat{y}_f(t)$ and $y(t)$, and $y_w(t)$ turned out to be real. There are cases, however, in which, even if a nonrational $U(s)$ exists and $y_f(t)$ may be computed (and is real), $y(t)$ cannot be determined and/or $y_w(t)$, $\hat{y}_f(t)$ are not real. For instance, the input

$$u(t) = \frac{1}{t + t_0}, \quad t > 0, \tag{8.53}$$

gives rise to the real finite forced response

$$y_f(t) = e^{-t}\int_0^t \frac{e^{\tau}}{\tau + t_0}\,d\tau, \quad t > 0, \tag{8.54}$$

whereas the integral from $-\infty$ does not converge.

Moreover, for the system with transfer function (8.37) and impulse response $w(t) = e^{-t}\,\delta_{-1}(t)$, the system component $y_w(t) = U(-1)e^{-t}\delta_{-1}(t)$ is complex if $U(-1)$ is complex. For example, if $u(t) = (1/\sqrt{t})\,\delta_{-1}(t)$, then $U(s) = \sqrt{\pi/s}$ and $U(-1)$ is imaginary. Similarly, if $u(t) = [\ln(t) + \gamma]\,\delta_{-1}(t)$, where $\gamma$ is the Euler constant ($\gamma \simeq 0.577$), then $U(s) = -(1/s)\ln(s)$, and again $U(-1)$ is imaginary. Obviously, when $y_w(t)$ is complex, $\hat{y}_f(t)$ must also be so since $y_f(t)$ is real. Similar considerations could be made for the case in which $U(s)$ is rational and $W(s)$ is nonrational (but has no poles): the forced response is then formed by a combination of input modes (input component) and by another term to be regarded as the system compenent. Such a decomposition of $y_f(t)$ is not possible if both $U(s)$ and $W(s)$ are nonrational.

## 8.5 Concluding remarks

A precise definition of steady–state, asymptotic and transient response for linear time–invariant systems has been provided. In particular, it has been shown that, when both the transfer function $W(s)$ and the Laplace transform $U(s)$ of the input are rational, the forced response can *uniquely* be subdivided, according to (8.16) and (8.17), into three parts that have been named the system component, the input component and the interaction component (Section 8.1).

The form of the system component $y_w(t)$ is similar that of the impulse response $w(t)$ and the form of the input component $y_u(t)$ is similar to that of the input $u(t)$, whereas the interaction component $y_c(t)$ contains terms that are not present in $w(t)$ and $u(t)$. Under suitable assumptions (see Section 8.2), $y_w(t)$ can reasonably be called transient response and $y_u(t)$ steady–state response. The expression "asymptotic response" seems to be more appropriate when $u(t)$ and, thus, $y_u(t)$ are unbounded.

The possibility of determining the steady–state response from the indefinite integral (8.25) has been considered as well as the use of linear combinations of $u(t)$ and its derivatives (Section 8.3). Possible extensions of the previous concepts to the cases where either $W(s)$ or $U(s)$ is nonrational have been discussed in Section 8.4.

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Often, short papers tend to be sharper than longer works because they focus on a single theme without lingering on unessential aspects, thus showing clearly the signicance of a contribution or an idea. The author of this book had the privilege of collaborating for over a quarter of a century with Antonio Lepschy (1931-2005), a recognized leader of the Italian control community.

Lepschy had a liking for the brief paper format, so that many results obtained by his research team were published in this way. The present compilation tells a few of these short stories, duly updated, trying to preserve their original flavour.

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