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Plastic torsion and tension of naturally uniformly twisted bars

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THE PAPER deals with the problem of plastic torsion and tension of twisted rods made of perfectly plastic materials. The considerations are based on the Huber-Mises-Hencky yield condition, the Hencky-Ilyushin (or Levy-Mises) plasticity theory and the de St. Venant principle. The problem is solved in an original, curvilinear and oblique reference frame ξ, η, ζ, the stresses and strains being considered in a locally orthogonal coordinate system ξ, η, z. Such procedure enables us to find a relatively simple solution. In the co-ordinate system introduced all the fundamental equations are written, i.e., the geometric relations, the compatibility conditions and the equations of equilibrium. The paper is illustrated by examples concerning the plastic torsion of a naturally twisted rod of square cross-section, and the plastic tension of such a rod.

W pracy zajęto się rozwiązaniem plastycznego skręcania z rozciąganiem prętów zwitych, wykonanych z materiału idealnie plastycznego. Skorzystano z warunku plastyczności Hubera-Misesa-Hencky'ego, teorii plastyczności Hencky'ego-Iliuszina (lub Levy-Misesa) oraz zasady de St. Venanta. Problem rozwiązano w nowo wprowadzonym, krzywoliniowym, ukośnokątnym układzie odniesienia ξ, η, ζ, przy czym naprężenia i odkształcenia rozpatrywano w lokalnie ortogonalnym układzie współrzędnych ξ, η, z. Takie postępowanie pozwoliło na stosunkowo proste rozwiązanie zagadnienia. Dla nowo wprowadzonego układu wyprowadzono wszystkie podstawowe równania, a więc związki geometryczne, warunki nierozdzielności i warunki równowagi. Pracę zilustrowano przykładami plastycznego skręcania pręta zwitego o przekroju kwadratowym, oraz plastycznego rozciągania takiego pręta.

В работе решены задачи пластического кручения с растяжением витых стержней, выполненных из идеально-пластического материала. Использованы условия пластичности Губера-Мизеса-Генки, теории пластичности Генки-Ильюшина или Леви-Мизеса и принцип Сен-Венана. Задачи решаются в нововведённой криволинейной неортогональной системе координат ξ, η, ζ, причём напряжения и деформации рассмотрены в локально ортогональной системе ξ, η, z. Такой подход позволил получить относительно простым путём решение задачи. В принятой системе выведены все основные уравнения, т. е. геометрические соотношения, условия неразрывности и уравнения равновесия. Иллюстрацией метода являются решения задач о пластическом кручении витого стержня квадратного сечения и о пластическом растяжении этого стержня. Решения получены по методу малого параметра. Показано, что даже применяя метод малого параметра, трудно получить эффективные результаты, учитывающие все уравнения. Поэтому даны приближённые решения, удовлетворяющие уравнениям равновесия, условию текучести, краевым условиям и условиям непрерывности, что соответствует статически допустимому решению. Даны эффективные формулы для расчёта несущей способности при кручении и при растяжении витого стержня квадратного сечения.

1. Introductory remarks

AS A NATURALLY uniformly twisted bar we understand a bar created by simultaneous shift and proportional rotation of an arbitrary cross-section about a straight axis. The corresponding coefficient of proportionality ϑ_0 will be called "unit angle of natural twist".

The problem under consideration may be applied to determine the limit carrying capacity of naturally twisted bars e.g., spiral drills (which — usually made of brittle materials — show at elevated temperatures tendencies to plastic collapse), airscrews etc. This problem gives also a certain approximation for straight prismatic bars, subject to torsion and with geometry changes taken into account, since most effects are here similar (cf. W. OLSZAK [9, 10].

Elastic torsion of naturally twisted bars was analyzed by G. YU. DZHANELIDZE [2], and G. YU. DZHANELIDZE and A. I. LURIE [3]. They derived the basic equation of the problem and solved some simple examples. DZHANELIDZE analyzed also the case of tension. He determined normal stresses accompanying torsion and shearing stresses accompanying tension of naturally twisted bars. Elastic stability of such bars was investigated by L. S. LEIBENZON [7], and A. I. LURIE [8]. N. A. CHERNYSHEV [1] considered a similar problem for helical springs.

The investigations in the plastic range are very scarce. We mention here only an approximate analysis of fully plastic state of a helical bar, given by D. D. IVLEV [5]. B. R. SETH [11] considered finite plastic torsion of a straight bar of circular cross-section.

The present paper gives general equations describing plastic simultaneous torsion and tension of naturally uniformly twisted bars, and some particular solutions. The assumptions are as follows: Huber-Mises-Hencky yield condition, incompressibility, theory of small elastic plastic deformations (Hencky-Ilyushin; the Levy-Mises theory leads here to the same results), finally de Saint-Venant's principle. A new, convenient curvilinear system of coordinates is introduced. The lines ξ, η, ζ (described subsequently by the equations $\eta = \text{const}$, $\zeta = \text{const}$; $\zeta = \text{const}$, $\xi = \text{const}$; $\xi = \text{const}$, $\eta = \text{const}$) are not locally orthogonal, but we refer stresses and strains to the locally orthogonal system ξ, η, z. All the basic equations are derived and solved using this approach.

2. System of coordinates

For a naturally uniformly twisted bar with unit angle of natural twist ϑ_0 we introduce the following curvilinear system of coordinates:

$$\xi = x\cdot\cos\vartheta_0 z + y\cdot\sin\vartheta_0 z, \quad \eta = -x\cdot\sin\vartheta_0 z + y\cdot\cos\vartheta_0 z, \quad \zeta = z. \tag{2.1}$$

In these coordinates the problem of torsion with tension is reduced to a two-dimensional case, since all the derivatives of stresses and strains with respect to ζ vanish.

To derive basic equations of the theory of plasticity in the coordinates ξ, η, ζ we transform the stresses, strains and displacements to the locally orthogonal system ξ, η, z and transform the differentiation with respect to x, y, z into differentiation with respect to ξ, η, ζ (Fig. 1). Such an approach, used e.g, by DZHANELIDZE, seems to be the simplest in the case under consideration. Thus we express σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx} in terms of σ_ξ, σ_η, σ_z, $\tau_{\xi\eta}$, $\tau_{\eta z}$, $\tau_{z\xi}$. The transformation of stresses and strains (or strain rates) is a two-dimensional tensorial one:

$$\begin{aligned} &\sigma_x = \sigma_\xi\cos^2\vartheta_0\zeta + \sigma_\eta\sin^2\vartheta_0\zeta - 2\tau_{\xi\eta}\sin\vartheta_0\zeta\cos\vartheta_0\zeta,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\sigma_z = \sigma_z \end{aligned} \tag{2.2}$$

and the displacements are transformed as vectors:

$$\begin{aligned} &u_x = u_\xi\cos\vartheta_0\zeta - u_\eta\sin\vartheta_0\zeta, \quad u_y = u_\xi\sin\vartheta_0\zeta + u_\eta\cos\vartheta_0\zeta,\\ &u_z = u_z. \end{aligned} \tag{2.3}$$

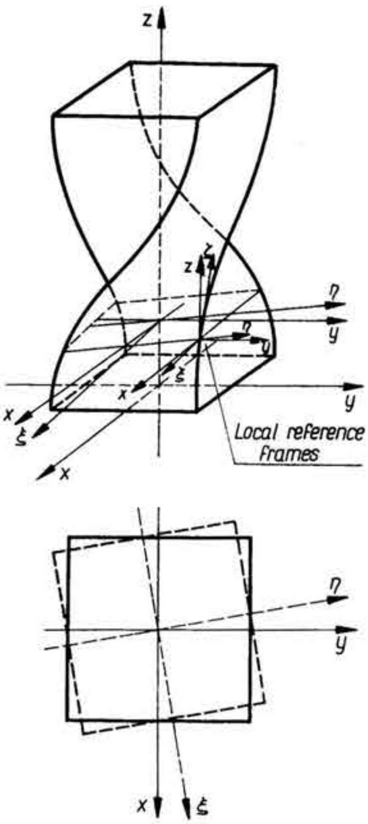

FIG. 1.

The differentiation is transformed as follows:

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}+\frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial x} = \cos\vartheta_0\zeta\frac{\partial}{\partial \xi}-\sin\vartheta_0\zeta\frac{\partial}{\partial \eta},\\
\frac{\partial}{\partial y} &= \sin\vartheta_0\zeta\frac{\partial}{\partial \xi}+\cos\vartheta_0\zeta\frac{\partial}{\partial \eta},\\
\frac{\partial}{\partial z} &= \vartheta_0\eta\frac{\partial}{\partial \xi}-\vartheta_0\xi\frac{\partial}{\partial \eta}+\frac{\partial}{\partial \zeta}.
\end{aligned}
\tag{2.4}
$$

In what follows, the derivatives of stresses and strains with respect to ζ will be omitted.

3. Basic equations of the continuum mechanics

Substituting (2.2) and (2.4) into the equations of internal equilibrium without body forces $\sigma_{ij,j} = 0$, we may write the two first equations in the general form

$$
\begin{aligned}
&\Omega_1[\sigma_{ij}]\sin\vartheta_0\zeta+\Omega_2[\sigma_{ij}]\cos\vartheta_0\zeta = 0,\\
&\Omega_1[\sigma_{ij}]\cos\vartheta_0\zeta-\Omega_2[\sigma_{ij}]\sin\vartheta_0\zeta = 0,
\end{aligned}
\tag{3.1}
$$

where Ω_1 and Ω_2 are some differential operators. Hence $\Omega_1[\sigma_{ij}] = \Omega_2[\sigma_{ij}] = 0$ and finally, with the third equation added, we arrive at the following system:

$$\begin{aligned}
&\frac{\partial\sigma_\xi}{\partial\xi} + \frac{\partial\tau_{\xi\eta}}{\partial\eta} + \vartheta_0\left(\eta\frac{\partial\tau_{\xi z}}{\partial\xi} - \xi\frac{\partial\tau_{\xi z}}{\partial\eta} - \tau_{\eta z}\right) = 0,\\
&\frac{\partial\tau_{\xi\eta}}{\partial\xi} + \frac{\partial\sigma_\eta}{\partial\eta} + \vartheta_0\left(\eta\frac{\partial\tau_{\eta z}}{\partial\xi} - \xi\frac{\partial\tau_{\eta z}}{\partial\eta} + \tau_{\xi z}\right) = 0,\\
&\frac{\partial\tau_{\xi z}}{\partial\xi} + \frac{\partial\tau_{\eta z}}{\partial\eta} + \vartheta_0\left(\eta\frac{\partial\sigma_z}{\partial\xi} - \xi\frac{\partial\sigma_z}{\partial\eta}\right) = 0.
\end{aligned} \tag{3.2}$$

Similarly, substituting the formulas analogous to (2.2) for strains, (2.3) and (2.4), into the Cauchy equations, $\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$ we compare the coefficients of the corresponding trigonometrical functions at both sides of subsequent equations and obtain

$$\begin{aligned}
&\varepsilon_\xi = \frac{\partial u_\xi}{\partial\xi}, \quad \varepsilon_\eta = \frac{\partial u_\eta}{\partial\eta}, \quad \gamma_{\xi\eta} = \frac{\partial u_\xi}{\partial\eta} + \frac{\partial u_\eta}{\partial\xi},\\
&\varepsilon_z = \frac{\partial u_z}{\partial\zeta} + \vartheta_0\left(\eta\frac{\partial u_z}{\partial\xi} - \xi\frac{\partial u_z}{\partial\eta}\right),\\
&\gamma_{\xi z} = \frac{\partial u_z}{\partial\xi} + \frac{\partial u_\xi}{\partial\zeta} + \vartheta_0\left(\eta\frac{\partial u_\xi}{\partial\xi} - \xi\frac{\partial u_\xi}{\partial\eta} - u_\eta\right),\\
&\gamma_{\eta z} = \frac{\partial u_z}{\partial\eta} + \frac{\partial u_\eta}{\partial\zeta} + \vartheta_0\left(\eta\frac{\partial u_\eta}{\partial\xi} - \xi\frac{\partial u_\eta}{\partial\eta} + u_\xi\right).
\end{aligned} \tag{3.3}$$

Continuing, we transform the compatibility conditions

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \tag{3.4}$$

to the form (the derivatives with respect to ζ being omitted)

$$\begin{aligned}
&\frac{\partial^2\varepsilon_\xi}{\partial\eta^2} + \frac{\partial^2\varepsilon_\eta}{\partial\xi^2} = \frac{\partial^2\gamma_{\xi\eta}}{\partial\xi\partial\eta},\\
&\frac{\partial^2\varepsilon_z}{\partial\xi^2} = \vartheta_0\left\{A[\gamma_{\xi z}] - \frac{\partial\gamma_{\eta z}}{\partial\xi}\right\} - \vartheta_0^2\left\{\eta A[\varepsilon_\xi] - \xi B[\varepsilon_\eta] - 2\left(\eta\frac{\partial\gamma_{\xi\eta}}{\partial\xi} - \xi\frac{\partial\gamma_{\xi\eta}}{\partial\eta} + \varepsilon_\xi - \varepsilon_\eta\right)\right\},\\
&\frac{\partial^2\varepsilon_z}{\partial\eta^2} = \vartheta_0\left\{B[\gamma_{\eta z}] + \frac{\partial\gamma_{\xi z}}{\partial\eta}\right\} - \vartheta_0^2\left\{\eta A[\varepsilon_\eta] - \xi B[\varepsilon_\xi] + 2\left(\eta\frac{\partial\gamma_{\xi\eta}}{\partial\xi} - \xi\frac{\partial\gamma_{\xi\eta}}{\partial\eta} + \varepsilon_\xi - \varepsilon_\eta\right)\right\},\\
&2\frac{\partial^2\varepsilon_z}{\partial\xi\partial\eta} = \vartheta_0\left\{A[\gamma_{\eta z}] + B[\gamma_{\xi z}] - \frac{\partial\gamma_{\eta z}}{\partial\eta} + \frac{\partial\gamma_{\xi z}}{\partial\xi}\right\} - \vartheta_0^2\left\{\eta A[\gamma_{\xi\eta}] - \xi B[\gamma_{\xi\eta}] + \right.\\
&\qquad\left. + 4\eta\left(\frac{\partial\varepsilon_\xi}{\partial\xi} - \frac{\partial\varepsilon_\eta}{\partial\xi}\right) - 4\xi\left(\frac{\partial\varepsilon_\xi}{\partial\eta} - \frac{\partial\varepsilon_\eta}{\partial\eta}\right) - 4\gamma_{\xi\eta}\right\},\\
&\frac{\partial}{\partial\xi}\left(\frac{\partial\gamma_{\eta z}}{\partial\xi} - \frac{\partial\gamma_{\xi z}}{\partial\eta}\right) = \vartheta_0\left\{-2B[\varepsilon_\xi] + A[\gamma_{\xi\eta}] + 2\frac{\partial\varepsilon_\xi}{\partial\xi} - 2\frac{\partial\varepsilon_\eta}{\partial\xi} + 2\frac{\partial\gamma_{\xi\eta}}{\partial\eta}\right\},\\
&\frac{\partial}{\partial\eta}\left(\frac{\partial\gamma_{\eta z}}{\partial\xi} - \frac{\partial\gamma_{\xi z}}{\partial\eta}\right) = \vartheta_0\left\{2A[\varepsilon_\eta] - B[\gamma_{\xi\eta}] - 2\frac{\partial\varepsilon_\xi}{\partial\eta} + 2\frac{\partial\varepsilon_\eta}{\partial\eta} + 2\frac{\partial\gamma_{\xi\eta}}{\partial\xi}\right\},
\end{aligned} \tag{3.5}$$

where the operators A and B are as follows:

$$A = \frac{\partial}{\partial \xi}\left(\eta \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \eta}\right), \quad B = \frac{\partial}{\partial \eta}\left(\eta \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial \eta}\right). \tag{3.6}$$

For small angles of natural twist ϑ_0, the first compatibility condition refers to the two-dimensional problem in the plane $\xi\eta$, the next three conditions — to the elongation ε_z, and the two last ones — to the problem of torsion.

Finally, we derive the boundary conditions. If the boundary surface of the body is described by the equation $F(x, y, z) = 0$, the stress boundary conditions may be written in Cartesian coordinates as follows (W. KRZYŚ, M. ŻYCZKOWSKI [5])

$$\pm p_{ni}\sqrt{F_{,j}\,F_{,j}} = F_{ik}\,\sigma_{,k}, \tag{3.7}$$

where p_{ni} denote surface tractions, $i = x, y, z$. Transforming these tractions according to (2.3), the stresses according to (2.2), the derivatives as indicated by (2.4), and bearing in mind that $\partial F/\partial \zeta = 0$, we arrive at:

$$\begin{aligned} \pm p_{n\xi}\sqrt{F_\xi'^2+F_\eta'^2+\vartheta_0^2(\eta F_\xi'-\xi F_\eta')^2} &= \sigma_\xi F_\xi' + \tau_{\xi\eta}F_\eta' + \vartheta_0(\eta F_\xi' - \xi F_\eta')\tau_{\xi z}, \\ \pm p_{n\eta}\sqrt{F_\xi'^2+F_\eta'^2+\vartheta_0^2(\eta F_\xi'-\xi F_\eta')^2} &= \tau_{\xi\eta} F_\xi' + \sigma_\eta F_\eta' + \vartheta_0(\eta F_\xi' - \xi F_\eta')\tau_{\eta z}, \\ \pm p_{nz}\sqrt{F_\xi'^2+F_\eta'^2+\vartheta_0^2(\eta F_\xi'-\xi F_\eta')^2} &= \tau_{\xi z} F_\xi' + \tau_{\eta z} F_\eta' + \vartheta_0(\eta F' - \xi F_\eta')\sigma_z. \end{aligned} \tag{3.8}$$

For a free surface the expressions on the right-hand side of these equations are equal to zero.

4. Theory of plasticity. Discontinuity lines

Since the stresses and strains are expressed in a locally orthogonal system, the physical equations of plasticity are written in their classical form

$$D_\varepsilon = \varphi D_\sigma, \tag{4.1}$$

where D_σ denotes the stress deviator; D_ε stands here for the strain deviator (the Hencky-Ilyushin theory) or, quite formally, for the strain-rate deviator (Levy-Mises theory).

The condition of incompressibility yields

$$\varepsilon_\xi + \varepsilon_\eta + \varepsilon_z = 0. \tag{4.2}$$

The Huber-Mises-Hencky yield condition will be written thus:

$$(\sigma_\xi - \sigma_\eta)^2 + (\sigma_\eta - \sigma_z)^2 + (\sigma_z - \sigma_\xi)^2 + 6(\tau_{\xi\eta}^2 + \tau_{\xi z}^2 + \tau_{z\eta}^2) = 2\sigma_0^2. \tag{4.3}$$

The problem of pure plastic torsion of prismatic bars results, as a rule, in a certain distribution of discontinuity lines. An analytical description of them is given by the present authors in [4]. It turns out, at least for bisymmetrical cross-sections, that this pattern of discontinuity lines remains unchanged in the case of uniformly naturally twisted bars with the same shape of the cross-section.

Consider, for example, a square bar (Fig. 2). For a prismatic bar the discontinuity lines coincide with the diagonals. Suppose that for a naturally twisted bar their position is changed as shown by the solid line. However, looking at the same cross-section from the other side we arrive at the same problem of torsion with the discontinuity lines shown

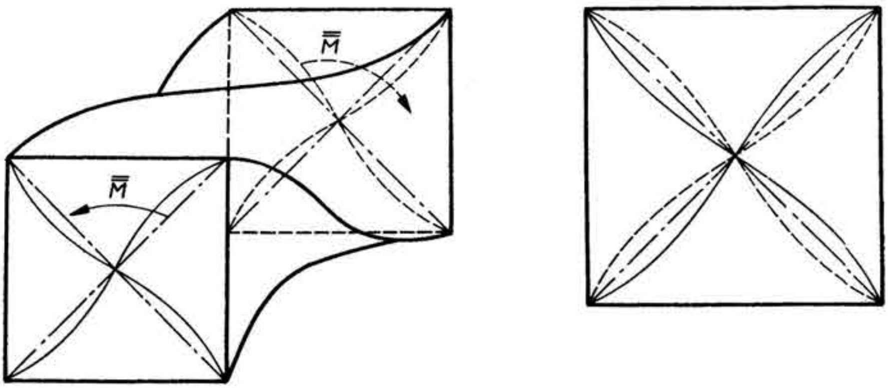

Fig. 2.

by the dashed line. This fact is contradictory, thus the discontinuity lines must coincide with the diagonals.

To derive the conditions of equilibrium along the discontinuity lines (discontinuity surfaces), we may use boundary conditions (3.8). Across the discontinuity lines the tractions must be continuous. Denoting the stresses in the two neighbouring zones (I) and (II) by $\sigma_\xi^{(I)}, \sigma_\eta^{(I)}$, ... and $\sigma_\xi^{(II)}\ \sigma_\eta^{(II)}$, ..., and the equation of the discontinuity line by $F(\xi, \eta) = 0$, we obtain from (3.8)

$$
\begin{aligned}
&(\sigma_\xi^{(I)}-\sigma_\xi^{(II)})F'_\xi+(\tau_{\xi\eta}^{(I)}-\tau_{\xi\eta}^{(II)})F'_\eta+\vartheta_0(\tau_{\xi z}^{(I)}-\tau_{\xi z}^{(II)})(\eta F'_\xi-\xi F'_\eta) = 0,\\
&(\tau_{\xi\eta}^{(I)}-\tau_{\xi\eta}^{(II)})F'_\xi+(\sigma_\eta^{(I)}-\sigma_\eta^{(II)})F'_\eta+\vartheta_0(\tau_{\eta z}^{(I)}-\tau_{\eta z}^{(II)})(\eta F'_\xi-\xi F'_\eta) = 0,\\
&(\tau_{\xi z}^{(I)}-\tau_{\xi z}^{(II)})F'_\xi+(\tau_{\eta z}^{(I)}-\tau_{\eta z}^{(II)})F'_\eta+\vartheta_0(\sigma_z^{(I)}-\sigma_z^{(II)})(\eta F'_\xi-\xi F'_\eta) = 0.
\end{aligned}
\tag{4.4}
$$

5. Application of the perturbation method

The problem under consideration is described by 16 equations [three equations of equilibrium (3.2), six geometrical Eqs. (3.3), six physical Eqs. (4.1) and (4.2) and the yield condition (4.3)] with 16 unknowns (stresses, strains or strain rates, displacements or velocities, and the function φ). In the limit state it is assumed that these equations hold for the body as a whole.

To obtain a relatively simple solution of the problem, we apply the perturbation method with ϑ_0 being a small parameter. Thus the zeroth-order approximation refers to the prismatic bar. The solution of the problems of pure torsion and pure tension are here well known; the problem of combined torsion with tension was examined in detail by M. Wnuk [12].

We present the solution in the form:

$$
\begin{aligned}
&\sigma_{ij} = \sum_{n=0}^{\infty} \sigma_{ijn}\vartheta_0^n, \qquad \varepsilon_{ij} = \sum_{n=0}^{\infty} \varepsilon_{ijn}\vartheta_0^n,\\
&u_i = \sum_{n=0}^{\infty} u_{in}\vartheta_0^n, \qquad \varphi = \sum_{n=0}^{\infty} \varphi_n\vartheta_0^n.
\end{aligned}
\tag{5.1}
$$

In the zeroth-order approximation, $\sigma_{\xi 0} = \sigma_{\eta 0} = \tau_{\xi\eta 0} = \gamma_{\xi\eta 0} = 0$; the equations may be reduced here to one second-order nonlinear partial differential equation for the displacement w, [12]. For the higher-order approximations $n \geqslant 1$, we obtain the following system of equations, linear with respect to subsequent unknowns:

$$\frac{\partial\sigma_{\xi n}}{\partial\xi} + \frac{\partial\tau_{\xi\eta n}}{\partial\eta} = -\eta\frac{\partial\tau_{\xi z(n-1)}}{\partial\xi} + \xi\frac{\partial\tau_{\xi z(n-1)}}{\partial\eta} + \tau_{\eta z(n-1)},$$

$$\frac{\partial\tau_{\xi\eta n}}{\partial\xi} + \frac{\partial\sigma_{\eta n}}{\partial\eta} = -\eta\frac{\partial\tau_{\eta z(n-1)}}{\partial\xi} + \xi\frac{\partial\tau_{\eta z(n-1)}}{\partial\eta} - \tau_{\xi z(n-1)},$$

$$\frac{\partial\tau_{\xi z n}}{\partial\xi} + \frac{\partial\tau_{\eta z n}}{\partial\eta} = -\eta\frac{\partial\sigma_{z(n-1)}}{\partial\xi} + \xi\frac{\partial\sigma_{z(n-1)}}{\partial\eta}, \qquad \frac{\partial^2\varepsilon_{\xi n}}{\partial\eta^2} + \frac{\partial^2\varepsilon_{\eta n}}{\partial\xi^2} - \frac{\partial^2\gamma_{\xi\eta n}}{\partial\xi\,\partial\eta} = 0,$$

$$\frac{\partial^2\varepsilon_{zn}}{\partial\xi^2} = A[\gamma_{\xi z(n-1)}] - \frac{\partial\gamma_{(\eta z n-1)}}{\partial\xi} - \eta A[\varepsilon_{\eta(n-2)}] - \xi B[\varepsilon_{\eta(n-2)}]$$
$$-2\left(\eta\frac{\partial\gamma_{\xi\eta(n-2)}}{\partial\xi} - \xi\frac{\partial\gamma_{\xi\eta(n-1)}}{\partial\eta} + \varepsilon_{\xi(n-2)} - \varepsilon_{\eta(n-2)}\right),$$

$$\frac{\partial^2\varepsilon_{zn}}{\partial\eta^2} = B[\gamma_{\eta z(n-1)}] + \frac{\partial\gamma_{\xi z(n-1)}}{\partial\eta} - \eta A[\varepsilon_{\eta(n-2)}] - \xi B[\varepsilon_{\xi(n-2)}]$$
$$+2\left(\eta\frac{\partial\gamma_{\xi\eta(n-2)}}{\partial\xi} - \xi\frac{\partial\gamma_{\xi\eta(n-2)}}{\partial\eta} + \varepsilon_{\xi(n-2)} - \varepsilon_{\eta(n-2)}\right),$$

$$2\frac{\partial^2\varepsilon_{zn}}{\partial\xi\partial\eta} = A[\gamma_{\eta z(n-1)}] + B[\gamma_{\xi z(n-1)}] - \frac{\partial\gamma_{\eta z(n-1)}}{\partial\eta} + \frac{\partial\gamma_{\xi z(n-1)}}{\partial\xi}$$
$$-\eta A[\gamma_{\xi\eta(n-2)}] + \xi B[\gamma_{\xi\eta(n-2)}] - 4\eta\left(\frac{\partial\varepsilon_{\xi(n-2)}}{\partial\xi} - \frac{\partial\varepsilon_{\eta(n-2)}}{\partial\xi}\right)$$
(5.2)
$$+4\xi\left(\frac{\partial\varepsilon_{\xi(n-2)}}{\partial\eta} - \frac{\partial\varepsilon_{\eta(n-2)}}{\partial\eta}\right) + 4\gamma_{\xi\eta(n-2)},$$

$$\frac{\partial^2\gamma_{\eta z n}}{\partial\xi^2} - \frac{\partial^2\gamma_{\xi z n}}{\partial\eta\partial\xi} = -2B[\varepsilon_{\xi(n-1)}] + A[\gamma_{\xi\eta(n-1)}] + 2\left(\frac{\partial\varepsilon_{\xi(n-1)}}{\partial\xi} - \frac{\partial\varepsilon_{\eta(n-1)}}{\partial\xi} + \frac{\partial\gamma_{\xi\eta(n-1)}}{\partial\eta}\right),$$

$$\frac{\partial^2\gamma_{\eta z n}}{\partial\xi\partial\eta} - \frac{\partial^2\gamma_{\xi z n}}{\partial\eta^2} = 2A[\varepsilon_{\eta(n-1)}] - B[\gamma_{\xi\eta(n-1)}] - 2\left(\frac{\partial\varepsilon_{\xi(n-1)}}{\partial\eta} - \frac{\partial\varepsilon_{\eta(n-1)}}{\partial\eta} - \frac{\partial\gamma_{\xi\eta(n-1)}}{\partial\xi}\right),$$

$$\varepsilon_{\xi n} + \varepsilon_{\eta n} + \varepsilon_{zn} = 0,$$

$$\varepsilon_{zn} - \varepsilon_{\xi n} = \sum_{i=0}^{n}\varphi_i(\sigma_{z(n-i)} - \sigma_{\xi(n-i)}),$$

$$\varepsilon_{zn} - \varepsilon_{\eta n} = \sum_{i=0}^{n}\varphi_i(\sigma_{z(n-i)} - \sigma_{\eta(n-i)}),$$

$$\gamma_{\xi\eta n} = 2\sum_{i=0}^{n}\varphi_i\tau_{\xi\eta(n-i)},$$

$$\gamma_{\eta z n} = 2\sum_{i=0}^{n} \varphi_i \tau_{\eta z(n-i)},$$

$$\gamma_{\xi z n} = 2\sum_{i=0}^{n} \varphi_i \tau_{\xi z(n-i)},$$

$$\sum_{i=0}^{n} [\sigma_{\xi i}\sigma_{\xi(n-i)} + \sigma_{\eta i}\sigma_{\eta(n-i)} + \sigma_{zi}\sigma_{z(n-i)} - \sigma_{\xi i}\sigma_{\eta(n-i)} - \sigma_{\eta i}\sigma_{z(n-i)}$$
$$- \sigma_{zi}\sigma_{\xi(n-i)} + 3\tau_{\xi\eta i}\tau_{\xi\eta(n-i)} + 3\tau_{\xi zi}\tau_{\xi z(n-i)} + 3\tau_{\eta zi}\tau_{\eta z(n-i)} = 0.$$

The operators A and B are here determined by (3.6). In some cases it is more convenient to replace the compatibility conditions by the relations (3.3), which may be expanded into power series without any trouble.

The boundary conditions (3.8) at the free lateral surface of the bar yield

$$\begin{aligned} \sigma_{\xi n}F'_\xi + \tau_{\xi\eta n}F'_\eta &= -(\eta F'_\xi - \xi F'_\eta)\tau_{\xi z(n-1)}, \\ \tau_{\xi\eta n}F'_\xi + \sigma_{\eta n}F'_\eta &= -(\eta F'_\xi - \xi F'_\eta)\tau_{\eta z(n-1)}, \\ \tau_{\xi z n}F'_\xi + \tau_{\eta z n}F'_\eta &= -(\eta F'_\xi - \xi F'_\eta)\sigma_{z(n-1)}. \end{aligned} \tag{5.3}$$

Similar conditions along the discontinuity lines may be obtained from (4.4). Effective solutions of the system (5.2) with boundary conditions (5.3) depend on the shape of the cross-section of the bar.

External loadings, determining the limit carrying capacity of the bar (plastic interaction curve in the combined case) are given by the formulas

$$\begin{aligned} N &= \sum_{n=0}^{\infty} \vartheta_0^n \iint_A \sigma_{zn}\, dA, \\ M &= \sum_{n=0}^{\infty} \vartheta_0^n \iint_A (\tau_{\eta z n}\xi - \tau_{\xi z n}\eta)\, dA, \end{aligned} \tag{5.4}$$

A being the cross-sectional area of the bar.

6. Example of pure torsion

6.1. Basic equations

As the first example, let us consider pure torsion of a bar with square cross-section with the side $2a$. The zeroth approximation, corresponding to a prismatic bar, is here very simple. In view of the fourfold symmetry of the cross-section, let us consider only the octant I, $0 \leqslant \xi \leqslant a$, $0 \leqslant \eta \leqslant \xi$, and conditions along the borders with zones II and III (Fig. 3). The symmetry conditions lead namely to the following relations between the stresses and strains inside the zones I, II, and III. The functions σ_ξ, σ_η, σ_z, $\tau_{z\eta}$, the corresponding strains and φ being symmetrical with respect to the axis ξ, thus fulfil the relation $f^{(\mathrm{II})}(\xi, \eta) = f^{(\mathrm{I})}(\xi, -\eta)$. The functions $\tau_{z\xi}$, $\tau_{\xi\eta}$, $\gamma_{z\xi}$ and $\gamma_{\xi\eta}$ are antisymmetrical ones, thus $f^{(\mathrm{II})}(\xi, \eta) = -f^{(\mathrm{I})}(\xi, \eta)$. Further, the solution for the zone III may be obtained from the zone II by a simple rotation of the bar. Thus $\sigma_z^{(\mathrm{III})}(\xi, \eta) = \sigma_z^{(\mathrm{II})}(\eta, -\xi)$, and

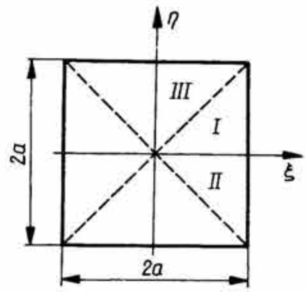

FIG. 3.

similarly ε_z and φ, $\sigma_\xi^{(III)}(\xi, \eta) = \sigma_\eta^{(II)}(\eta, -\xi)$, and similarly σ_η, $\tau_{z\eta}$, ε_ξ, ε_η, and $\gamma_{z\eta}$, finally $\tau_{\xi\eta}^{(III)}(\xi, \eta) = -\tau_{\xi\eta}^{(II)}(\eta, -\xi)$, and similarly $\tau_{z\xi}$, $\gamma_{\xi\eta}$ and $\gamma_{z\xi}$.

For the octant I of the prismatic bar we have in this case

$$\tau_{\xi z0} = 0, \quad \tau_{\eta z0} = \frac{\sigma_0}{\sqrt{3}} = k,$$

(6.1) $$\gamma_{\xi z0} = 0, \quad \gamma_{\eta z0} = 2\vartheta(\xi-\eta), \quad \varphi_0 = \frac{\vartheta}{k}(\xi-\eta),$$

$$\sigma_{\xi 0} = \sigma_{\eta 0} = \sigma_{z0} = \varepsilon_{\xi 0} = \varepsilon_{\eta 0} = \varepsilon_{z0} = 0.$$

In these formulas ϑ denotes the unit angle of twist, due to the action of the twisting moment M (Hencky-Ilyushin theory; in the limit state we have to assume here $\vartheta \to \infty$), or the rate of the unit angle of twist (Levy-Mises theory).

Discussing now the first approximation, $n = 1$, we may split the system (5.2) into three parts. The third, eigth, ninth, fourteenth, fifteenth and sixteenth equations form a linear homogeneous system with respect to the unknowns: $\tau_{\xi z1}$, $\tau_{\eta z1}$, $\gamma_{\xi z1}$, $\gamma_{\eta z1}$ and φ_1; the corresponding third boundary condition (5.3) is also homogeneous, thus $\tau_{\xi z1} = \tau_{\eta z1} = \gamma_{\xi z1} = \gamma_{\eta z1} = \varphi_1 = 0$. This result is very important, since it turns out that the natural twist has no influence on the limit torque in the first approximation, its effect will be seen only through the second-order terms. The fifth, sixth and seventh equations determine the strain ε_{z1} — namely:

$$\frac{\partial^2 \varepsilon_{z1}}{\partial \xi^2} = A[\gamma_{\xi z0}] - \frac{\partial \gamma_{\eta z0}}{\partial \xi},$$

(6.2) $$\frac{\partial^2 \varepsilon_{z1}}{\partial \eta^2} = B[\gamma_{\eta z0}] + \frac{\partial \gamma_{\xi z0}}{\partial \eta},$$

$$2\frac{\partial^2 \xi_{z1}}{\partial \xi \partial \eta} = A[\gamma_{\eta z0}] + B[\gamma_{\xi z0}] - \frac{\partial \gamma_{\eta z0}}{\partial \eta} + \frac{\partial \gamma_{\xi z0}}{\partial \xi}.$$

Hence, after substitution and integration:

(6.3) $$\varepsilon_{z1} = (-\xi^2 + 2\xi\eta + \eta^2 + C_1\xi + C_2\eta + C_3)\vartheta.$$

The remaining seven equations form a system analogous to that describing the plane problem of the theory of elasticity with body forces:

$$(6.4)\qquad \begin{gathered} \frac{\partial\sigma_{\xi 1}}{\partial\xi}+\frac{\partial\tau_{\xi\eta 1}}{\partial\eta}=k,\quad \frac{\partial\tau_{\xi\eta 1}}{\partial\xi}+\frac{\partial\sigma_{\eta 1}}{\partial\eta}=0,\quad \frac{\partial^2\varepsilon_{\xi 1}}{\partial\eta^2}+\frac{\partial^2\varepsilon_{\eta 1}}{\partial\xi^2}=\frac{\partial^2\gamma_{\xi\eta 1}}{\partial\xi\partial\eta},\\ \varepsilon_{z1}-\varepsilon_{\xi 1}=\varphi_0(\sigma_{z1}-\sigma_{\xi 1}),\quad \varepsilon_{z1}-\varepsilon_{\eta 1}=\varphi_0(\sigma_{z1}-\sigma_{\eta 1}),\\ \gamma_{\xi\eta 1}=2\varphi_0\tau_{\xi\eta 1},\quad \varepsilon_{\xi 1}+\varepsilon_{\eta 1}+\varepsilon_{z1}=0. \end{gathered}$$

We introduce the stress function ϕ_1,

$$(6.5)\qquad \sigma_{\xi 1}=\frac{\partial^2\phi_1}{\partial\eta^2},\quad \sigma_{\eta 1}=\frac{\partial^2\phi_1}{\partial\xi^2},\quad \tau_{\xi\eta 1}=-\frac{\partial^2\phi_1}{\partial\xi\partial\eta}+k\eta$$

and express σ_{z1} and the strains in terms of ϕ_1; substituting them into the compatibility condition, we obtain the following linear partial differential equation of the fourth order:

$$(6.6)\qquad (\xi-\eta)\nabla^4\phi_1+2\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)\nabla^2\phi_1=4k.$$

This equation must be satisfied within the triangle $0\leqslant\xi\leqslant a$, $0<\eta<\xi$. The boundary conditions (5.3) are here as follows:

$$(6.7)\qquad \frac{\partial^2\phi_1}{\partial\eta^2}=0,\quad \frac{\partial^2\phi_1}{\partial\xi\partial\eta}=2k\eta,\quad \text{along } \xi=a.$$

The continuity conditions (4.4) along $\eta=0$ yield at first

$$(6.8)\qquad \tau^{(\mathrm{I})}_{\xi\eta 1}=\tau^{(\mathrm{II})}_{\xi\eta 1},\quad \sigma^{(\mathrm{I})}_{\eta 1}=\sigma^{(\mathrm{II})}_{\eta 1};$$

now, $\tau_{\xi\eta}$ is an antisymmetrical function of η, thus simply $\tau_{\xi\eta 1}=0$ along $\eta=0$, and the second condition is fulfilled automatically in view of the symmetry of σ_η. The conditions (6.8) may be rewritten in the form

$$(6.9)\qquad \frac{\partial^2\phi_1}{\partial\xi\partial\eta}=0\quad \text{along } \eta=0.$$

The continuity conditions (4.4) along $\xi-\eta=0$ yield at first

$$(6.10)\qquad \begin{gathered} \sigma^{(\mathrm{I})}_{\xi 1}-\sigma^{(\mathrm{III})}_{\xi 1}-\tau^{(\mathrm{I})}_{\xi\eta 1}+\tau^{(\mathrm{III})}_{\xi\eta 1}+2\eta(\tau^{(\mathrm{I})}_{\xi z0}-\tau^{(\mathrm{III})}_{\xi z0})=0,\\ \tau^{(\mathrm{I})}_{\xi\eta 1}-\tau^{(\mathrm{III})}_{\xi\eta 1}-\sigma^{(\mathrm{I})}_{\eta 1}+\sigma^{(\mathrm{III})}_{\eta 1}+2\eta(\tau^{(\mathrm{I})}_{\eta z0}-\tau^{(\mathrm{III})}_{\eta z0})=0. \end{gathered}$$

Expressing the stresses in the zone III by those in the zone II, and subsequently in the zone I, we find $\sigma^{(\mathrm{III})}_{\xi 1}(\xi,\eta)=\sigma^{(\mathrm{II})}_{\eta 1}(\eta,-\xi)=\sigma^{(\mathrm{I})}_{\eta 1}(\eta,\xi)$, similarly $\sigma^{(\mathrm{III})}_{\eta 1}$, and $\tau^{(\mathrm{III})}_{\xi\eta 1}(\xi,\eta)=-\tau^{(\mathrm{II})}_{\xi\eta 1}(\eta,-\xi)=\tau^{(\mathrm{I})}_{\xi\eta 1}(\eta,\xi)$;further $\tau^{(\mathrm{I})}_{\xi z0}=\tau^{(\mathrm{III})}_{\eta z0}=0$, $\tau^{(\mathrm{III})}_{\xi z0}=-k$, $\tau^{(\mathrm{I})}_{\eta z0}=k$, and both Eqs. (6.10) give the same result

$$(6.11)\qquad \sigma^{(\mathrm{I})}_{\xi 1}-\sigma^{(\mathrm{I})}_{\eta 1}+2\eta k=0$$

or

$$(6.12)\qquad \frac{\partial^2\phi_1}{\partial\xi^2}-\frac{\partial^2\phi_1}{\partial\eta^2}=2\eta k\quad \text{along } \xi=\eta.$$

The Eq. (6.6) with the boundary conditions (6.7), (6.9) and (6.12) determines the distribution of the first-order corrections in the naturally twisted bar.

As already indicated these corrections have no influence on the limit torque. To evaluate this influence we have to discuss the second approximation, at least. Considering this approximation, we find the situation quite opposite to that at the stage of the first one. Now, the system of the first, second, fourth, fifth, sixth, seventh, tenth, eleventh, twelfth and thirteenth Eqs. (5.2) is a linear homogeneous system with respect to the unknowns $\sigma_{\xi 2}$, $\sigma_{\eta 2}$, σ_{z2}, $\tau_{\xi\eta 2}$, $\varepsilon_{\xi 2}$, $\varepsilon_{\eta 2}$, ε_{z2} and $\gamma_{\xi\eta 2}$; the corresponding boundary conditions are also homogeneous thus all these unknowns are equal to zero.

The remaining second-order corrections are determined by the remaining equations. The last equation — the expanded yield condition — makes it possible to evaluate directly the stress $\tau_{z\eta 2}$, namely:

$$\tau_{z\eta 2} = -\frac{1}{6k}(\sigma_{\xi 1}^2+\sigma_{\eta 1}^2+\sigma_{z1}^2-\sigma_{\xi 1}\sigma_{\eta 1}-\sigma_{\eta 1}\sigma_{z1}-\sigma_{z1}\sigma_{\xi 1}+3\tau_{\xi\eta 1}^2). \tag{6.13}$$

The third equation of equilibrium may now be used to determine $\tau_{\xi z2}$:

$$\frac{\partial\tau_{\xi z2}}{\partial\xi}-\frac{\partial\tau_{z\eta 2}}{\partial\eta} = -\eta\frac{\partial\sigma_{z1}}{\partial\xi}+\xi\frac{\partial\sigma_{z1}}{\partial\eta}. \tag{6.14}$$

This equation is furnished with the boundary condition

$$\tau_{z\xi 2} = -\eta\sigma_{z1} \quad \text{for} \quad \xi = a. \tag{6.15}$$

The remaining compatibility conditions may be rearranged (making use of the compatibility condition joining $\varepsilon_{\xi 1}$, $\varepsilon_{\eta 1}$, and $\gamma_{\xi\eta 1}$) and integrated subsequently with respect to ξ and η. Then they are reduced to one common equation

$$\frac{\partial\gamma_{\eta z2}}{\partial\xi}-\frac{\partial\gamma_{\xi z2}}{\partial\eta} = -2\eta\frac{\partial\varepsilon_{\xi 1}}{\partial\eta}-2\xi\frac{\partial\varepsilon_{\eta 1}}{\partial\xi}+\xi\frac{\partial\gamma_{\xi\eta 1}}{\partial\eta}+\eta\frac{\partial\gamma_{\xi\eta 1}}{\partial\xi}+C, \tag{6.16}$$

which together with the remaining physical equations

$$\gamma_{\xi z2} = \frac{2\vartheta}{k}(\xi-\eta)\tau_{\xi z2}, \quad \gamma_{\eta z2} = \frac{2\vartheta}{k}(\xi-\eta)\tau_{\eta z2}+2\varphi_2 k \tag{6.17}$$

determine the unknowns $\gamma_{\xi z2}$, $\gamma_{\eta z2}$, and φ_2.

The formulas for $\tau_{\xi z2}$ and $\tau_{\eta z2}$ make it possible to find the second-order correction for the limit twisting moment $\overline{\overline{M}}$.

In the odd higher-order terms we have always $\tau_{\xi zn} = \tau_{z\eta n} = \gamma_{\xi zn} = \gamma_{\eta zn} = \varphi_n = 0$, and in the even ones we have $\sigma_{\xi n} = \sigma_{\eta n} = \sigma_{zn} = \tau_{\xi\eta n} = \varepsilon_{\xi n} = \varepsilon_{\eta n} = \varepsilon_{zn} = \gamma_{\xi\eta n} = 0$.

6.2. A statically admissible solution

Even in spite of the perturbation method applied, the solution of the system of linear equations obtained is complicated. For example, the fourth-order partial differential Eq. (6.6), singular along the line $\xi-\eta = 0$, is to be solved with the boundary condition (6.12) along this line. To obtain a simple analytical solution we apply here a statically admissible approach, solving only the equations of equilibrium, yield condition, the boundary conditions, and the conditions of continuity along the discontinuity lines. At each stage we have four equations with six unknown functions (stresses), thus two functions will be chosen arbitrarily. The limit twisting moment $\overline{\overline{M}}$ obtained then gives a lower estimation of the exact solution.

The zeroth-order approximation is determined exactly, (6.1). Proceeding to the first-order approximation we find first the only statically admissible distribution of $\tau_{\xi z1}$ and $\tau_{\eta z1}$ — namely, $\tau_{\xi z1} = \tau_{\eta z1} = 0$. The remaining equations do not contain the unknown σ_{z1}, which will appear only in the equations of the second approximation. Thus we may choose here only one function arbitrarily. We assume $\tau_{\xi\eta 1}$ in the form:

$$\tau_{\xi\eta 1} = -\eta k\left[\alpha+(1-\alpha)\frac{\xi}{a}\right], \tag{6.18}$$

where α is a free dimensionless parameter, to be found later from the condition of maximal twisting moment; the corresponding boundary condition (6.7) is already satisfied.

Integrating the equations of equilibrium and making use of the boundary condition for $\sigma_{\xi 1}$ and of the continuity condition along $\xi-\eta = 0$, we find

$$\begin{aligned} \sigma_{\xi 1} &= k\left[(1+\alpha)(\xi-a)+(1-\alpha)\frac{\xi^2-a^2}{2a}\right], \\ \sigma_{\eta 1} &= 2k\xi+k\left[(1+\alpha)(\xi-a)+(1-\alpha)\frac{\eta^2-a^2}{2a}\right]. \end{aligned} \tag{6.19}$$

Starting with the second-order approximation, we assume first $\sigma_{\xi 2} = \sigma_{\eta 2} = \tau_{\xi\eta 2} = 0$. Although these functions are determined only by the two first equations of equilibrium and one of them may be chosen arbitrarily, the assumed choice is the best one, since — as we found in sec. 6.1. — it is exact. The remaining two equations contain σ_{z1}, $\tau_{\xi z2}$, and $\tau_{\eta z2}$. The distribution of the stress σ_{z1} must correspond to zero normal force $N_1 = 0$. We assume, quite arbitrarily, $\sigma_{z1} = 0$, then the yield condition (6.13) determines $\tau_{\eta z2}$:

$$\begin{aligned} \tau_{\eta z2} = -\frac{k}{6}\Bigg[4\xi^2 &+ \frac{(1-\alpha)^2}{4a^2}(\xi^4+\eta^4+a^4+11\xi^2\eta^2-\xi^2a^2-\eta^2a^2) \\ &+ \frac{1-\alpha^2}{2a}(\xi^3+\eta^2\xi-\xi^2a-\eta^2a-2\xi a^2+2a^3)+2(1+\alpha)(\xi^2-\xi a) \\ &+ \frac{1-\alpha}{a}(2\xi\eta^2-\xi a^2-\xi^3)+6\alpha\frac{1-\alpha}{a}\eta^2\xi+3\alpha^2\eta^2+(1+\alpha)^2(\xi-a)^2\Bigg]. \end{aligned} \tag{6.20}$$

The third equation of equilibrium and the boundary condition (6.15) determines finally $\tau_{\xi z2}$:

$$\begin{aligned} \tau_{\xi z2} = \frac{k}{6}\Bigg[\frac{(1-\alpha)^2}{4a^2}&\left(4\eta^3\xi-4\eta^3a+\frac{22}{3}\eta\xi^3-\frac{22}{3}\eta a^3-2\eta\xi a^2+2\eta a^3\right) \\ &+\frac{1-\alpha^2}{2a}(\eta\xi^2-\eta a^2-2\eta\xi a+2\eta a^2)+\frac{1-\alpha}{a}(2\eta\xi^2-2\eta a^2) \\ &+6\alpha\frac{1-\alpha}{a}(\eta\xi^2-\eta a^2)+6\alpha^2(\eta\xi-\eta a)\Bigg]. \end{aligned} \tag{6.21}$$

Thus the second-order approximation of the statically admissible solution is fully determined. The most interesting stresses $\tau_{\eta z}$ and $\tau_{\xi z}$ are shown in Fig. 4 in the zeroth and in the second approximation for $\vartheta_0 a = 0.3$.

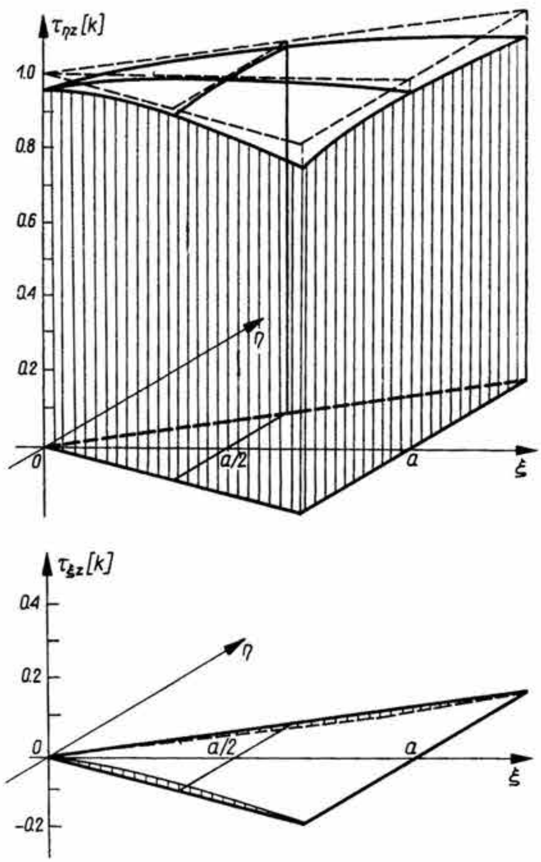

FIG. 4.

The limit twisting moment M is given by the integral (5.4) taken over the considered octant and multiplied by 8:

$$M = \frac{8}{3}ka^3 - \frac{2}{3}ka^5(1.128+0.4734\alpha+0.0634\alpha^2)\vartheta_0^2 + \dots \tag{6.22}$$

The optimal value of the parameter α corresponds to the maximal moment M, thus to the minimal value is in the bracket; we obtain $\alpha = -3.74$, and the best approximation is as follows:

$$\overline{\overline{M}} = \frac{8}{3}ka^3 - 0.124ka^5\vartheta_0^2 + \dots \tag{6.23}$$

This moment cannot be smaller than the limit moment for the circular bar with the radius a (inscribed in the considered square bar), $M_0 = \frac{2}{3}\pi ka^3$. The last value may be regarded as asymptotic, good for $\vartheta_0 \to \infty$. To improve the accuracy of the series (6.23) we propose a certain approximation, which takes into account also the asymptotic value. Assume, namely, the following approximate formula:

$$\overline{\overline{M}} = \frac{8}{3} ka^3 \frac{1+B\vartheta_0^2}{1+C\vartheta_0^2}. \tag{6.24}$$

The requirements of the agreement of the expansion of (6.24) with (6.23) and of the given asymptotic value determine B and C. Thus for practical application we propose the formula

$$\overline{\overline{M}} = \frac{8}{3} ka^3 \frac{1+0.170\,\vartheta_0^2 a^2}{1+0.216\,\vartheta_0^2 a^2}. \tag{6.25}$$

The dependency of the limit twisting moment $\overline{\overline{M}}$ on the unit angle of natural twist ϑ_0 is shown in Fig. 5.

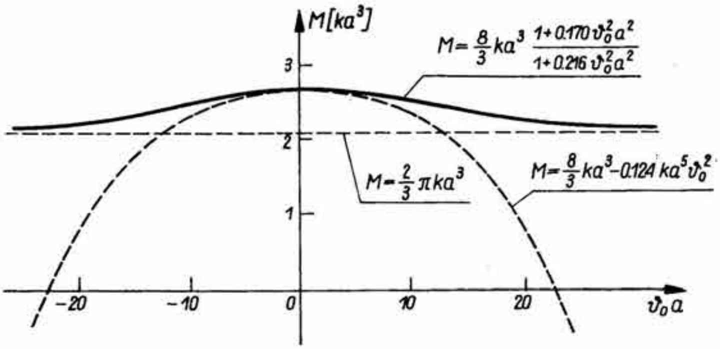

Fig. 5.

7. Example of pure tension

As the second example we consider pure tension of a naturally twisted bar with square cross-section. The perturbation method combined with the statically admissible approach will be used.

The zeroth approximation refers to a prismatic bar, thus $\sigma_{z0} = \sigma_0$, and the remaining stresses vanish.

The first approximation in stresses is determined by four homogeneous equations

$$\begin{gathered}
\frac{\partial \sigma_{\xi 1}}{\partial \xi} + \frac{\partial \tau_{\xi\eta 1}}{\partial \eta} = 0, \qquad \frac{\partial \tau_{\xi z 1}}{\partial \xi} + \frac{\partial \tau_{\eta z 1}}{\partial \eta} = 0, \\
\frac{\partial \tau_{\xi\eta 1}}{\partial \xi} + \frac{\partial \sigma_{\eta 1}}{\partial \eta} = 0, \qquad 2\sigma_{z1} - \sigma_{\xi 1} - \sigma_{\eta 1} = 0,
\end{gathered} \tag{7.1}$$

with the boundary conditions, some of which are non-homogeneous:

$$\begin{aligned}
&\sigma_{\xi 1} = 0, \quad \tau_{\xi\eta 1} = 0, \quad \tau_{\xi z 1} = -\eta\sigma_0 \quad \text{for} \quad \xi = \pm a, \\
&\sigma_{\eta 1} = 0, \quad \tau_{\xi\eta 1} = 0, \quad \tau_{\eta z 1} = \xi\sigma_0 \quad \text{for} \quad \eta = \pm a.
\end{aligned} \tag{7.2}$$

We assume $\sigma_{\xi 1} = \sigma_{\eta 1} = \sigma_{z1} = \tau_{\xi\eta 1} = 0$ (this assumption may be proved to be exact), further

$$\tau_{\xi z1} = \eta\sigma_0\left(-1-\alpha\frac{\xi^2-a^2}{a^2}\right),$$
(7.3)
$$\tau_{\eta z1} = \xi\sigma_0\left(1+\alpha\frac{\eta^2-a^2}{a^2}\right).$$

The coefficient α cannot be regarded here as free, since we have to impose the condition of no twisting moment $M_1 = 0$ (condition of external equilibrium) hence $\alpha = 3/2$. The first approximation has no influence on the limit normal force N.

The second approximation in stresses is determined by the following four equations:

$$\frac{\partial\sigma_{\xi 2}}{\partial\xi}+\frac{\partial\tau_{\xi\eta 2}}{\partial\eta} = \frac{3}{2}\sigma_0\xi\frac{3\eta^2-\xi^2}{a^2},$$
(7.4)
$$\frac{\partial\tau_{\xi\eta 2}}{\partial\xi}+\frac{\partial\sigma_{\eta 2}}{\partial\eta} = \frac{3}{2}\sigma_0\eta\frac{3\xi^2-\eta^2}{a^2},$$
$$\frac{\partial\tau_{\xi z2}}{\partial\xi}+\frac{\partial\tau_{\eta z2}}{\partial\eta} = 0,$$
$$2\sigma_{z2}-\sigma_{\xi 2}-\sigma_{\eta 2} = \sigma_0\left[\frac{9\xi^2\eta^2}{a^2}-\frac{1}{4}\left(\frac{27\xi^2\eta^2}{a^4}-3\right)(\xi^2+\eta^2)\right],$$

with the boundary conditions

$$\sigma_{\xi 2} = \eta^2\sigma_0,\quad \tau_{\xi\eta 2} = -\eta a\sigma_0\left(1+\frac{3}{2}\frac{\eta^2-a^2}{a^2}\right),\quad \tau_{\xi z2} = 0\quad \text{for}\quad \xi = \pm a,$$
(7.5)
$$\sigma_{\eta 2} = \xi^2\sigma_0,\quad \tau_{\xi\eta 2} = -\xi a\sigma_0\left(1+\frac{3}{2}\frac{\xi^2-a^2}{a^2}\right),\quad \tau_{\eta z2} = 0\quad \text{for}\quad \eta = \pm a.$$

We assume $\tau_{\xi z2} = \tau_{\eta z2} = 0$ (this assumption may be proved to be exact), further, to satisfy the boundary conditions,

(7.6)
$$\tau_{\xi\eta 2} = -\xi\eta\sigma_0\left(\frac{3}{2}\frac{\xi^2+\eta^2}{a^2}-2\right).$$

The equilibrium equations with the remaining boundary conditions determine now the stresses $\sigma_{\xi 2}$ and $\sigma_{\eta 2}$:

$$\sigma_{\xi 2} = \sigma_0\left[\frac{\eta^2}{2a^2}(9\xi^2-7a^2)-(\xi^2-a^2)\right],$$
(7.7)
$$\sigma_{\eta 2} = \sigma_0\left[\frac{\xi^2}{2a^2}(9\eta^2-7a^2)-(\eta^2-a^2)\right],$$

and the yield condition gives the distribution of σ_{z2}:

(7.8)
$$\sigma_{z2} = \sigma_0\left[9\frac{\xi^2\eta^2}{a^2}-\left(\frac{27}{8}\frac{\xi^2\eta^2}{a^4}+\frac{21}{8}\right)(\xi^2+\eta^2)+a^2\right].$$

Integrating the stresses σ_z over the cross-sectional area we find the limit normal force

$$\overline{\overline{N}} = \iint\limits_A \sigma_z dA = 4a^2\sigma_0\left(1-\frac{1}{5}\vartheta_0^2a^2+\ldots\right). \tag{7.9}$$

This force cannot be smaller that the limit force for the circular bar with the radius a (inscribed in the square bar considered), $N_0 = \pi a^2\sigma_0$. Treating the last value as an asymptotic one, we assume the following approximation

$$\overline{\overline{N}} = 4a^2\sigma_0\frac{1+B\vartheta_0^2}{1+C\vartheta_0^2}; \tag{7.10}$$

the requirements of the agreement of the expansion of (7.10) with (7.9) and of the given asymptotic value determine B and C. Thus for practical applications we propose the formula

$$\overline{\overline{N}} = 4a^2\sigma_0\frac{1+0.733\vartheta_0^2a^2}{1+0.933\vartheta_0^2a^2}. \tag{7.11}$$

The dependency of the limit normal force N on the unit angle of natural twist ϑ_0 is presented in Fig. 6.

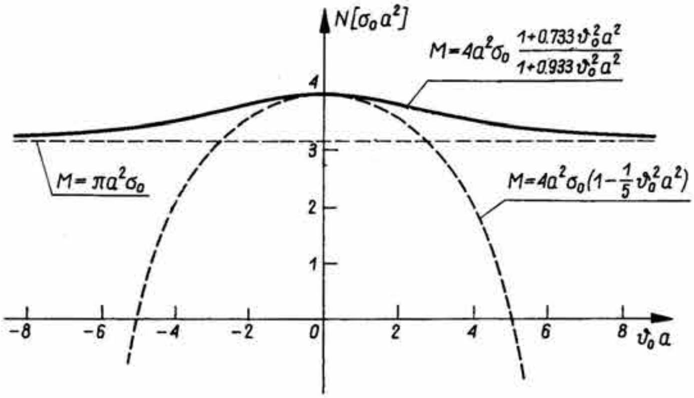

FIG. 6.

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