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Convexity in plasticity

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CONVEXITY of the admissible domain for a rigid, perfectly plastic continuum is considered. Notions of the distance function of the admissible domain, the supporting function of the admissible domain and the directional derivative are defined and their importance to the discussion of convex domains is shown. As examples the linear combination of convex domains and dual convex domains are investigated.

Rozważono wypukłość dopuszczalnego obszaru dla sztywno, idealnie plastycznego ośrodka ciągłego. Zdefiniowano pojęcia funkcji odległości obszaru dopuszczalnego i funkcji nośnej obszaru dopuszczalnego i kierunkowej pochodnej oraz pokazano ich znaczenie w dyskusji obszaru wypukłego. Jako przykłady zbadano liniową kombinację obszarów wypukłych oraz podwójny obszar wypukły.

Исследована выпуклость области допустимых напряжений в жестко-идеально пластическом материале. Даны определения функций расстояния в допустимой области и несущей функции в этой области, а также производной по направлению. Показано значение этих понятий в исследовании свойств выпуклой области. В качестве примеров изучены линейные комбинации выпуклых областей и двойная выпуклая область.

1. Distance function

THE MECHANICAL behavior of a rigid, perfectly plastic continuum is described by relations between generalized stresses $Q_1, Q_2, \ldots, Q_n$ and the associated generalized strain rates $q_1, q_2, \ldots, q_n$, the specific power of dissipation being $Q_1q_1+Q_2q_2+\ldots+Q_nq_n$ [1]. In an arch, for instance, the state of stress at a cross-section is specified by two generalized-stresses, the bending moment M and the axial force N. The associated generalized strain rates are the rate of change $\varkappa$ of the curvature of the center line and the rate of extension ε of this line. Note that the shear force is treated not as a generalized stress but as the reaction to the kinematic constraint of the technical theory of arches, which stipulates that the center line remain normal to the cross-sections.

According to the theory of plastic potential [2, 3], the mechanical behavior of a rigid, perfectly plastic continuum is completely defined by its *yield condition*. In the n-dimensional stress space with rectangular Cartesian coordinates Q_i, the yield condition specifies the convex domain of admissible states of stress (*admissible domain*), which is bounded by the yield locus. Interior points of this domain represent states of stress below the yield limit for which all strain rates vanish. Points on the yield locus represent states of stress at the yield limit under which nonvanishing strain rates are possible. Points outside the admissible domain represent states of stress that cannot be attained in the considered continuum.

Assume that the origin O of stress space is an interior point of the admissible domain. If $\mathbf{Q}$ is the position vector of an arbitrary point Q of stress space (*stress point*), the ray OQ

intersects the yield locus at a single point Q° with position vector $\mathbf{Q}^\circ$. For varying $\mathbf{Q}$, the value of the function

$$F(\mathbf{Q}) = |\mathbf{Q}|/|\mathbf{Q}^\circ| \tag{1.1}$$

is the reciprocal of the *load factor* by which the state of stress represented by $\mathbf{Q}$ must be multiplied to transform it into a state of stress at the yield limit. In the theory of convex domains, the function (1.1) is known as the *distance function* of the admissible domain, which is characterized by $F(\mathbf{Q}) \leqslant 1$. The distance function of a convex domain has the following properties:

$$\begin{aligned} &\text{(a)} \quad F(\mathbf{Q}) > 0 \quad \text{if} \quad \mathbf{Q} \neq \mathbf{0}, \quad F(\mathbf{0}) = 0; \\ &\text{(b)} \quad F(\mu\mathbf{Q}) = \mu F(\mathbf{Q}) \quad \text{for} \quad \mu > 0; \\ &\text{(c)} \quad F(\mathbf{Q}+\mathbf{R}) \leqslant F(\mathbf{Q})+F(\mathbf{R}). \end{aligned} \tag{1.2}$$

While the propositions (a) and (b) are immediate consequences of the definition of the distance function, (c) may be established as follows. If $\mathbf{Q}$ and $\mathbf{R}$ are the position vectors of points of the admissible domain, $\mathbf{Q}/F(\mathbf{Q})$ and $\mathbf{R}/F(\mathbf{R})$ are the position vectors of points on the yield locus. It therefore follows from the convexity of this locus that

$$\mathbf{S} = (1-\lambda)\{\mathbf{Q}/F(\mathbf{Q})\}+\lambda\mathbf{R}/F(\mathbf{R}), \quad 0 \leqslant \lambda \leqslant 1, \tag{1.3}$$

is a point of the admissible domain. Accordingly,

$$F(\mathbf{S}) \leqslant 1. \tag{1.4}$$

The substitution of

$$\lambda = F(\mathbf{R})/\{F(\mathbf{Q})+F(\mathbf{R})\} \tag{1.5}$$

into (1.3) and use of (1.4) and $(1.2)_b$ then yield $(1.2)_c$.

We mention without proof that any function with the properties (1.2) is the distance function of a convex domain that has the origin as interior point. The points of this domain satisfy $F(\mathbf{Q}) \leqslant 1$ (see, for instance, [4], p. 22).

Examples of distance functions are:

$F(\mathbf{Q}) = |\mathbf{Q}|$ for the unit sphere about the origin; and

$F(\mathbf{Q}) = 2\max|Q_i|$ for the unit cube centered at the origin.

Another use of the distance function in the theory of perfectly plastic solids will be discussed in Sec. 5.

2. Supporting function

While a state of stress $\mathbf{Q}$ specifies the *position* of the stress point, a strain rate $\mathbf{q}$ will be viewed as specifying a *direction*, namely the direction of the ray from the origin to the point with the position vector $\mathbf{q}$. Note that, for positive λ, the directions specified by $\mathbf{q}$ and $\lambda\mathbf{q}$ are identical.

For a given constant c and a fixed direction $\mathbf{q}$, the equation

$$\mathbf{Q}\cdot\mathbf{q} = c, \tag{2.1}$$

in which the dot indicates the scalar product, specifies an oriented plane. Points with position vectors $\mathbf{Q}$ satisfying $\mathbf{Q}\cdot\mathbf{q} \leqslant c$ fill the negative half-space bounded by the plane (2.1), and the strict inequality applies to the interior points of this half-space. Note that the equation of a plane with the orientation $\mathbf{q}$ will always be written in the form (2.1) and not in the form $-\mathbf{Q}\cdot\mathbf{q} = -c$, which represents the same plane with the opposite orientation.

In the following, it will be assumed that the admissible domain of the considered rigid, perfectly plastic solid does not extend to infinity. An oriented plane through a boundary point $\mathbf{Q}$ of the admissible domain is called a *supporting plane* of this domain, if the negative half-space bounded by this plane contains the admissible domain. This supporting plane is called *regular* or *singular* according to whether $\mathbf{Q}$ is or is not the only common point of admissible domain and supporting plane.

For a given direction $\mathbf{q}(\neq \mathbf{0})$, there is exactly one supporting plane of the admissible domain that has the orientation $\mathbf{q}$. For varying $\mathbf{q}$, the equations of these supporting planes will be written in the form

$$\mathbf{Q}\cdot\mathbf{q} = H(\mathbf{q}). \tag{2.2}$$

To evaluate $H(\mathbf{q})$ for a given $\mathbf{q}$, we may choose $\mathbf{Q}$ in (2.2) as a common point of admissible domain and supporting plane. For a regular supporting plane, this choice yields a unique $\mathbf{Q}$, and for a singular supporting plane, the scalar product $\mathbf{Q}\cdot\mathbf{q}$ has the same value regardless of which common point of admissible domain and supporting plane is chosen. Thus, $H(\mathbf{q})$ in (2.2) depends only on $\mathbf{q}$.

The function $H(\mathbf{q})$ is called the *supporting function* of the admissible domain whose points $\mathbf{Q}$ satisfy the relations

$$\mathbf{Q}\cdot\mathbf{q} \leqslant H(\mathbf{q}) \quad \text{for all } \mathbf{q}. \tag{2.3}$$

For a given $\mathbf{q}$, there exists at least one point $\mathbf{Q}$ such that the equality sign holds in (2.3). In other words, the value of $H(\mathbf{q})$ for a given $\mathbf{q}$ is the maximum value of the scalar product $\mathbf{Q}\cdot\mathbf{q}$ for all positions of $\mathbf{Q}$ in the admissible domain.

The function $H(\mathbf{q})$ has the following properties:

$$\begin{aligned} &\text{(a)} \quad H(\mathbf{0}) = 0;\\ &\text{(b)} \quad H(\mu\mathbf{q}) = \mu H(\mathbf{q}) \quad \text{for} \quad \mu > 0;\\ &\text{(c)} \quad H(\mathbf{q}+\mathbf{r}) \leqslant H(\mathbf{q})+H(\mathbf{r}). \end{aligned} \tag{2.4}$$

The properties $(2.4)_a$ and $(2.4)_b$ follow from the definition of the supporting function and from the fact that, for positive μ, the directions $\mathbf{q}$ and $\mu\mathbf{q}$ yield the same supporting plane. The property $(2.4)_c$ is established as follows. *All* points $\mathbf{Q}$ of the admissible domain satisfy the inequalities $\mathbf{Q}\cdot\mathbf{q} \leqslant H(\mathbf{q})$ and $\mathbf{Q}\cdot\mathbf{r} \leqslant H(\mathbf{r})$, where $\mathbf{q}$ and $\mathbf{r}$ are fixed directions. Summation of these inequalities yields

$$\mathbf{Q}\cdot(\mathbf{q}+\mathbf{r}) \leqslant H(\mathbf{q})+H(\mathbf{r}). \tag{2.5}$$

By the maximum characterization of the supporting function, there exists at least one point $\mathbf{Q}$ in the admissible domain for which the left side of (2.5) has the value $H(\mathbf{q}+\mathbf{r})$, and this establishes the property $(2.4)_c$.

It can be shown that any function $H(\mathbf{q})$ that is defined for all directions $\mathbf{q}$ and has the properties (2.4) is the supporting function of a convex domain (see, for instance, [4], p. 26).

Examples of supporting functions are:

$H(\mathbf{q}) = \mathbf{Q}^{(0)}\cdot\mathbf{q}$ for the point with the position vector $\mathbf{Q}^{(0)}$;

$H(\mathbf{q}) = \mathbf{Q}^{(0)}\cdot\mathbf{q}+|\mathbf{Q}^{(1)}\cdot\mathbf{q}|$ for the line segment with the endpoints $\mathbf{Q}^{(0)}\pm\mathbf{Q}^{(1)}$;

$H(\mathbf{q}) = \dfrac{1}{2}\sum_i |q_i|$ for the unit cube centered at the origin.

The value of the supporting function $H(\mathbf{q})$ of the admissible domain of a rigid, perfectly plastic solid is the specific power of dissipation for the strain rate $\mathbf{q}$. The maximum characterization of the supporting function yields the principle of maximum specific power of dissipation. The common points of the admissible domain and its supporting plane for the direction $\mathbf{q}$ represent the states of stress that are capable of producing the strain rate $\mathbf{q}$.

3. Directional derivative

Let $H(\mathbf{q})$ be the supporting function of the admissible domain of a rigid, perfectly plastic solid. For a fixed strain rate $\mathbf{r}$, the function

$$H'(\mathbf{q};\mathbf{r}) = \lim_{h\to+0}\{H(\mathbf{q}+h\mathbf{r})-H(\mathbf{q})\}/h \tag{3.1}$$

is called the *directional derivative* of H in the direction $\mathbf{r}$. When $\mathbf{q}$ is kept fixed, while $\mathbf{r}$ is allowed to vary, the function $H'(\mathbf{q};\mathbf{r})$ has the following properties:

$$\begin{aligned} &\text{(a)}\quad H'(\mathbf{q};\mathbf{0}) = 0;\\ &\text{(b)}\quad H'(\mathbf{q};\mu\mathbf{r}) = \mu H'(\mathbf{q};\mathbf{r});\\ &\text{(c)}\quad H'(\mathbf{q};\mathbf{r}+\mathbf{s}) \leqslant H'(\mathbf{q};\mathbf{r})+H'(\mathbf{q};\mathbf{s}). \end{aligned} \tag{3.2}$$

The properties $(3.2)_a$ and $(3.2)_b$ follow from the definition of the directional derivative. To establish the property $(3.2)_c$, note that, according to $(2.4)_c$,

$$\begin{aligned} \frac{1}{h}H[\mathbf{q}+h(\mathbf{r}+\mathbf{s})] &\equiv \frac{1}{h}H\left[\frac{1}{2}(\mathbf{q}+2h\mathbf{r})+\frac{1}{2}(\mathbf{q}+2h\mathbf{s})\right]\\ &\leqslant \frac{1}{2h}H(\mathbf{q}+2h\mathbf{r})+\frac{1}{2h}H(\mathbf{q}+2h\mathbf{s}). \end{aligned} \tag{3.3}$$

Use of (3.3) in the definition of the directional derivative yields $(3.2)_c$.

According to the relations (3.2), the function $H'(\mathbf{q};\mathbf{r})$ for fixed $\mathbf{q}$ and variable $\mathbf{r}$ has the characteristic properties (2.4) of a supporting function. It can, in fact, be shown that $H'(\mathbf{q};\mathbf{r})$ is the supporting function of the convex domain that the admissible domain and its supporting plane for the direction $\mathbf{q}$ have in common (see [4], p. 26).

Assume, for instance, that the admissible domain is the unit cube centered at the origin. In three dimensions, with $\mathbf{q} = (q_1, q_2, q_3)$, this cube has the supporting function

$$H(\mathbf{q}) = \frac{1}{2}(|q_1|+|q_2|+|q_3|). \tag{3.4}$$

For a fixed $\mathbf{q}$ with, say $q_1 > 0$, $q_2 = 0$, $q_3 < 0$, and sufficiently small positive h,

$$|q_1+hr_1| = q_1+hr_1,\quad |q_2+hr_2| = h|r_2|,\quad |q_3+hr_3| = q_3-hr_3.$$

Accordingly,

(3.5) $$H'(\mathbf{q}; \mathbf{r}) = \frac{1}{2}(r_1 + |r_2| - r_3),$$

and this is the supporting function of the edge of the unit cube that has the endpoints $\frac{1}{2}(1, \pm 1, -1)$. Any point on this edge represents a state of stress that is capable of producing the assumed strain rate $\mathbf{q}$.

It follows from the definition of the directional derivative and $(3.2)_b$ that, to within higher order terms in h,

(3.6) $$H(\mathbf{q}+h\mathbf{r}) - H(\mathbf{q}) = hH'(\mathbf{q}; \mathbf{r}) = H'(\mathbf{q}; h\mathbf{r}).$$

Accordingly,

(3.7) $$H'(\mathbf{q}; \mathbf{r}) = \mathbf{r} \cdot \operatorname{grad} H$$

provided that the gradient with the components $\partial H/\partial q_i$ exists for the direction $\mathbf{q}$. For a state of stress $\mathbf{Q}$ that can produce the strain rate $\mathbf{q}$, we have not only $\mathbf{Q} \cdot \mathbf{q} = H(\mathbf{q})$, but also

(3.8) $$\mathbf{Q} \cdot \mathbf{r} = H'(\mathbf{q}; \mathbf{r}).$$

Comparison of (3.7) and (3.8) finally yields

$$\mathbf{Q} = \operatorname{grad} H.$$

4. Linear combination of convex domains

Let D_1 and D_2 be convex domains with the supporting functions $H^{(1)}(\mathbf{q})$ and $H^{(2)}(\mathbf{q})$, and let $\mathbf{Q}^{(1)}$ and $\mathbf{Q}^{(2)}$ be the generic points of these domains. If λ_1 and λ_2 are fixed, nonnegative numbers, the points

(4.1) $$\mathbf{Q} = \lambda_1 \mathbf{Q}^{(1)} + \lambda_2 \mathbf{Q}^{(2)},$$

where $\mathbf{Q}^{(1)}$ and $\mathbf{Q}^{(2)}$ assume all positions in D_1 and D_2, are the points of a convex domain D, which is called the *linear combination* of the domains D_1 and D_2 with the *weights* λ_1 and λ_2. Indeed, $\mathbf{Q}^{(1)}$ and $\mathbf{Q}^{(2)}$ satisfy the inequalities

(4.2) $$\mathbf{Q}^{(1)} \cdot \mathbf{q} \leqslant H^{(1)}(\mathbf{q}), \qquad \mathbf{Q}^{(2)} \cdot \mathbf{q} \leqslant H^{(2)}(q)$$

for arbitrary $\mathbf{q}$. Accordingly, the vector $\mathbf{Q}$ in (4.1) satisfies

(4.3) $$\mathbf{Q} \cdot \mathbf{q} \leqslant \lambda_1 H^{(1)}(\mathbf{q}) + \lambda_2 H^{(2)}(\mathbf{q}).$$

By the maximum property of supporting functions, there exist points $\mathbf{Q}^{(1)}$ and $\mathbf{Q}^{(2)}$ for which (4.2) holds with the equality sign. For the corresponding point $\mathbf{Q}$ given by (4.1), the equality sign holds in (4.3). It follows that the right-hand side of (4.3) is the supporting function $H(\mathbf{q})$ of the domain D.

$H'(\mathbf{q}; \mathbf{r})$, considered as function of $\mathbf{r}$, is the supporting function of the convex domain D' that the convex domain D and its supporting plane for the direction $\mathbf{q}$ have in common. It therefore follows from

(4.4) $$H'(\mathbf{q}; \mathbf{r}) = \lambda_1 H^{(1)\prime}(\mathbf{q}; \mathbf{r}) + \lambda_2 H^{(2)\prime}(\mathbf{q}; \mathbf{r})$$

that D' is obtained from the similarly defined convex domains D'_1 and D'_2 by linear combination with the weights λ_1 and λ_2.

The generalization of the concept of the linear combination of convex domains to more than two domains is immediate, as is the transition to a continuous linear combination of convex domains $D(\eta)$ whose supporting functions $H(\mathbf{q}, \eta)$ depend on a parameter η. If $\lambda(\eta) \geqslant 0$ is the weighting function for the linear combination D, the supporting function of D is defined as the integral

$$H(\mathbf{q}) = \int \lambda(\eta) H(\mathbf{q}, \eta) d\eta \tag{4.5}$$

extended over a certain interval of η.

As example for the linear combination of admissible domains in plasticity, consider the sandwich cross-section in Fig. 1a, which is subject to an axial force N (positive if tensile)

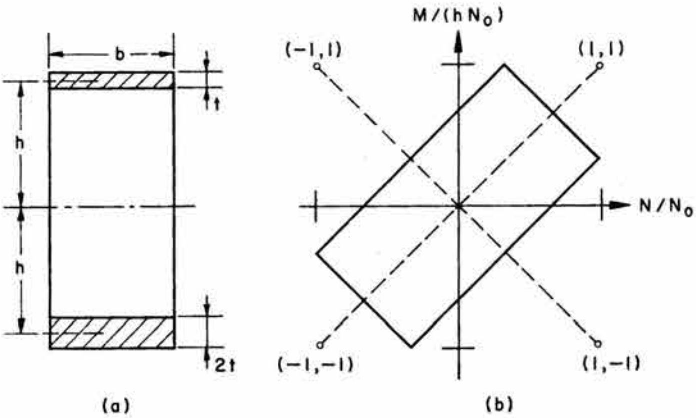

FIG. 1.

and a bending moment M (positive if the lower cover plate is stressed in tension). The material of the cover plates is rigid, perfectly plastic with the uniaxial yield stresses $\pm\sigma_0$, and the core does not carry axial stresses. If q_1 denotes the rate of extension of the central fiber and q_2 is the product of h and the rate of curvature, the specific rates of dissipation in the lower and upper cover plates are $\sigma_0|q_1+q_2|$ and $\sigma_0|q_1-q_2|$, and the total power of dissipation in these plates is $\sigma_0 bt\{2|q_1+q_2|+|q_1-q_2|\}$. Dividing by the maximum axial force $N_0 = 3\sigma_0 bt$, we obtain the reduced power of dissipation

$$H(\mathbf{q}) = \frac{1}{3}\{2|q_1+q_2|+|q-q_2|\}. \tag{4.6}$$

The generalized stresses corresponding to this reduced dissipation function are $Q_1 = N/N_0$ and $Q_2 = M/(hN_0)$.

The function (4.6) is the linear combination of the functions

$$H^{(1)} = |q_1+q_2|, \quad H^{(2)} = |q_1-q_2| \tag{4.7}$$

with the weights $\lambda_1 = 2/3$, $\lambda_2 = 1/3$. The supporting functions (4.7) represent the line segments with the endpoints $\pm(1, 1)$ and $\pm(1, -1)$ shown by the dashed lines in Fig. 1b.

Their linear combination with the weights 2/3 and 1/3 is the full-line rectangle in Fig. 1b, which is the admissible domain for the considered sandwich section.

Figure 2a shows an example involving a continuous linear combination of convex domains. The rectangular section is subject to a bending moment M and an axial force N

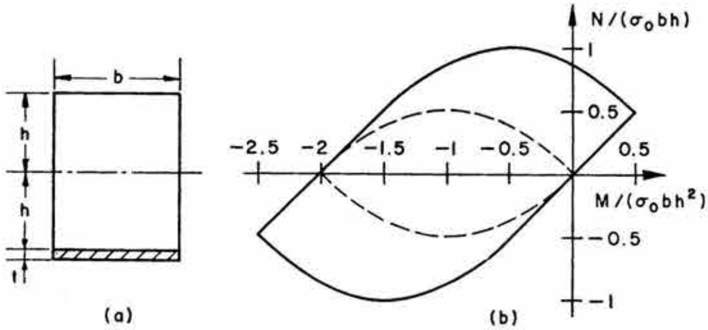

FIG. 2.

and consists of two rigid, perfectly plastic materials with the uniaxial yield stresses $-\sigma_0$, 0 for the solid upper part and $\pm\sigma_1$ for the thin reinforcing plate at the bottom.

We first determine the supporting function $H^{(1)}(\mathbf{q})$ for the upper part. If q_1 and q_2 have the same meaning as before, a layer of thickness dy at the distance y below the central fiber contributes the amount $\frac{1}{2}\,\sigma_0 bh.\{-(q_1+\eta q_2)+|q_1+\eta q_2|\}d\eta$ to the power of dissipation, where $\eta = y/h$. Using $\sigma_0 bh$ as the reduction factor, we obtain the reduced power of dissipation of the upper part as

$$H^{(1)}(\mathbf{q}) = -q_1+\frac{1}{2}\int_{-1}^{1}|q_1+\eta q_2|\,d\eta. \tag{4.8}$$

Here, $-q_1$ is the supporting function of the point $(-1, 0)$, while the expression under the integral sign is the supporting function of the line element with the endpoints $\pm\frac{1}{2}(d\eta, \eta d\eta)$. These line elements must be linearly combined with unit weights. Since each of them is symmetric with respect to the origin, their linear combination has the same central symmetry. Finally, the resulting convex domain must be given a unit translation in the negative q_1-direction as is expressed by the additive term $-q_1$ in (4.8).

To obtain the *form* of the yield locus, we string together the infinitesimal vectors represented by the line elements above in a sequence that furnishes a convex locus. If we start at the origin and use the elements $-(d\eta, \eta d\eta)$ from $\eta = -1$ to $\eta = \eta^*$, we arrive at the point with the coordinates

$$Q_1^* = -\int_{-1}^{\eta^*} d\eta = -(\eta^*+1), \quad Q_2^* = -\int_{-1}^{\eta^*}\eta d\eta = \frac{1}{2}(1-\eta^{*2}). \tag{4.9}$$

In particular, for $\eta^* = 1$, we arrive at the point $(-2, 0)$. A unit translation in the positive q_1-direction is therefore required to make the origin a center of symmetry, but this translation will be cancelled by the equal and opposite translation required by the additive term $-q_1$ in (4.8). Eliminating η^* from the Eqs. (4.9), dropping the asterisks, and completing the yield locus by symmetry with respect to the origin, one obtains

$$Q_2 = \pm \frac{1}{2} Q_1(Q_1+2), \quad -1 \leqslant Q_1 \leqslant 1. \tag{4.10}$$

This yield locus is shown by the dashed line in Fig. 2b.

The reduced power of dissipation for the cover plate at the bottom of the section in Fig. 2a is

$$H^{(2)}(\mathbf{q}) = \frac{\sigma_1 t}{\sigma_0 h} |q_1 + q_2|. \tag{4.11}$$

This is the supporting function of the line segment with the endpoints $\pm(1, 1)\sigma_1 t/(\sigma_0 h)$. For $\sigma_1 t/(\sigma_0 h) = 1/2$, the linear combination of the convex domains with the supporting functions (4.8) and (4.11) is bounded by the full line in Fig. 2b, which is the yield locus for the composite section in Fig. 2a.

The integral in (4.8) can be evaluated as follows. The position η_0 of the neutral axis is found from $q_1 + \eta_0 q_2 = 0$. Thus,

$$H^{(1)}(\mathbf{q}) = -q_1 + \frac{1}{2} \int_{-1}^{1} |q_1 + \eta q_2| d\eta = -q_1 + \frac{1}{2} |q_2| \int_{-1}^{1} |\eta - \eta_0| d\eta. \tag{4.12}$$

With η restricted to $-1 \leqslant \eta \leqslant 1$, the absolute value $|\eta - \eta_0|$ equals $\eta - \eta_0$ for $n_0 < -1$ and $\eta_0 - \eta$ for $\eta_0 > 1$. For $-1 \leqslant \eta_0 \leqslant 1$, the interval of integration in (4.12) consists of the subintervals $-1 \leqslant \eta \leqslant \eta_0$ where $|\eta - \eta_0| = \eta_0 - \eta$ and $\eta_0 \leqslant \eta \leqslant 1$ where $|\eta - \eta_0| = \eta - \eta_0$. Accordingly,

$$H^{(1)}(\mathbf{q}) = \begin{cases} -q_1 - q_1 \operatorname{sgn} q_2 & \text{for} \quad q_1/q_2 < -1, \\ -q_1 - \dfrac{1}{2} |q_2| \left(1 + \dfrac{q_1^2}{q_2^2}\right) & \text{for} \quad -1 \leqslant q_1/q_2 \leqslant 1, \\ -q_1 + q_1 \operatorname{sgn} q_2 & \text{for} \quad q_1/q_2 > 1. \end{cases} \tag{4.13}$$

For $q_2 > 0$, the first and third expressions in (4.13) are the supporting functions of the points $(-2, 0)$ and $(0, 0)$; for $q_2 < 0$, these points change roles. For $q_2 > 0$, the second expression in (4.13) is differentiable and furnishes

$$Q_1 = \partial H^{(1)}/\partial q_1 = -1 + \frac{q_1}{q_2}, \quad Q_2 = \partial H^{(1)}/\partial q_2 = \frac{1}{2}\left(1 - \frac{q_1^2}{q_2^2}\right). \tag{4.14}$$

For $q_2 < 0$, the sign of Q_2 in (4.14) must be changed. Elimination of q_1/q_2 between the equations for Q_1 and Q_2 again yields (4.10).

5. Dual convex domains

The built-in beam in Fig. 3 has a constant cross-section with the yield moment Y. It carries the loads Q_1, Q_2, which will be taken as the generalized stresses. The corresponding

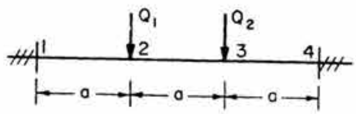

FIG. 3.

generalized strain rates are the rates of deflection q_1, q_2 of the points of application of the loads. Table 1 shows the rates of rotation in the plastic hinges at the cross-sections 1 through 4 that are caused by $q_1 = 1$, $q_2 = 0$ or $q_1 = 0$, $q_2 = 1$.

Table 1

Section				Factor
1	2	3	4	
-1	2	-1	0	q_1/a
0	-1	2	-1	q_2/a

If Y/a is used as the reduction factor, the reduced power of dissipation is

$$H(\mathbf{q}) = |q_1|+|2q_1-q_2|+|q_1-2q_2|+|q_2|. \tag{5.1}$$

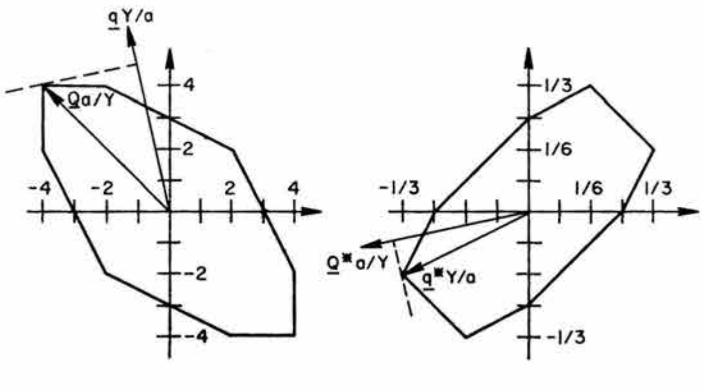

FIG. 4.

Figure 4a shows the admissible domain for the beam, which is the linear combination, with unit weights, of the line segments with the endpoints $\pm(1, 0)$, $\pm(2, -1)$, $\pm(1, -2)$, $\pm(0, 1)$.

The load factor μ that will bring a given stress point $\mathbf{Q}$ into the supporting plane for a given direction $\mathbf{q}$ is found from

$$\mu\mathbf{Q}\cdot\mathbf{q} = H(\mathbf{q}). \tag{5.2}$$

The load factor λ for plastic collapse is the smallest possible value of μ. Thus,

$$\lambda = \min\{H(\mathbf{q})/(\mathbf{Q}\cdot\mathbf{q})\}. \tag{5.3}$$

Note that, in evaluating λ, we need only consider the directions $\mathbf{q}$ that are normal to the line segments whose linear combination is the admissible region. These directions are: (0, 1), (1, 2), (2, 1), and (1, 0). Since the distance function $F(\mathbf{Q})$ of the admissible domain is the reciprocal of the load factor for plastic collapse, we have

$$F(\mathbf{Q}) = \max\{|Q_2|/4, \quad |Q_1+2Q_2|/6, \quad |2Q_1+Q_2|/6, \quad |Q_1|/4\}. \tag{5.4}$$

In view of (1.2) and (2.4), the distance function $F(\mathbf{Q})$ has the characteristic properties of a supporting function. The convex domain with the supporting function $F(\mathbf{Q})$ is called the *dual* of the admissible domain with the distance function $F(\mathbf{Q})$. The points $\mathbf{q}$ that this dual (Fig. 4b) has in common with its supporting plane for the direction $\mathbf{Q}$ represent strain rates that can occur under a yield point stress of the direction $\mathbf{Q}$ and that result in unit reduced power of dissipation.

The coordinates of the vertices in Fig. 4b are obtained as the factors of Q_1 and Q_2 in the four terms of (5.4). For example, to the term $|Q_1+2Q_2|/6$ there correspond the vertices $\pm(1/6, 1/3)$. Having determined the vertices of the dual in this manner, we may construct an alternative form of the supporting function of the dual by regarding it as a linear combination of line segments. This supporting function is

$$F(\mathbf{Q}) = \frac{1}{24}\{2|Q_1-Q_2|+3|Q_1+Q_2|+|Q_1+2Q_2|+|2Q_1+Q_2|\}; \tag{5.5}$$

it provides a closed-form expression for the load factor for plastic collapse, $\lambda = 1/F(\mathbf{Q})$, in dependence on the loads Q_1, Q_2.

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