Polska Akademia Nauk • Instytut Badań Systemowych

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ZARZĄDZANIE

Książka jubileuszowa
z okazji
70-lecia urodzin

PROFESORA KAZIMIERZA MAŃCZAKA

pod redakcją
Jakuba Gutenbauma

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Redaktor
prof. dr hab. inż. Jakub Gutenbaum

Warszawa 2002

ISBN 83-85847-78-2

Wydawca: Instytut Badań Systemowych PAN
ul. Newelska 6 01-447 Warszawa
http://www.ibspan.waw.pl

Opracowanie składopisu: Anna Gostyńska, Jadwiga Hartman

Druk: KOMO-GRAF, Warszawa
nakład 200 egz., 34 ark. wyd., 31 ark. druk.

EQUIVALENT STANDARD SYSTEMS FOR SINGULAR DISCRETE-TIME SYSTEMS WITH DELAYS

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Abstract: *A standard system which has the same solution for admissible initial conditions as a given singular discrete-time linear system with delays in state and input is derived. Some basic properties of the fundamental matrices of the singular discrete-time linear systems with delays are characterised. A method of determination of the standard system based on the shuffle algorithm is proposed.*

Keywords: *singular systems, discrete-time systems, equivalent standard systems, delays.*

1. Introduction

The analysis and synthesis of linear systems with delays has been considered in many papers and books (Górecki et al. 1989, Medvedev 1997, Olbrot 1978). A survey of linear singular systems is given in (Lewis 1986). Recently in (Kaczorek 1998) a standard system which is equivalent to (has the same solution as) the singular (regular) linear system has been derived.

The main purpose this paper is to extend this approach for discrete-time linear systems with delays in state and input. Some basic properties of the fundamental matrices of the singular discrete-time linear systems with delays will be characterised. A standard system which has the same solution for admissible initial conditions as a given singular discrete-time linear system will be derived. A method of determination of the standard system based on the shuffle algorithm (Luenberger 1978, Kaczorek 1992) will be proposed.

2. Preliminaries and fundamental matrices

Let Z_+ be the set of nonnegative integers. The set of $m \times n$ real matrices will be denoted by $R^{m \times n}$ and $R^m := R^{m \times 1}$.
Consider the linear discrete-time system with delays:

$$Ex_{i+1} = A_1 x_i + A_2 x_{i-d_1} + B_1 u_i + B_2 u_{i-d_2} \quad i, d_1, d_2 \in Z_+ \tag{1}$$

where $x_i \in R^n$ is the state vector, $u_i \in R^m$ is the input vector, $E, A_1, A_2 \in R^{n \times n}$, $B_1, B_2 \in R^{n \times m}$ with E possibly singular and d_1 and d_2 is the delay in state and input, respectively.
To compute x_i for $i \in Z_+$ we have to know u_i for $i \geq -d_2$ and the initial conditions:

$$x_{-d_1}, x_{-d_1+1}, \dots, x_0 \tag{2}$$

The system (1) is called standard if $E = I_n$ (the identity matrix) and it is called singular if $\det E = 0$.
It is assumed that:

$$\det\left[Ez - A_1 - A_2 z^{-d_1}\right] \neq 0 \text{ for some } z \in \mathbf{C} \text{ (the field of complex numbers)} \tag{3}$$

The system (1) is called regular if (3) holds.
If (3) is satisfied then:

$$\left[Ez - A_1 - A_2 z^{-d_1}\right]^{-1} = \sum_{i=-\mu}^{\infty} \Phi_i z^{-(i+1)} \tag{4}$$

where μ is the index of nilpotence and Φ_i is the fundamental matrix.

Substitution of (4) into the equality
$\left[Ez - A_1 - A_2 z^{-d_1}\right]^{-1}\left[Ez - A_1 - A_2 z^{-d_1}\right] = I_n$ yields:

$$\left(\sum_{i=-\mu}^{\infty} \Phi_i z^{-(i+1)}\right)\left[Ez - A_1 - A_2 z^{-d_1}\right] = \left[Ez - A_1 - A_2 z^{-d_1}\right]\left(\sum_{i=-\mu}^{\infty} \Phi_i z^{-(i+1)}\right) = I_n \tag{5}$$

Comparison of the coefficients at the same power of z in (5) yields:

$$E\Phi_i - A_1\Phi_{i-1} - A_2\Phi_{i-d_1-1} = \Phi_i E - \Phi_{i-1}A_1 - \Phi_{i-d_1-1}A_2 = \\ = I_n\delta_i := \begin{cases} I_n \ for\ i=0 \\ 0\ for\ i=-\mu+1,-\mu+2,...,-1,1,2,... \end{cases} \tag{6}$$

where δ_i is the Kronecker delta and $E\Phi_{-\mu} = \Phi_{-\mu}E = 0$.

Lemma. The fundamental matrices Φ_i and E, A_2 satisfy the equalities:

$$\Phi_j E\Phi_i + \Phi_{i-d_1-1}A_2\Phi_j + \Phi_{i-d_1-2}A_2\Phi_{j+1} + \cdots + \Phi_{j-d_1-1}A_2\Phi_{i-2} + \\ + \Phi_{j-d_1}A_2\Phi_{i-1} = \Phi_i E\Phi_j + \Phi_{i-1}A_2\Phi_{j-d_1} + \Phi_{i-2}A_2\Phi_{j-d_1-1} + \cdots \\ \cdots + \Phi_{j+1}A_2\Phi_{i-d_1-2} + \Phi_j A_2\Phi_{i-d_1-1} \quad \text{for} \quad i,j \in Z_+ \tag{7}$$

$$\Phi_1 E\Phi_i + \Phi_{1-d_1}A_2\Phi_{i-1} + \Phi_{2-d_1}A_2\Phi_{i-2} + \cdots + \Phi_0 A_2\Phi_{i-d_1} = \Phi_{i+1} \text{for } i \in Z_+ \tag{8}$$

Proof. From (6) we have:

$$E\Phi_i = A_1\Phi_{i-1} + A_2\Phi_{i-d_1-1} \quad \text{for} \quad i > 0 \tag{9a}$$

and:

$$\Phi_{i-1}A_1 = \Phi_i E - \Phi_{i-d_1-1}A_2 \quad \text{for} \quad i > 0 \tag{9b}$$

Using successively (9a) and (9b) we obtain for $i \in Z_+$:

$$\Phi_1 E\Phi_i = \Phi_1 A_1\Phi_{i-1} + \Phi_1 A_2\Phi_{i-d_1-1} = \Phi_2 E\Phi_{i-1} - \Phi_{1-d_1}A_2\Phi_{i-1} + \\ + \Phi_1 A_2\Phi_{i-d_1-1} = \Phi_2 A_1\Phi_{i-2} + \Phi_2 A_2\Phi_{i-d_1-2} - \Phi_{1-d_1}A_2\Phi_{i-1} + \Phi_1 A_2\Phi_{i-d_1-1} = \\ = \Phi_3 E\Phi_{i-2} - \Phi_{i-d_1-2}A_2\Phi_{i-2} + \Phi_2 A_2\Phi_{i-d_1-2} - \Phi_{1-d_1}A_2\Phi_{i-1} + \\ + \Phi_1 A_2\Phi_{i-d_1-1} = \cdots = \Phi_i E\Phi_1 - \Phi_{i-d_1-1}A_2\Phi_1 + \Phi_{i-1}A_2\Phi_{i-d_1} - \Phi_{1-d_2-2}A_2\Phi_2 + \cdots \\ \cdots + \Phi_{i-2}A_2\Phi_{2-d_1} - \cdots + \Phi_1 A_2\Phi_{i-d_1-1} - \Phi_{1-d_1}A_2\Phi_{i-1}$$

Similarly:

$$\Phi_2 E\Phi_i = \Phi_2 A_1\Phi_{i-1} + \Phi_2 A_2\Phi_{i-d_1-1} = \Phi_3 E\Phi_{i-1} - \Phi_{2-d_1}A_2\Phi_{i-1} + \Phi_2 A_2\Phi_{i-d_1-1} = \cdots \\ \cdots = \Phi_3 A_1\Phi_{i-2} + \Phi_3 A_2\Phi_{i-d_1-2} - \Phi_{2-d_1}A_2\Phi_{i-1} + \Phi_2 A_2\Phi_{i-d_1-1} = \\ = \Phi_4 E\Phi_{i-2} - \Phi_{3-d_1}A_2\Phi_{i-2} + \Phi_3 A_2\Phi_{i-d_1-2} - \Phi_{2-d_1}A_2\Phi_{i-1} + \Phi_2 A_2\Phi_{i-d_1-1} = \cdots \\ \cdots = \Phi_i E\Phi_2 - \Phi_{i-d_1-1}A_2\Phi_2 + \Phi_{i-1}A_2\Phi_{2-d_1} - \Phi_{1-d_1-2}A_2\Phi_3 + \Phi_{i-2}A_2\Phi_{3-d_1} - \cdots \\ \cdots + \Phi_2 A_2\Phi_{i-d_1-1} - \Phi_{2-d_1}A_2\Phi_{i-1}$$

and for any $j \in Z_+$:

$$\Phi_j E\Phi_i = \Phi_j A_1 \Phi_{i-1} + \Phi_j A_2 \Phi_{i-d_1-1} = \Phi_{j+1} E\Phi_{i-1} - \Phi_{j-d_1} A_2 \Phi_{i-1} + \Phi_j A_2 \Phi_{i-d_1-1} =$$
$$= \Phi_{j+1} A_1 \Phi_{i-2} + \Phi_{j+1} A_2 \Phi_{i-d_1-2} - \Phi_{j-d_1} A_2 \Phi_{i-1} + \Phi_j A_2 \Phi_{i-d_1-1} =$$
$$= \Phi_{j+2} E\Phi_{i-2} - \Phi_{j-d_1+1} A_2 \Phi_{i-2} + \Phi_{j+1} A_2 \Phi_{i-d_1-2} - \Phi_{j-d_1} A_2 \Phi_{i-1} +$$
$$\Phi_j A_2 \Phi_{i-d_1-1} = \cdots = \Phi_i E\Phi_j - \Phi_{i-d_1-1} A_2 \Phi_j + \Phi_{i-1} A_2 \Phi_{j-d_1} - \Phi_{i-d_1-2} A_2 \Phi_{j+1} +$$
$$\Phi_{i-2} A_2 \Phi_{j-d_1+1} - \cdots - \Phi_{j-d_1} A_2 \Phi_{i-1} + \Phi_j A_2 \Phi_{i-d_1-1}$$

The equality (8) may be proved in a similar way using (6),(7) and (9).

Remark 1. For $A_2 = 0$ from (7) we obtain the equality $\Phi_j E\Phi_i = \Phi_i E\Phi_j$ which was shown in Lewis and Metrzios (1990).

3. Solution of the system

Let $P \in R^{n \times n}$ be a nonsingular matrix of elementary row operations such that:

$$PE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \tag{10}$$

where $E_1 \in R^{r \times n}$ has full row rank.
Premultiplying (1) by P and using (10) we obtain:

$$E_1 x_{i+1} = A_{11} x_i + A_{21} x_{i-d_1} + B_{11} u_i + B_{21} u_{i-d_2} \qquad i \in Z_+ \tag{11a}$$

and:

$$0 = A_{12} x_i + A_{22} x_{i-d_1} + B_{12} u_i + B_{22} u_{i-d_2} \qquad i \in Z_+ \tag{11b}$$

where:

$$PA_1 = \begin{bmatrix} A_{11} \\ A_{12} \end{bmatrix}, PA_2 = \begin{bmatrix} A_{21} \\ A_{22} \end{bmatrix}, PB_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, PB_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}$$

$$A_{11} \in R^{r \times n}, A_{21} \in R^{r \times n}, B_{11} \in R^{r \times m}, B_{21} \in R^{r \times m}$$

Definition 1. A set X_a of initial conditions (2) satisfying (11b) for $i = 0,1,...,d_1$ for a given input sequence u_i for $i > -d_2$ is called the set of admissible initial conditions of the system (1).

Definition 2. A set U_a of input sequences u_i for $i > -d_2$ satisfying (11b) for $i = 0,1,...,d_2$ for given input conditions (2) is called the set of acceptable input sequences of the system (1).

Note that the system (1) has a solution $x_i, i \in Z_+$ for a given input sequence u_i for $i > -d_2$ only if initial conditions (2) belong to the set X_a. Similarly, the system (1) has a solution $x_i, i \in Z_+$ for given initial conditions (2) only if the input sequence $u_i, i \in Z_+$ belongs to the set U_a.

In what follows it is assumed that the input sequences are given and the initial conditions (2) belong to X_a.

Application of the Z-transform to (1) yields:

$$X(z) = \left[Ez - A_1 - A_2 z^{-d_1}\right]^{-1} \Big\{ zEx_0 + A_2 z^{-d_1}\left(x_{-1}z + x_{-2}z^2 + \cdots + x_{-d_1} z^{d_1}\right) + \\ + \left(B_1 + B_2 z^{-d_2}\right)U(s) + B_2 z^{-d_2}\left(u_{-1}z + u_{-2}z^2 + \cdots + u_{-d_2} z^{d_2}\right)\Big\} \quad (12)$$

where $X(z)$ and $U(z)$ are the Z-transform of x_i and u_i, respectively.

By substituting (4) into (12) we obtain:

$$X(z) = \sum_{i=-\mu}^{\infty} \Phi_i z^{-(i+1)} \Big[zEx_0 + A_2\left(x_{-d_1} + x_{-d_1+1}z^{-1} + \cdots + x_{-1}z^{1-d_1}\right) + \\ + \left(B_1 + B_2 z^{-d_2}\right)U(s) + B_2\left(u_{-d_2} + u_{-d_2+1}z^{-1} + \cdots + u_{-1_2} z^{1-d_2}\right)\Big] \quad (13)$$

Using the convolution theorem to (13) we obtain for $i > 0$:

$$x_i = \Phi_i E x_0 + \Phi_{i-1}A_2 x_{-d_1} + \Phi_{i-2}A_2 x_{1-d_1} + \cdots + \Phi_{i-d_1}A_2 x_{-1} + \\ + \sum_{k=0}^{i+\mu-1} \Phi_{i-k-1}B_1 u_k + \sum_{k=0}^{i+\mu-d_2-1} \Phi_{i-k-d_2-1}B_2 u_k + \Phi_{i-1}B_2 u_{-d_2} + \\ + \Phi_{i-2}B_2 u_{1-d_2} + \cdots + \Phi_{i-d_2}B_2 u_{-1} \quad (14)$$

Therefore, we have proved the following theorem.

Theorem 1. Let (2) hold. Then the solution x_i to the equation (1) with admissible initial conditions (2) is given by (14).

4. Equivalent standard system

Knowing the fundamental matrices Φ_i for $i=-\mu,1-\mu,...,0,1,...$ of (1) we may compute the matrices T_j for $j \in Z_+$ which are defined by the expansion:

$$\left[I_n z-\Phi_1 E-\Phi_{1-d_1} A_2 z^{-1}-\Phi_{2-d_1} A_2 z^{-2}-\cdots-\Phi_0 A_2 z^{-d_1}\right]\left[\sum_{k=0}^{\infty} T_k z^{-(k+1)}\right]=$$

$$=\left[\sum_{k=0}^{\infty} T_k z^{-(k+1)}\right]\left[I_n z-\Phi_1 E-\Phi_{1-d_1} A_2 z^{-1}-\Phi_{2-d_1} A_2 z^{-2}-\cdots-\Phi_0 A_2 z^{-d_1}\right]=I_n \tag{15}$$

From (15) it follows that:

$$\begin{aligned}
&T_0=I_n, \quad T_1=\Phi_1 E, \quad T_2=\Phi_1 E T_1+\Phi_{1-d_1} A_2, \\
&T_3=\Phi_1 E T_2+\Phi_{1-d_1} A_2 T_1+\Phi_{2-d_1} A_2 ; \ldots \\
&T_{d_1}=\Phi_1 E T_{d_1-1}+\Phi_{1-d_1} A_2 T_{d_1-2}+\cdots+\Phi_{-1} A_2 ; \\
&T_{d_1+1}=\Phi_1 E T_{d_1}+\Phi_{1-d_1} A_2 T_{d_1-1}+\cdots+\Phi_0 A_2 ; \ldots
\end{aligned} \tag{16}$$

and for $k \geq 2$:

$$T_{d_1+k}=\Phi_1 E T_{d_1+k-1}+\Phi_{1-d_1} A_2 T_{d_1+k-2}+\cdots+\Phi_0 A_2 T_{k-1}$$

Consider the standard discrete-time system described by the equation:

$$\begin{aligned}
x_{i+1}=F_0 x_i+F_1 x_{i-1}+F_2 x_{i-2}+\cdots+F_{d_1} x_{i-d_1}+H_{-d_2} u_{i-d_2}+H_{1-d_2} u_{i-d_2+1}+\cdots \\
\ldots+H_{\mu-d_2} u_{i-d_2+\mu}+H_0 u_i+H_1 u_{i+1}+\cdots+H_\mu u_{i+\mu}
\end{aligned} \tag{17}$$

where:

$$F_0:=\Phi_1 E,\ F_1:=\Phi_{1-d_1} A_2 ; \qquad F_2:=\Phi_{2-d_1} A_2, \ldots, F_{d_1}:=\Phi_0 A_2$$

$$H_\mu:=\Phi_{-\mu} B_1,\ H_{\mu-1}:=\Phi_{1-\mu} B_1-T_1 H_\mu$$

$$H_{\mu-2}:=\Phi_{2-\mu} B_1-T_1 H_{\mu-1}-T_2 H_\mu, \ldots,$$

$$H_{\mu-d_2} := \Phi_{-\mu}B_2 + \Phi_{d_2-\mu}B_1 - T_{d_2-\mu}H_0 - \cdots - T_{d_2}H_\mu$$

$$H_{\mu-d_2-1} := \Phi_{-\mu+1}B_2 + \Phi_{\mu-d_2+1}B_1 - T_{\mu-d_2+1}H_0 - \cdots - T_{d_2+1}H_\mu - T_1 H_{\mu-d_2}$$

....

$$H_{1-d_2} := \Phi_{-1}B_2 + \Phi_{d_2-1}B_1 - T_{d_2-1}H_0 - \cdots - T_{\mu+d_2-1}H_\mu - \\ - T_1 H_{2-d_2} - \cdots - T_{\mu-1}H_{\mu-d_2}$$

$$H_{-d_2} := \Phi_0 B_2 + \Phi_{d_2}B_1 - T_{d_2}H_0 - \cdots - T_{\mu+d_2}H_\mu - T_1 H_{1-d_2} - \cdots - T_\mu H_{\mu-d_2} \tag{18}$$

Theorem 2. Equations (1) and (17) with admissible initial conditions (2) have the same solution (14).

Proof. We shall show that (14) is also the solution of (17) with admissible initial conditions (2). Using (17), (14), (7) and (8) we obtain:

$$F_0 x_i + F_1 x_{i-1} + F_2 x_{i-2} + \cdots + F_{d_1} x_{i-d_1} + H_{-d_2}u_{i-d_2} + H_{1-d_2}u_{i-d_2+1} + \cdots \\ \ldots + H_{\mu-d_2}u_{i-d_2+\mu} + H_0 u_i + H_1 u_{i+1} + \cdots + H_\mu u_{i+\mu} = \\ = F_0 \Big[\Phi_i E x_0 + \Phi_{i-1}A_2 x_{-d_1} + \Phi_{i-2}A_2 x_{1-d_1} + \cdots + \Phi_{i-d_1}A_2 x_{-1} + \\ + \sum_{k=0}^{i+\mu-1}\Phi_{i-k-1}B_1 u_k + \sum_{k=0}^{i+\mu-d_2-1}\Phi_{i-k-d_2-1}B_2 u_k + \\ + \Phi_{i-1}B_2 u_{-d_2} + \Phi_{i-2}B_2 u_{1-d_2} + \cdots + \Phi_{i-d_2}B_2 u_{-1}\Big] + \\ + F_1 \Big[\Phi_{i-1}E x_0 + \Phi_{i-2}A_2 x_{-d_1} + \Phi_{i-3}A_2 x_{1-d_1} + \cdots + \Phi_{i-d_1-1}A_2 x_{-1} + \\ + \sum_{k=0}^{i+\mu-2}\Phi_{i-k-2}B_1 u_k + \sum_{k=0}^{i+\mu-d_2-2}\Phi_{i-k-d_2-2}B_2 u_k + \Phi_{i-2}B_2 u_{-d_2} + \\ + \Phi_{i-3}B_2 u_{1-d_2} + \cdots + \Phi_{i-d_2-1}B_2 u_{-1}\Big] + \cdots + F_{d_1}\Big[\Phi_{i-d_1}E x_0 + \Phi_{i-d_1-1}A_2 x_{-d_1} + \\ + \Phi_{i-d_1-2}A_2 x_{1-d_1} + \cdots + \Phi_{i-2d_1}A_2 x_{-1} + \sum_{k=0}^{i+\mu-d_1-1}\Phi_{i-d_1-k-1}B_1 u_k + \\ + \sum_{k=0}^{i+\mu-d_1-d_2-1}\Phi +_{i-k-d_1-d_2-1} B_2 u_k + \Phi_{i-d_1-1}B_2 u_{-d_2} + \Phi_{i-d_1-2}B_2 u_{1-d_1-d_2} + \cdots \\ \cdots + \Phi_{i-d_1-d_2}B_2 u_{-1}\Big] + H_{-d_2}u_{i-d_2} + H_{1-d_2}u_{i-d_2+1} + \cdots + H_{\mu-d_2}u_{i-d_2+\mu} + H_0 u_i + \\ + H_1 u_{i+1} + \cdots + H_\mu u_{i+\mu} = \Phi_{i+1}E x_0 + \Phi_i A_2 x_{-d_1} + \Phi_{i-1}A_2 x_{1-d_1} + \cdots$$

$$\cdots + \Phi_{i-d_1+1}A_2x_{-1} + \sum_{k=0}^{i+\mu}\Phi_{i-k}B_1u_k + \sum_{k=0}^{i+\mu-d_2}\Phi_{i-k-d_2}B_2u_k + \Phi_iB_2u_{-d_2} +$$

$$+\Phi_{i-1}B_2u_{1-d_2+1} + \cdots + \Phi_{i-d_2+1}B_2u_{-1} = x_{i+1}$$

Therefore (14) is also the solution of (17).

Remark 2. For $A_2 = 0$ and $B_2 = 0$ the equation (17) has the form (Kaczorek 1998):

$$x_{i+1} = F_0x_i + H_0u_i + H_1u_{i+1} + \cdots + H_\mu u_{i+\mu} =$$

$$= \Phi_0A_1x_i + \Phi_0B_1u_1 + \Phi_{-1}B_1u_{i+1} + \cdots + \Phi_{-\mu}B_1u_{i+\mu} \tag{19}$$

where the fundamental matrices $\Phi_0, \Phi_{-1}, ..., \Phi_{-\mu}$ are defined by Lewis (1990), Mertzios (1989), Kaczorek (1998):

$$[Ez - A_1]^{-1} = \sum_{i=-\mu}^{\infty}\Phi_iz^{-(i+1)} \tag{20}$$

Example 1

Consider the system (1) with:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ d_1 = 2,\ d_2 = 3 \tag{21}$$

The system is regular since:

$$\det\left[Ez - A_1 - A_2z^{-d_1}\right] = \begin{vmatrix} z-1, -z^{-2} \\ -z^{-2}, -1 \end{vmatrix} = 1 - z - z^{-4} \tag{22}$$

and:

$$\left[Ez - A_1 - A_2z^{-d_1}\right]^{-1} = \begin{bmatrix} z-1, -z^{-2} \\ -z^{-2}, -1 \end{bmatrix}^{-1} = \frac{1}{1-z-z^{-4}}\begin{bmatrix} -1, -z^{-2} \\ -z^{-2}, z-1 \end{bmatrix} =$$

$$= \Phi_{-1} + \Phi_0z^{-1} + \Phi_1z^{-2} + \Phi_2z^{-3} + \Phi_3z^{-4} + \cdots \tag{23}$$

where:

$$\Phi_{-1}=\begin{bmatrix}0&0\\0&-1\end{bmatrix},\ \Phi_0=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad \Phi_1=\begin{bmatrix}1&0\\0&0\end{bmatrix},$$
$$\Phi_2=\begin{bmatrix}1&-1\\-1&0\end{bmatrix},\ \Phi_3=\begin{bmatrix}1&-1\\-1&0\end{bmatrix},\ \Phi_4=\begin{bmatrix}1&-1\\-1&1\end{bmatrix},\dots \tag{24}$$

In this case the equality (8) has the form:

$$\Phi_1E\Phi_i+\Phi_{-1}A_2\Phi_{i-1}+\Phi_0A_2\Phi_{i-2}=\Phi_{i+1}\qquad \text{for } i\in Z_+ \tag{25a}$$

or:

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}\Phi_i+\begin{bmatrix}0&0\\-1&0\end{bmatrix}\Phi_{i-1}+\begin{bmatrix}0&1\\0&0\end{bmatrix}\Phi_{i-2}=\Phi_{i+1} \tag{25b}$$

It is easy to check that the matrices (24) satisfy the equality (25b) for $i\in Z_+$.

Using (18), (16) and (21) we obtain:

$$F_0=\Phi_1E=\begin{bmatrix}1&0\\0&0\end{bmatrix},$$

$$F_1=\Phi_{1-d_1}A_2=\Phi_{-1}A_2=\begin{bmatrix}0&0\\-1&0\end{bmatrix},$$

$$F_2=\Phi_{2-d_1}A_2=\Phi_0A_2=\begin{bmatrix}0&1\\0&0\end{bmatrix}$$

$$T_0=I_n=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ T_1=\Phi_1E=F_0=\begin{bmatrix}1&0\\0&0\end{bmatrix},$$

$$T_2=\Phi_1ET_1+\Phi_{1-d_1}A_2=F_0T_1+F_1=\begin{bmatrix}1&0\\-1&0\end{bmatrix},$$

$$T_3=\Phi_1ET_2+\Phi_{1-d_1}A_2T_1+\Phi_{2-d_1}A_2=F_0T_2+F_1T_1+F_2=\begin{bmatrix}1&1\\-1&0\end{bmatrix},$$

$$T_4=F_0T_3+F_1T_2+F_2T_1=\begin{bmatrix}1&1\\-1&0\end{bmatrix}$$

$$H_1 = \Phi_{-1}B_1 = \begin{bmatrix} 0 \\ -2 \end{bmatrix},$$

$$H_0 = \Phi_0 B_1 - T_1 H_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$H_{-2} = \Phi_{-1}B_2 + \Phi_2 B_1 - T_2 H_0 - T_3 H_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

$$H_{-3} = \Phi_0 B_2 + \Phi_3 B_1 - T_3 H_0 - T_4 H_1 - T_1 H_{-2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence, in this case equation (17) has the form:

$$\begin{aligned} x_{i+1} = &\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x_{i-1} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{i-2} + \\ &+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ -2 \end{bmatrix} u_{i+1} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u_{i-2} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_{i-3} \end{aligned} \tag{26}$$

and the set of admissible initial conditions (2) for given $u_i, i \geq d_2$ is defined by the equality:

$$[0 \quad 1]x_i + [1 \quad 0]x_{i-2} + 2u_i + u_{i-3} = 0 \ \text{ for } \ i = 0,1,2 \tag{27}$$

5. Application of the shuffle algorithm to determination of the standard system

The shuffle algorithm was proposed by Luenberger (1978), Lewis (1986) for time invariant linear systems. In this section the shuffle algorithm will be applied to determination of the equivalent standard system (17) for the singular discrete-time system (1).

Knowing the matrices E, A_1, A_2, B_1, B_2 of (1) we construct the array:

$$E, A_1, A_2, B_1, B_2 \tag{28}$$

By performing the elementary row operations on the array (28) we reduce it to the form:

$$\begin{matrix} E_1, A_{11}, A_{21}, B_{11}, B_{21} \\ 0, \ A_{12}, A_{22}, B_{12}, B_{22} \end{matrix} \tag{29}$$

where $E_1 \in R^{n_1 \times n}$ is of full row rank and $A_{11}, A_{21} \in R^{n_1 \times n}, B_{11}, B_{21} \in R^{n_1 \times m}$.
Shuffle the array (29) to the form:

$$\begin{matrix} E_1, & A_{11}, & A_{21}, & 0, & B_{11}, & 0, & B_{21}, & 0 \\ A_{12}, & 0, & 0, & -A_{22}, & 0, & -B_{12}, & 0, & -B_{22} \end{matrix} \tag{30}$$

If the matrix:

$$\begin{bmatrix} E_1 \\ A_{12} \end{bmatrix} \tag{31}$$

is nonsingular then:

$$\begin{aligned} x_{i+1} = \begin{bmatrix} E_1 \\ A_{12} \end{bmatrix}^{-1} \Bigg\{ & \begin{bmatrix} A_{11} \\ 0 \end{bmatrix} x_i + \begin{bmatrix} A_{21} \\ 0 \end{bmatrix} x_{i-d_1} + \begin{bmatrix} 0 \\ -A_{22} \end{bmatrix} x_{i-d_1+1} + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} u_i + \\ & + \begin{bmatrix} 0 \\ -B_{12} \end{bmatrix} u_{i+1} + \begin{bmatrix} B_{21} \\ 0 \end{bmatrix} u_{i-d_2} + \begin{bmatrix} 0 \\ -B_{22} \end{bmatrix} u_{i-d_2+1} \Bigg\} \end{aligned} \tag{32}$$

If the matrix (31) is singular, by performing elementary row operations on (30) we obtain:

$$\begin{matrix} E_2, & A_{11}^1, & A_{21}^1, & A_{31}, & B_{11}^1, & B_{21}^1, & B_{31}, & B_{41} \\ 0, & A_{12}^1, & A_{22}^1, & A_{32}, & B_{12}^1, & B_{22}^1, & B_{32}, & B_{42} \end{matrix} \tag{33}$$

where E_2 is of full row rank and $rank E_2 \geq rank E_1$.

Shuffle the array (33) to the form:

$$\begin{matrix} E_2, A_{11}^1, A_{21}^1, A_{31}, 0, B_{11}^1, B_{21}^1, 0, B_{31}, B_{41}, 0 \\ A_{12}^1, 0, 0, -A_{22}^1, -A_{32}, 0, -B_{12}^1, -B_{22}^1, 0, -B_{32}, -B_{42} \end{matrix} \tag{34}$$

If the matrix:

$$\begin{bmatrix} E_2 \\ A_{12}^1 \end{bmatrix} \tag{35}$$

is nonsingular then:

$$x_{i+1} = \begin{bmatrix} E_1 \\ A_{12}^1 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} A_{11} \\ 0 \end{bmatrix} x_i + \begin{bmatrix} A_{21}^1 \\ 0 \end{bmatrix} x_{i-d_1} + \begin{bmatrix} A_{13} \\ -A_{22}^1 \end{bmatrix} x_{i-d_1+1} + \right.$$

$$+ \begin{bmatrix} 0 \\ -A_{32} \end{bmatrix} x_{i-d_1+2} + \begin{bmatrix} B_{11}^1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} B_{21}^1 \\ -B_{12}^1 \end{bmatrix} u_{i+1} +$$

$$\left. + \begin{bmatrix} B_{31} \\ 0 \end{bmatrix} u_{i-d_2} + \begin{bmatrix} B_{41} \\ -B_{32} \end{bmatrix} u_{i-d_2+1} + \begin{bmatrix} 0 \\ -B_{42} \end{bmatrix} u_{i-d_2+2} \right\} \tag{36}$$

If the matrix (35) is singular we repeat the procedure for (34). If the system (1) is regular and (3) holds then after k steps we obtain (Kaczorek 1992) a nonsingular matrix $\begin{bmatrix} E_k \\ A_{12}^{k-1} \end{bmatrix}$.

The shuffle algorithm can be justified as follows. The elementary row operations do not change the rank of $\left[Ez - A_1 - A_2 z^{-d_1} \right]$ and the shuffle of the array is equivalent to multiplication of the lower rows by -z (Luenberger 1978, Kaczorek 1992).

Example 2 (continuation of Example 1)

To find the equivalent standard system (26) for the singular system (1) with (21), we apply the shuffle algorithm. In this case the array (28) has already the form (29):

$$E, A_1, A_2, B_1, B_2 = \left[\begin{array}{cc:cc:cc:c:c} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & 1 \end{array} \right] \tag{37}$$

with:

$$E_1 = [1 \quad 0],\ A_{11} = [1 \quad 0],\ A_{12} = [0 \quad 1],\ A_{21} = [0 \quad 1],\ A_{22}[1 \quad 0],$$
$$B_{11} = [1],\quad B_{12} = [2],\quad B_{21} = [1],\quad B_{22} = [1]$$

The shuffle of (37) yields:

$$\begin{bmatrix} E_1, A_{12}, A_{21}, 0, B_{11}, 0, B_{21}, 0 \\ A_{21}, 0, 0, -A_{22}, 0, -B_{12}, 0, -B_{22} \end{bmatrix} =$$

$$= \left[\begin{array}{cc:cc:cc:cc:c:c:c:c} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & 0 & -1 \end{array} \right]$$

The matrix $\begin{bmatrix} E_1 \\ A_{21} \end{bmatrix} = \begin{bmatrix} 10 \\ 01 \end{bmatrix}$ is nonsingular and the equivalent standard system (32) has the form:

$$x_{i+1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{i-2} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x_{i-1} +$$
$$+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ -2 \end{bmatrix} u_{i+1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_{i-3} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u_{i-2}$$

6. Concluding remarks

A standard system (17) which has the same solution (14) for admissible initial conditions (2) as the singular system (1) has been derived. Some basic properties of the fundamental matrices defined by (4) of the system (1) have been characterised. A method based on the shuffle algorithm for determination of the matrices of the standard system (17) has been proposed. Using the equivalent standard system (17) all results based on the solution, concerning for example reachability, controllability, observability, and minimum energy control obtained for the standard discrete-time linear systems (17) can be immediately extended to singular system (1). An open problem is to extend this approach to singular 2D linear systems described by the singular 2D Roesser type model and the singular 2D Fornasini-Marchesini type models (Kaczorek 1992).

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ISBN 83-85847-78-2