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Optimum design of vibrating beams under axial compression

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A solution is found to the problem of determining the shape of a simply supported laterally vibrating beam that has the highest possible value of the first fundamental frequency under a given axial compressive load. The length, volume and the material properties of the beam are assumed to be given. The cross-section is allowed to vary in such a way that the bending rigidity is proportional to the first, second or third power of the mass per unit length of the beam.

Znaleziono rozwiązania zagadnienia wyznaczania takiego kształtu swobodnie podpartych belek drgających w kierunku poprzecznym, dla którego przy danej sile ściskającej podstawowa częstość drgań osiąga wielkość maksymalną. Długość, objętość i własności materiałowe belki przyjęto jako wielkości ustalone. Uwzględniono takie zmienności przekroju poprzecznego belki, przy których sztywność zginania jest proporcjonalna do pierwszej, drugiej lub trzeciej potęgi gęstości liniowej belki.

Найдены решения проблемы определения такой формы свободно подпертых колеблющихся балок в поперечном направлении, для которой — при заданной сжимающей силе — основная частота колебаний достигает максимальной величины. Длина, объем и материальные свойства балки приняты как установленные величины. Учтены такие изменения поперечного сечения балки, при которых жесткость изгиба пропорциональна первой, второй или третей степени линейной плотности балки.

1. Introduction

THE PRESENT paper is devoted to determining the shape of a simply supported vibrating beam that has the highest possible value of the first fundamental frequency ω_0. The beam, which is made of a linearly elastic material, is undergoing small, harmonic vibrations in the lateral direction and is subjected to a given axial compressive load that may be less than, equal to or greater than the Euler buckling load for a corresponding beam of uniform cross-section. In particular, we are interested in evaluating the effect of axial compression on the shape of the beam, optimally designed in the above sense.

Problems of the optimum design of vibrating elastic elements have received considerable attention in recent years. NIORDSON [1], considering a simply supported beam with geometrically similar cross-sections, maximized the fundamental frequency of transverse vibrations by appropriate tapering of the beam. BRACH [2] considered a whole group of Bernoulli-Euler beams with all sets of homogeneous boundary conditions and optimized their shape with respect to the fundamental frequency of transverse oscillation. The cross-section in these case was allowed to vary in such a way that the second area moment was linearly related to the area. In particular, it was shown that the fundamental frequency of a cantilever and a free-free beam is unbounded. This result has been verified for a cantilever beam by more elaborate mathematical and physical reasoning by KARIHALOO and NIORDSON [3] and VEPA [4]. In [3] the authors optimized the shape of a cantilever beam with respect to its fundamental frequency, assuming the bending rigidity to be proportional to the first, second or third power of the mass per unit length of the beam. The first case was shown to be degenerate in the absence of non-structural mass, which confirmed the above-mentioned result obtained by BRACH. Mention should also be made of the classical problem of finding the optimal shape of a Bernoulli-Euler column subjected to a compressive load. The correct solution to this problem was first arrived at by CLAUSEN and then, independently, by KELLER [5]. It was found that by appropriately tapering a column of circular cross-section its critical buckling load could be increased by one-third in relation to that of a uniform column of the same material, volume, length and cross-sectional shape. The case solved in [5] lies at one extremity of the present study, while the other is formed by [1]. Our analysis follows closely that of ref. [1] and [3], though the extension, we believe, is by no means trivial. However, some occasional repetition is unavoidable for the clarity of presentation.

It is easy to show that if a rectangular cross-section has a fixed width and variable height, the bending rigidity is proportional to the cube of the mass per unit length. If the height and width vary with constant ratio (geometrically similar cross-sections), the bending rigidity will be proportional to the square of the mass per unit length. Finally, if only the width varies, the proportion will be linear. We shall thus assume in the sequel that the bending rigidity EI and the mass per unit length of the beam are related as follows:

$$I = cA^n, \tag{1.1}$$

where A is the area of the cross-section and c a constant depending implicitly upon n and $n = 1, 2$ or 3. It will subsequently become clear that for $n = 1$ and in the absence of axial compressive load, the optimum shape corresponds to the uniform beam, i.e. no increase in the fundamental frequency in relation to that of the uniform beam is possible.

2. Transverse vibrations of a tapered beam

From classical vibration theory, we know that a linear elastic beam subjected to an axial compression can vibrate harmonically in any one (or linear combination) of an infinite number of characteristic shapes. The differential equation of motion and the boundary conditions can be written in a non-dimensional form as:

$$(\alpha^n Y'')'' + Ky'' - \alpha y = 0, \tag{2.1}$$

$$\begin{gathered} y(0) = y(1) = 0, \\ \alpha^n y''(0) = \alpha^n y''(1) = 0. \end{gathered} \tag{2.2}$$

Here, y denotes the amplitude of the lateral deflection, and a dash indicates differentiation with respect to the non-dimensional coordinate along the beam $x = \xi/l$, where l is the length of the beam. The non-dimensional area function is denoted $\alpha = Al/V$, where V is the total volume of the beam and is assumed to be given. Furthermore, the following non-dimensional parameters have been introduced:

$$\lambda = \omega^2 \frac{\gamma l^{3+n}}{EcV^{n-1}}, \tag{2.3}$$

(2.4)
$$K = \frac{Pl^{2+n}}{EcV^n}\left(= \frac{Pl^2}{EI}\right),$$

where c is the constant defined by the relation (1.1).

Observe that for a uniform beam, K corresponding to the Euler buckling load equals $K_{cr} = \pi^2$. It is thus expedient to introduce another non-dimensional parameter in place of K, viz, K^* defined by $K^* = K/K_{cr}$.

From the definition of dimensionless area function α, it follows that

(2.5)
$$\int_0^1 \alpha dx = 1.$$

The variational method used here for solving the optimization problem is similar to that used in Ref. [1] and [3]. Our aim is to determine that non-negative function $\alpha(x)$, if one exists, which makes the first fundamental eigenvalue λ a maximum.

If (2.1) is multiplied by y and integrated over the range 0 and 1, an expression of the following form is obtained for λ:

(2.6)
$$\lambda = \frac{\int_0^1 \alpha^n (y'')^2 dx - K\int_0^1 (y')^2 dx}{\int_0^1 \alpha y^2 dx}.$$

To obtain the numerator in the above expression, integration by parts was carried out and the boundary conditions (2.2) were invoked.

Note that the maximum attainable value of K is given by

(2.7)
$$K_{\max} = \frac{\int_0^1 \alpha^n (y'')^2 dx}{\int_0^1 (y')^2 dx}.$$

In determining an expression for α, we follow the variational procedure outlined in [1] and arrive at

(2.8)
$$n\alpha^{n-1}(y'')^2 - \lambda y^2 = \lambda a^2$$

which can be rewritten as

(2.9)
$$n\alpha^n (y'')^2 - \lambda\alpha y^2 = \lambda\alpha a^2,$$

where a^2 is a number independent of x.

At this stage let us study the case when $n = 1$, i.e. when the bending rigidity of the beam is linearly proportional to its mass per unit length. In this case Eq. (2.9) takes the form

$$(y'')^2 - \lambda y^2 = \lambda a^2$$

which, after integrating from 0 and 1 and together with (2.6), leads to

$$\int_0^1 \lambda a^2 dx = K\int_0^1 (y')^2 dx.$$

It is thus clear that when $K = 0$, i.e. in the absence of axial compression, $a^2 = 0$ and hence

$$y'' = \pm \sqrt{\lambda} y.$$

In other words, for $n = 1$ and in the absence of axial compressive load, the optimum shape corresponds to the uniform beam, i.e. no increase in the fundamental frequency over that of the uniform beam is possible.

This result obviously does not hold good if $K \neq 0$. The problem in this case is identical to that of a cantilever beam with a non-structural mass at its end. This question has been studied in detail in [3].

Solving Eq. (2.8) for α, we get

$$\alpha = \left[\frac{\lambda}{n}\frac{a^2+y^2}{(y'')^2}\right]^{1/(n-1)}, \quad n > 1. \tag{2.10}$$

Multiplying both sides of (2.8) by α and integrating over the range 0 and 1, we find

$$a^2 = \frac{(n-1)\int\limits_0^1 \alpha^n (y'')^2 dx + K\int\limits_0^1 (y')^2 dx}{\lambda} \tag{2.11}$$

from which it follows that a^2 is positive for all $n \geqslant 1$. Substitution of α [Eq. (2.10)] into Eq. (2.1) yields the following nonlinear differential equation for y:

$$\left\{\left[\frac{a^2+y^2}{n(y'')^2}\right]^{n/(n-1)} y''\right\}'' + \frac{K}{\lambda^{n/(n-1)}} y'' - \left[\frac{a^2+y^2}{n(y'')^2}\right]^{1/(n-1)} y = 0. \tag{2.12}$$

Let us assume that the solution $y(x)$ is symmetric with respect to $x = 1/2$. Instead of the boundary conditions at $x = 1$, we then have the following at the mid point:

$$y'(1/2) = \left\{\left[\frac{a^2+y^2}{n(y'')^2}\right]^{1/(n-1)} y''\right\}'_{x=1/2} = 0. \tag{2.13}$$

The nonlinear differential Eq. (2.12), together with the boundary conditions at $x = 0$ and $x = 1/2$, constitutes the transformed (nonlinear) eigenvalue problem for the new parameter a^2 (the eigenvalue).

Substitution of α from (2.10) into (2.5) and (2.12) leads to the following system of implicit equations for determining a^2 and λ $(n > 1)$

$$\begin{gathered}\int\limits_0^1 \left\{\frac{\lambda}{n}\frac{a^2+y^2}{(y'')^2}\right\}^{1/(n-1)} dx = 1\\ \frac{(n-1)\int\limits_0^1 \left\{\frac{\lambda}{n}\frac{a^2+y^2}{(y'')^2}\right\}^{1/(n-1)} (y'')^2 dx + K(y')^2 dx}{\lambda} = a^2.\end{gathered} \tag{2.14}$$

Here we will consider two specific cases, namely those corresponding to $n = 2$ and 3.

It is expedient in such optimization problems to analyse the behaviour of the solution near $x = 0$ before attempting to solve the differential Eq. (2.12). The procedure is similar to that used in [3]. The solution $y(x)$ near $x = 0$ is assumed to be expandable in a power series of x with a characteristic term bx^k. This term is substituted in the differential Eq. (2.12), and the coefficient of the leading term is equated to zero for evaluating the smallest non-integer value of k. The calculations give

$$(-nk+2p-k+2)(-nk+p-k+3) = 0, \quad n > 1.$$

Specifically for $n = 2$ and 3, we have, respectively

$$k = 5/3 \quad \text{and} \quad 3/2.$$

It is easily verified that a solution of the form

$$y = a_1 x + a_2 x^2 + \ldots + bx + \ldots \tag{2.15}$$

satisfies the boundary conditions (2.2) at $x = 0$.

3. Solution by successive iterations

It is not possible, in general, to obtain a solution of the nonlinear differential Eq. (2.12) in a closed form, and hence some sort of a numerical technique has to be used. The method of successive iterations applied here is essentially similar to that employed in [3] and is based on a formal integration of the differential equation, satisfying one of the boundary conditions at each integration stage. The iteration procedure is elucidated here with respect to $n = 3$, while the results are reported for both $n = 3$, $n = 2$.

Integrating (2.12) twice and satisfying the boundary conditions at $x = 1/2$, we get

$$y''(x) = \frac{\left(\dfrac{a^2+y^2}{3}\right)^{3/4}}{\left[\displaystyle\int_0^x \int_x^{1/2} \left\{\frac{\left(\dfrac{a^2+y^2}{3}\right)^{1/2}}{y''} y - \frac{K}{\lambda^{3/2}} y\right\} dx\,dx\right]^{1/2}}. \tag{3.1}$$

Care has been taken here to separate the differential operator of the highest order on the left-hand side in order to assure convergence of the successive iterates.

Because of the square root type of singularity exhibited by y'' near $x = 0$, the iterations cannot be carried out directly, but as indicated by the expansion formula (2.15) we can define a finite function $g(x)$ in the closed interval $0 \leqslant x < 1/2$ as follows:

$$g(x) = -x^{1/2} y''(x). \tag{3.2}$$

The iterative procedure to give $\alpha(x)$ and λ is carried out in the following sequence

i $$y_n'(x) = \int_x^{1/2} g(x) x^{-1/2} dx,$$

ii $$y_n(x) = \int_0^x y_n'(x)\, dx,$$

iii $$\lambda = \frac{0.25}{\left[\displaystyle\int_0^{1/2} A_n^*(x) x^{1/2} dx\right]^{1/2}},$$

$$a_n^2 = \frac{4\lambda^{3/2}\int\limits_0^{1/2} A^*(x)x^{1/2}\frac{a^2+y_n^2}{3}dx+2K\int\limits_0^{1/2} y_n' dx}{\lambda},$$

where

$$A^*(x) = \left[\frac{a_n^2+y_n^2}{3}\right]^{1/2}\frac{1}{g_n}. \tag{3.3}$$

Observe that the expressions for a^2 and λ are implicit and the parameters a^2 and λ are therefore determined in a substage within the main iteration loop.

iv
$$M_{n+1}(x) = \int\limits_0^x\int\limits_x^{1/2}\left\{A_n^*(x)x^{1/2}y_n(x)+\frac{K}{\lambda^{3/2}}g_n(x)x^{-1/2}\right\}dx^2,$$

v
$$g_{n+1}(x) = -y_n''(x)x^{1/2} = -x^{1/2}\frac{\left[\dfrac{a^2+y_n^2}{3}\right]^{3/4}}{\pm\{M_{n+1}(x)\}^{1/2}}.$$

In the above expression, the sign before the braces is chosen such as to make $g_{n+1}(x)$ positive in accordance with our sign convention.

The iterations were started with an arbitrary regular function $g(x)$. The functions g, A^*, y' and y were obtained within a few iterations. y'' was calculated from the relation (3.2).

Finally, the area function $\alpha(x)$ (which to a suitable scale also represents the depth variation along the length) of the optimum beam was computed for various values of the non-dimensional axial compression K from the following expression [obtained by putting (3.3) into (2.10)]:

$$\alpha(x) = (\lambda)^{1/2}A^*(x)x^{1/2}. \tag{3.4}$$

We derive below certain expressions that will be useful subsequently.

Substitution of (2.15) into (2.10) shows that the area function α is proportional to $x^{2/3}$ for small values of x when $n = 2$, and the linear dimension — or diameter — of the cross-section is thus proportional to $x^{1/3}$ near $x = 0$. Similarly, for $n = 3$, the linear dimension (or height) is proportional to $x^{1/2}$. The proportionality factor can be evaluated from Eq. (3.4) for $n = 3$ and from a similar expression for $n = 2$. In fact, for $n = 3$, the factor is directly given by

$$C = (\lambda)^{1/2}A^*(x) = \left(\frac{\lambda}{3}\right)^{1/2}\frac{(a^2+y^2(x))^{1/2}}{g(x)}. \tag{3.5}$$

For $n = 2$, the corresponding proportionality factor of the linear dimension ($\alpha^{1/2}$) is expressed by

$$C = \left(\frac{\lambda}{2}\right)^{1/2}\frac{(a^2+y^2(x))^{1/2}}{g(x)}. \tag{3.6}$$

Furthermore, let us express C_K^* in terms of its percentage deviation relative to C_0 corresponding to $K^* = 0$. Subscript attached to C refers to the particular value of K^*. The expression for the percentage variation in C_K^* near $x = 0$ has the following form:

$$(3.7)\qquad \left\{\frac{C_K^*-C_0}{C_0}\right\}\times 100 = 100\left[\left(\frac{\lambda_K^*}{\lambda_0}\right)^{1/2}\left\{\frac{a_K^{2*}+y_K^{2*}}{g_K^{2*}}\cdot\frac{g_0^2}{a_0^2+y_0^2}\right\}^{1/2}-1\right],\quad n \geqslant 1.$$

In particular, at $x = 0$, (3.7) reduces to

$$(3.8)\qquad \left\{\frac{C_K^*-C_0}{C_0}\right\}_{x=0}\times 100 = 100\left[\left(\frac{\lambda_K^*}{\lambda_0}\right)^{1/2}\frac{a_K^*}{a_0}\cdot\frac{g_0}{g_K^*}-1\right]_{x=0},\quad n \geqslant 1.$$

As in Ref. [1] and [3], numerical integration was carried out by sub-dividing the interval $0 \leqslant x \leqslant 1/2$ into a sufficient number of equal parts and applying the trapezoidal rule.

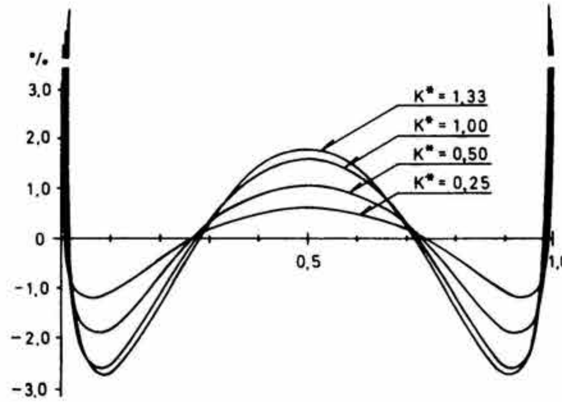

FIG. 1. Percentage variation in radius for various values of K^* relative to reference shape $K^* = 0{,}0$ for $n = 2$ (Fig 2).

The mesh length d was suitably varied so that the result could be extrapolated to $d = 0$ by means of Newton's rule.

Figures 2 and 4 show the variation of one half of the linear dimension of the cross-section (radius or one half of the depth depending upon n) as a function of the non-dimensional x for $K = 0$. The parameter K has been scaled with respect to $K_{cr}(=\pi^2)$ corresponding to the Euler buckling load of a uniform beam with the same volume, material and length (indicated by dashed lines in the figures) as the optimum beam. Similar diagrams could be drawn for other values of K^*. However, the variation in the optimal shape with a change in K^* proves to be small in relation to the optimal shape for $K^* = 0$. Thus, it was found convenient to exhibit the percentage change in one half of the linear dimension of cross-section relative to the reference shape corresponding to $K^* = 0$. These percentage

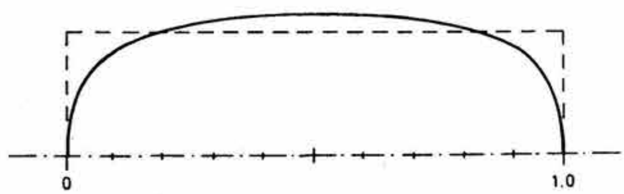

FIG. 2. Reference shape $K^* = 0{,}0$ $n = 2$.

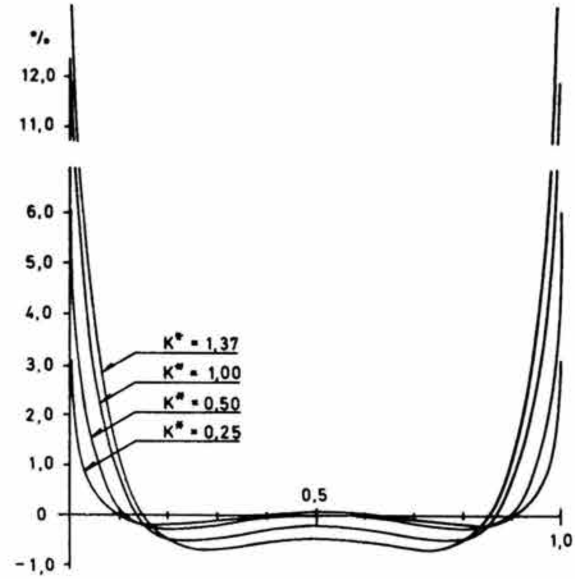

FIG. 3. Percentage variation in depth for various values of K^* relative to reference shape $K^* = 0{,}0$ for $n = 3$.

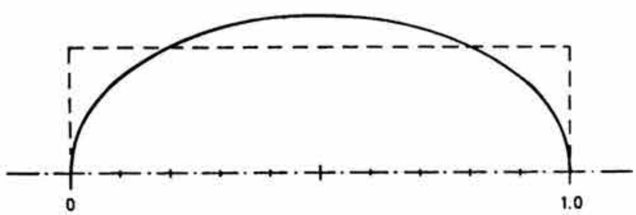

FIG. 4. Reference shape $K^* = 0{,}0$ for $n = 3$.

variations are shown in Figs. 1 and 3, respectively. Here it is necessary to add a word of caution regarding the percentage variation in the linear dimension near $x = 0$ and, in particular, at $x = 0$. It should be noted that, although the absolute value of the linear dimension tends to approach 0 as $x \to 0$, its percentage variation relative to the linear dimension for $K^* = 0$ increases to a finite value as $x \to 0$. This follows from an analysis of the Eq. (3.7). In order to appreciate this fact, the respective percentage variations at $x = 0$ have been tabulated in Table 2 for $n = 2$ and 3 and for various values of non-dimensional axial load parameter K^*.

Table 1. Ratio of $\sqrt{\dfrac{\lambda}{\lambda_{cr}}}$ ($\lambda_{cr} = \pi^4$) for various values of K* (value of K_{max} noted in each case)

n \ K^*	0	0.25	0.5	1.0	1.33	1.37
2	1.066	0.960	0.842	0.533	(k_{max})	—
3	1.119	1.012	0.901	0.607	0.257	(k_{max})

The ratio of the optimal lowest frequency ω_0 to that of the uniform beam with the same volume, material and length as the optimal beam is presented in Table 1 for $n = 2$ and 3 and for various values of K^*.

Table 2. Percentage variation in one half of the linear dimension for various values of K^* at $x = 0$

K^* / n	0	0.25	0.5	1.0	1.33	1.37
2	0	6.19	12.25	24.05	—	—
3	0	3.04	6.02	11.42	15.31	—

Finally we note that optimum shapes corresponding to $K = 0$ and $K = K_{max}$ for the beam with geometrically similar cross-sections ($n = 2$) approximate almost exactly the results of Refs. [1] and [5], respectively. However, for $n = 3$ the maximum value of K ($K_{max} = 1.37$) attained by us in a successive incremental procedure is less than that obtained in [6] by about 3%. In [6] the result was arrived at in a closed form by a direct analysis of an expression of the type (2.7).

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