CHAPTER XL.

SUPPLEMENTARY NOTES.

NOTE A. DEFINITION OF INTEGRATION. RIEMANN.

1875. The definition of the integral $\int_a^b \phi(x)\,dx$, given in Art. 11, for the case where $\phi(x)$ is single-valued, finite and continuous for the range $a \to b$, is an analytical expression of Newton's Second Lemma. It is pointed out in Art. 13 that the several subintervals $h_1, h_2, h_3, \ldots$ of the range a-b need not be taken as equal so long as it is understood that the greatest of them is ultimately taken as indefinitely small; and Cauchy adopted this modification as the basis of his investigations (see Art. 1266). But in dividing the range a-b into an infinite number of subdivisions,

$$\delta_1 \equiv x_1 - a,\quad \delta_2 \equiv x_2 - x_1, \ldots \delta_n \equiv b - x_{n-1},$$

the definition has still kept to the idea that each of these intervals is to be multiplied by the value of $\phi(x)$ at the beginning or at the end of the interval, that the sum of such products is to be formed, and then, if such sum has an existent limit and converges to a definite quantity, that limit is defined as $\int_a^b \phi(x)\,dx$. And it has been seen in Chapter V. how Cauchy proposed to exclude from the definition any element or elements in which $\phi(x)$ becomes infinite or discontinuous.

For the class of functions met with in elementary analysis and with which this treatise has been mainly concerned, this treatment will suffice, and has been adopted as offering an adequate scope for the beginner, with fewest difficulties in the initial conception of the processes to be followed.

But it is evident that the multipliers of the several subdivisions need not have been taken as the values of $\phi(x)$ at either *end* of the interval, but might equally well have been taken as any of its values intermediate between the greatest and least values which $\phi(x)$ is capable of assuming in each interval.

1876. Starting with this idea, Riemann in a memoir (*Ueber die Darstellbarkeit einer Function durch eine Trigonometrische Reihe*) has given a definition of integration which does not require that the function considered shall be continuous in the interval $a \to b$. Let a and b be two finite quantities between which a real variable x ranges. Let $\phi(x)$ be a function of x which remains finite, but not necessarily continuous in the interval. Take d a definite given small positive quantity, which is called the Norm, of any mode of division of the interval a-b into sub-elements or segments $\delta_1, \delta_2, \ldots \delta_n$, viz. $\delta_1 = x_1 - a$, $\delta_2 = x_2 - x_1$, $\ldots$ $\delta_n = b - x_{n-1}$, each of these elements being not greater than the norm d of that mode of division. Then evidently there is an infinite number of modes of division corresponding to any particular norm d, and each of these is also a possible mode of division for any greater norm. Let $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ be positive proper fractions, and let S stand for $\sum_1^n \delta_r f(x_{r-1} + \epsilon_r \delta_r)$. Then, if S converges to a definite limit whatever mode of division be chosen and whatever the fractions $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ may be when the norm d is made to diminish indefinitely, this limit is represented by $\int_a^b f(x)\,dx$, and the function is said to admit of integration for the range $a \to b$. (See Prof. H. J. S. Smith, *Proc. Lond. Math. Soc.*, vi., p. 140.)

1877. A formal proof of the convergence of the series S under certain conditions is given by Riemann, and amended by Prof. Smith in one or two particulars in which Riemann's demonstration is wanting in formal accuracy. The values of $\phi(x)$, corresponding to the values of x for any segment, are called the "ordinates" of the segment. The difference between the greatest and least ordinates of a segment is termed the "ordinate difference" or the "oscillation" of $\phi(x)$ for that segment. Let $D_1, D_2, \ldots D_n$ be the oscillations in the several segments. Then the greatest and least values of S for any particular mode of division are respectively attained by taking the greatest and least ordinates of the several segments, and the difference of these sums, viz. θ, is given by $\theta = \sum_1^n \delta_r D_r$. But for any definite norm d the greatest and least values of S do not in general result from the same mode of subdivision. Therefore the difference Θ between the greatest and least values of S for all modes of division corresponding to a given norm d will in general be greater than θ, which is the difference for a particular mode of division. And to be sure of the convergency of S it will be necessary to show that Θ in any case diminishes without limit when d diminishes without limit.

1878. Professor Smith enunciates Riemann's Theorem as follows:

Let σ be any given quantity, however small. Then, if in every division of norm d the sum of the segments for which the oscillations surpass σ diminishes without limit when d diminishes without limit, the function admits of integration, and conversely.

Let $G(d)$ and $L(d)$ be the greatest and least values of S corresponding to a given norm d, not necessarily arising from the same system of subdivisions for that norm.

Then taking any two norms d_1 and d_2 ($d_1 > d_2$), since every mode of division for norm d_2 is one for norm d_1, we have $G(d_1) \not< G(d_2)$ and $L(d_1) \not> L(d_2)$. Moreover, for every norm d_1 another norm d_2 can always be found which is less than d_1, such that $G(d_1) > G(d_2)$ and $L(d_1) < L(d_2)$, unless the max. and min. ordinates of the several segments are the same throughout the interval, however small the segments may be taken, in which case $G(d)$ and $L(d)$ are respectively $h_1(b-a)$ and $h_2(b-a)$, where h_1 and h_2 are the greatest and least ordinates common to all the segments. And therefore, except in this case, a series of norms $d_1, d_2, d_3, \ldots$ of decreasing magnitude can be found so that $G(d_1), G(d_2), G(d_3), \ldots$ forms a decreasing series, and $L(d_1), L(d_2), L(d_3) \ldots$ an increasing one.

And $G(d_1) > L(d_2)$, except in the case where the function can be represented by a series of segments of lines parallel to the x-axis, when we may have $G(d_1)=L(d_2)$. For if the two systems of division which respectively furnish $G(d_1)$ and $L(d_2)$ be superimposed, then to find the value of $G(d)$ for the new system of division, each resulting segment will have to be multiplied either by the same ordinate which multiplied it before or by a still greater one from a neighbouring segment; and to find the value of $L(d)$ for the new system, each segment must be multiplied either by the same ordinate which multiplied it before or by a still smaller ordinate from a neighbouring segment. So that the least value of S obtainable by taking the greatest ordinate for each segment in any mode of division whatever is not less than the greatest value of S obtainable in any division whatever by taking the least ordinate of each segment.

If then, for any given norm d, $L'(d)$ be the least value of S for the mode of division which yields $G(d)$, and $G'(d)$ be the greatest value of S for the mode of division which yields $L(d)$,

$$G(d) > G'(d); \quad G'(d) \geqslant L'(d) \quad \text{and} \quad L(d) < L'(d);$$

$$\therefore\ G(d)-L(d)=[G(d)-L'(d)]+[G'(d)-L(d)]-[G'(d)-L'(d)]$$
$$\not> [G(d)-L'(d)]+[G'(d)-L(d)].$$

But if s_1 be the sum of the segments which in the division $\{G(d), L'(d)\}$ have oscillations $>\sigma$, s_2 the sum of the segments which in the division $\{G'(d), L(d)\}$ have oscillations $>\sigma$, and Ω be the greatest oscillation for any division of norm d, which is by supposition finite; then

$$G(d)-L'(d)=\text{contribution from } s_1$$
$$+\text{contribution from } (b-a-s_1)$$
$$\not> s_1\Omega+\sigma(b-a-s_1)$$

and $$G'(d)-L(d) \not> s_2\Omega+\sigma(b-a-s_2);$$

$$\therefore \text{ adding, } G(d)-L(d) \not> (s_1+s_2)(\Omega-\sigma)+2\sigma(b-a),$$

and therefore, as σ is as small as we please and d can be taken so small that s_1+s_2 is as small as we please, $G(d)-L(d)$, that is Θ, diminishes without limit as d diminishes without limit and $f(x)$ admits of integration for the range a to b.

1879. Conversely, if $f(x)$ admits of integration in the interval a to b, S converges to a definite limit, and Θ diminishes indefinitely as d is made indefinitely small, and therefore also each of the differences θ must do the same. But if s be the sum of the segments in which the oscillations exceed σ in any mode of division, we have $\sigma s \ngtr \theta$. And however small σ may have been taken, we can, by taking d small enough, make θ/σ less than any assignable quantity, however small. Hence if S converges to a definite limit, s must also diminish without limit as d is indefinitely decreased.*

1880. Prof. Smith (*loc. cit.*) points out also that Riemann's criterion of integrability is applicable in the case of any multiple integral extended over a finite region.

1881. It is incidentally assumed that the interval a-b is one which extends from a given value of x, viz. $x=a$, to a greater one, $x=b$, and the interval a-b has been divided into subsections x_1-a, x_2-x_1, x_3-x_2, etc. If we reverse the order of the array of points $a, x_1, x_2, \ldots x_{n-1}, b$, the only difference in the argument will be that the sign of each of the partial products formed in constructing the maximum and minimum values of S has been changed; the new sums formed for the reversed order do not differ in absolute value from the values before considered, but are of opposite sign. It therefore follows that

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

1882. Moreover, if we add to the array several other points of division $x=c_1, x=c_2, \ldots x=c_{n-1}$, the maximum and minimum values of S have not been respectively increased and decreased, for the norm of the mode of division with the additional points in the array cannot have been increased by their introduction. But the sums corresponding to the maximum and minimum values of S for the several intervals a to c_1, c_1 to c_2, etc., are respectively

$$\nless \text{ and } \ngtr \int_a^{c_1} f(x)\,dx, \quad \int_{c_1}^{c_2} f(x)\,dx, \text{ etc.,}$$

and modes of division of these intervals can be found for which their maxima and minima differ from these respective quantities by less than any assignable quantities, however small. Also the aggregate of any of these modes of division

* *Proc. Lond. Math. Soc.*, vi., p. 143.

of these partial intervals forms a mode of division of the whole interval a-b. Hence $\int_a^b f(x)dx$ must be equal to the sum of the integrals $\int_a^{c_1} f(x)dx, \int_{c_1}^{c_2} f(x)dx, \ldots, \int_{c_{n-1}}^{b} f(x)dx.$

1883. In the same way other general propositions such as those of Chapter IX. may be reconsidered for Riemann's generalised definition.

Note B. Convergence of an Integral.

1884. An infinite integral is one in which either of the limits is $+\infty$ or $-\infty$, or in which the integration extends from $-\infty$ to $+\infty$. In what follows we shall assume that a is a positive quantity, *i.e.* $a>0$, and that $f(x)$ is a finite function of x for all values of x from a given value $x=a$ to another value $x=b$ which is greater than a, and that $f(x)$ is integrable in this range.

The integral $\int_a^\infty f(z)\,dz$ is defined as the limit, supposing such limit to exist, when x becomes infinitely large, of the integral $I \equiv \int_a^x f(z)\,dz$. If such limit be finite the integral is said to converge to that limit. If there be no finite limit to the increase in the value of I as x tends to $+\infty$, then, according as I tends to $\pm\infty$, the integral is said to diverge to $\pm\infty$. Integrals in which the integrand changes sign periodically in the march of x from a to ∞, such as

$$\int_a^\infty \sin x\,dx \quad \text{or} \quad \int_a^\infty x^2 \sin(bx+c)\,dx,$$

are said to oscillate, and such oscillations may be either finite or infinite by virtue of the growth of the multiplier of the factor of the integrand which causes the changes of sign during the march of x.

1885. If $f(x)$ be a function which changes sign during the march of x, the integral $\int_a^\infty f(z)\,dz$ is said to be absolutely convergent when $\int_a^\infty |f(z)|\,dz$ is convergent. But such an integral may be convergent even when not absolutely convergent.

The integral $\int_{-\infty}^{\infty} f(z)\,dz$ is defined as the sum of the integrals $\int_{-\infty}^{c} f(z)\,dz$ and $\int_{c}^{\infty} f(z)\,dz$, where c is a finite constant, and is said to be convergent when each of these integrals is convergent. Moreover, this definition is independent of the particular value of c. For, let c and c' be two values of x on the range of its values, $c' > c$.

Then $$\int_{x}^{c'} f(z)\,dz = \int_{x}^{c} f(z)\,dz + \int_{c}^{c'} f(z)\,dz \quad (x < c)$$

and $$\int_{c'}^{x} f(z)\,dz = \int_{c'}^{c} f(z)\,dz + \int_{c}^{x} f(z)\,dz \quad (x > c').$$

Hence, as $\int_{c}^{c'} f(z)\,dz$ and $\int_{c'}^{c} f(z)\,dz$ are finite, $\int_{x}^{c'} f(z)\,dz$ and $\int_{x}^{c} f(z)\,dz$ are both convergent or both divergent as $x \to -\infty$ and $\int_{c'}^{x} f(z)\,dz$ and $\int_{c}^{x} f(z)dz$ are both convergent or both divergent as $x \to \infty$.

Therefore, supposing $\int_{-\infty}^{c'} f(z)\,dz$ and $\int_{c'}^{\infty} f(z)\,dz$ to be both convergent integrals, we have

$$\int_{-\infty}^{c'} f(z)\,dz + \int_{c'}^{\infty} f(z)\,dz = \int_{-\infty}^{c} f(z)\,dz + \int_{c}^{\infty} f(z)\,dz,$$

which establishes the independence of the definition with respect to the particular value of c used.

1886. If $f_1(x)$, $f_2(x)$ be two positive finite functions of x, both integrable for the range a to b, $b > a > 0$, and such that $f_2(x) \not> f_1(x)$ for all values of x for that range, then, when b becomes infinitely large, $\int_{a}^{\infty} f_2(z)dz$ is convergent if $\int_{a}^{\infty} f_1(z)dz$ be convergent. And if $f_2(x) \not< f_1(x)$ for all values of x from a to b, then, when b becomes infinitely large, $\int_{a}^{\infty} f_2(z)\,dz$ is divergent if $\int_{a}^{\infty} f_1(z)\,dz$ be divergent.

In many cases comparison with a known convergent or divergent integral will suffice to determine the convergency or divergency of an integral.

For example, if $a>0$, $\int_a^\infty \frac{dz}{z^n}$ is convergent or divergent according as n is $>$ or $\not>1$.

Hence $\int_a^\infty \frac{dx}{x^2\sqrt{a^2+x^2}} < \int_a^\infty \frac{dx}{x^3}$ and is convergent, whilst

$$\int_b^\infty \frac{x^{\frac{3}{2}}dx}{\sqrt{x^4-a^4}} > \int_b^\infty \frac{dx}{\sqrt{x}}$$

and is divergent $(b>a)$.

1887. If then an index n can be assigned which is >1, and for which $x^n f(x)$ is finite for all values of x from $x=a$ to $x=\infty$, where $a>0$, it will follow that $|x^n f(x)|$ does not exceed some finite positive limit λ, and therefore that

$$\int_a^\infty |f(z)|\,dz \not> \lambda\int_a^\infty \frac{dz}{z^n}, \quad \textit{i.e.} \not> \frac{\lambda}{n-1}\frac{1}{a^{n-1}},$$

and is therefore convergent. Hence in such case $\int_a^\infty f(z)\,dz$ is absolutely convergent.

But if an index n can be assigned which is $\not>1$, and for which $x^n f(x)$ is never less than some finite positive limit λ (excluding zero) for all values of x from a to ∞, $(a>0)$, or if it becomes infinitely large when x increases indefinitely, it will follow that

$$\int_a^\infty f(x)\,dx \not< \lambda\int_a^\infty \frac{dx}{x^n}, \quad \textit{i.e.} \not< \frac{\lambda}{1-n}\Big[x^{1-n}\Big]_a^\infty \text{ or } \not< \lambda\Big[\log x\Big]_a^\infty,$$

and therefore in either case becomes positively infinite, and the integral diverges to $+\infty$.

And if an index n can be assigned which is $\not>1$ for which $x^n f(x)$ is negative, and its numerical value is never less than some finite limit λ (excluding zero) for all values of x from a to ∞, $(a>0)$, it will follow that $\int_a^\infty f(x)\,dx$ diverges to $-\infty$.

It appears therefore that under the conditions specified as to the integrability of $f(x)$, and as to its remaining finite for the range of integration, a to ∞, where $a>1$, if n can be assigned >1, such that a finite limit of $x^n f(x)$ exists when x becomes infinitely great, then $\int_a^\infty f(z)\,dz$ is convergent; and if n can be assigned $\not>1$, such that $x^n f(x)$ does not become zero when x is increased indefinitely, but whether it approaches a finite limit or becomes either positively or negatively infinite, the integral $\int_a^\infty f(z)dz$ is divergent.

For instance the integrals $I_1 \equiv \int_a^\infty \frac{x^2}{x^4+a^4}dx$; $I_2 = \int_a^\infty \frac{x^3}{x^4+a^4}dx$ are respectively convergent and divergent, for the indices 2 and 1 can be assigned for these respective cases for which

$$Lt_{x\to\infty}\, x^2 \frac{x^2}{x^4+a^4} = 1 \quad \text{and} \quad Lt_{x\to\infty}\, x \frac{x^3}{x^4+a^4} = 1,$$

and is finite in each case.

1888. Again the integral $\int_a^\infty \frac{\sin\theta}{\theta}d\theta$ is convergent, a being positive and >0. For by Art. 340,

$$\int_a^b \frac{\sin\theta}{\theta}d\theta = \frac{1}{a}\int_a^\xi \sin\theta\,d\theta + \frac{1}{b}\int_\xi^b \sin\theta\,d\theta, \quad a < \xi < b,$$

$$= \frac{1}{a}(\cos a - \cos\xi) + \frac{1}{b}(\cos\xi - \cos b),$$

which for any values of a, ξ, b cannot be greater than $\frac{2}{a}+\frac{2}{b}$, and, when b increases without limit, cannot be $> \frac{2}{a}$. Similarly $\int_a^\infty \frac{\cos\theta}{1+\theta^2}d\theta$ is convergent.

Also these integrals taken from 0 to a are obviously both finite. Hence the integrals from 0 to ∞ are finite. Their values have been found in Arts. 994, 1048.

1889. For other tests for Convergency, the reader may refer to Prof. Carslaw's *Fourier's Series*, pages 98-121.

Note C. Standard Forms.

1890. In such standard integrals as those of Arts. 44, 71, etc., viz. $\int \frac{dz}{\sqrt{a^2-z^2}}$, $\int \frac{dz}{\sqrt{z^2+a^2}}$, etc., which it is usual to give simply as $\sin^{-1}\frac{x}{a}$, $\sinh^{-1}\frac{x}{a}$, etc., it is to be noted that the left-hand members are even functions of a, whilst the right-hand members are odd functions of a. To be strictly accurate, such results should be written as $\sin^{-1}\frac{x}{|a|}$, $\sinh^{-1}\frac{x}{|a|}$, etc., where $|a|$ is the positive numerical value of $\sqrt{a^2}$, and where the inverse function is understood to have its principal value. Similarly

$$\int \frac{dz}{\sqrt{z^2-a^2}} = \log\frac{z+\sqrt{z^2-a^2}}{|a|}.$$

For in such cases the integral does not change its sign with a. And for exactness there must be a corresponding understanding as to all deduced results. In the same way in any other of the integrals discussed, and in which a constant is to be found with an even index in the integrand, and with an odd one in the result of integration a corresponding modification is to be understood; *e.g.* in the integral $\int_0^\infty \frac{\log(1+a^2z^2)}{1+b^2z^2}\,dz$, Art. 1044, the result of which is usually written as $\frac{\pi}{b}\log\frac{a+b}{b}$, but which is itself manifestly unaltered by a change of sign of a or of b, the value should strictly be written as

$$\frac{\pi}{|b|}\log\frac{|a|+|b|}{|b|}.$$

And similarly in any like case.

NOTE D. RATIONAL FRACTIONAL FORMS. HERMITE'S PROCESS.

1891. In the integration of rational algebraic fractional forms, viz. $f(z)/\phi(z)$ (Chap. V.), where f and ϕ are polynomials, rational as regards z, it has been assumed that the factorisation of $\phi(z)$ could be effected. This depends upon the possibility of solving $\phi(z)=0$.

It is a well-known fact, established by Abel and Wantzel, that it is impossible to solve algebraically the *general* equation of degree higher than the fourth. Hermite has given a solution of the quintic by aid of Elliptic Integrals (Burnside and Panton, *Th. Eq.*, p. 435). In consequence, the integration of such algebraic fractional forms as involve an unfactorisable denominator of the fifth or higher degree can only be completely performed for special forms of the numerator. But in any case, as we know that the equation $\phi(x)=0$ does possess as many roots as indicated by its degree, although there may be no means of discovering them, we are entitled to assert that the integral of $f(x)/\phi(x)$ does in every case consist of two portions, the one a rational algebraic function, and the other the sum of a set of simple logarithms with constant coefficients in which such pairs of terms as involve complementary imaginary roots may combine to form real terms by aid of the inverse symbols $\tan^{-1}$ or $\tanh^{-1}$.

1892. It has been shown by Hermite that the algebraic portion of such integrals can be always found, whether $\phi(x)$ be factorisable or not, and in cases where no logarithmic portion is present, or if the residual numerator happens to be a constant multiple of $\phi'(x)$ the whole integration can be effected. But in the general case no means of discovery of the Logarithmic portion is available for the reason stated.

An examination of the ordinary process for obtaining the H.C.F. of two polynomials in x, A and B, will disclose the fact that each of the successive "remainders" is of the form $\lambda A+\mu B$, where λ and μ are themselves polynomial expressions, and that when A and B are prime to each other the final remainder which is then merely numerical is also of the same form. It follows therefore that it is always possible in such case to find two polynomials λ and μ such that $\lambda A+\mu B$ is independent of x, and therefore also to find two polynomials λ' and μ' such that $\lambda' A+\mu' B=C$, where C is any given third polynomial in x. Moreover, supposing the degrees of A and B in x to be respectively the p^{th} and q^{th}, and that of C to be not more than $p+q-1$, we may note that it may be assumed that the degrees of λ' and μ' do not exceed the $(q-1)^{\text{th}}$ and $(p-1)^{\text{th}}$ respectively. For if we take their degrees to be greater than $q-1$ and $p-1$, we could by division write $\lambda'=\lambda'' B+\lambda'''$, $\mu'=\mu'' A+\mu'''$, where λ'', λ''', μ'', μ''' are other polynomials such that the degrees of λ''', μ''' do not respectively exceed $q-1$ and $p-1$, and thus $(\lambda''+\mu'')AB+\lambda''' A+\mu''' B=C$, and by equating coefficients of terms of higher degree than the highest in C, *i.e.* of the $(p+q)^{\text{th}}$, $(p+q+1)^{\text{th}}$, etc., degrees, it will appear that $\lambda''+\mu''$ must vanish identically.

1893. In the discussion of the integration of $f(x)/\phi(x)$, where $\phi(x)$ is unfactorisable, we may assume

(1) That $\phi(x)$ contains no repeated factor; otherwise the H.C.F. process upon $\phi(x)$ and $\phi'(x)$ would disclose that factor.

(2) That $f(x)$ is of lower degree than $\phi(x)$, by Art. 140, and that in this case the result is purely logarithmic.

(3) But if $\phi(x)$ be itself the square of an irreducible polynomial u, and $f(x)$ of lower degree than u, we may find polynomials λ and μ such that

$$f(x)=\lambda\frac{du}{dx}+\mu u,$$

$$i.e. \int\frac{f(x)}{\phi(x)}dx=\int\frac{\lambda}{u^2}\frac{du}{dx}dx+\int\frac{\mu}{u}dx=-\frac{\lambda}{u}+\int\frac{\mu+\frac{d\lambda}{dx}}{u}dx;$$

and supposing u of degree p, $\frac{du}{dx}$ is of degree $p-1$, so that λ and μ are of respective degrees $\not> p-1$ and $p-2$, so that $\mu+\frac{d\lambda}{dx}$ is of lower degree than u, and therefore the unintegrated portion is entirely logarithmic, but vanishing if $\mu+\frac{d\lambda}{dx}$ vanishes.

(4) If $\phi(x)$ be the r^{th} power of an irreducible polynomial u, we may find λ and μ such that $f(x)=\lambda\frac{du}{dx}+\mu u^{r-1}$, and then

$$\int\frac{f(x)}{\phi(x)}dx=\int\frac{\lambda}{u^r}\frac{du}{dx}dx+\int\frac{\mu}{u}dx=-\frac{1}{r-1}\frac{\lambda}{u^{r-1}}+\frac{1}{r-1}\int\frac{\frac{d\lambda}{dx}}{u^{r-1}}dx+\int\frac{\mu}{u}dx,$$

in which the index of the u in the integrand has been lowered by unity; and by repetitions of this process we may obtain a result in which the only unintegrated part is of the form

$$\int\frac{\chi(x)}{u}dx.$$

(5) If $\phi(x)$ be the product of positive integral powers of such irreducible factors, say $\phi(x)=u_1^{\alpha}u_2^{\beta}u_3^{\gamma}\ldots$, the separate prime factors u_1, u_2 ... may be discovered by the usual process employed in finding the H.C.F. for $\phi(x)$ and its differential coefficients, and thus, supposing $\alpha<\beta<\gamma\ldots$, if we determine λ and μ so that $\lambda_1u_2^{\beta}u_3^{\gamma}\ldots+\mu u_1^{\alpha}\equiv f(x)$, we can write $f(x)/\phi(x)$ in the form $\frac{\lambda_1}{u_1^{\alpha}}+\frac{\mu}{u_2^{\beta}u_3^{\gamma}\ldots}$, and repetitions of the process will separate out the fraction $\frac{f(x)}{\phi(x)}$ into the form $\frac{\lambda_1}{u_1^{\alpha}}+\frac{\lambda_2}{u_2^{\beta}}+\frac{\lambda_3}{u_3^{\gamma}}+\ldots$, to each of which portions we can apply the foregoing rules. Hence in all cases the algebraic portion of $\int\frac{f(x)}{\phi(x)}dx$ can be discovered.

Ex. To integrate $I=\int\frac{2+x+5x^4+2x^5+5x^9}{(1+x+x^5)^2}dx.$

Here $I=\int\frac{(1+x+x^5)(-3+5x^4)+4x+5}{(1+x+x^5)^2}dx$, and finding λ, μ such that $\lambda(1+5x^4)+\mu(1+x+x^5)\equiv 5+4x$, we may take λ of degree 4, μ of degree 3, and

$$(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4)(1+5x^4)$$
$$+(b_0+b_1x+b_2x^2+b_3x^3)(1+x+x^5)\equiv 5+4x,$$

giving $a_1=-1$, $b_0=5$ and the rest zero, whence

$$-x(1+5x^4)+5(1+x+x^5)=5+4x,$$

and $I=\int\frac{(1+x+x^5)(-3+5x^4)-x(1+5x^4)+5(1+x+x^5)}{(1+x+x^5)^2}dx$

$$=\int\frac{5x^4+2}{1+x+x^5}dx-\int x\frac{1+5x^4}{(1+x+x^5)^2}dx$$

$$=\int\frac{5x^4+2}{1+x+x^5}dx+\frac{x}{1+x+x^5}-\int\frac{dx}{1+x+x^5}=\frac{x}{1+x+x^5}+\log(1+x+x^5).$$

The same process will be helpful even in simple cases.

E.g. (i) $I=\int\frac{dx}{(x^2+1)^2}$. Writing $(a_0+a_1x)2x+b_0(x^2+1)\equiv 1$, we have

$$a_0=0,\quad a_1=-\tfrac{1}{2},\quad b_0=1\;;$$

$$I=\int\frac{(-\frac{1}{2}x)2x+(x^2+1)}{(x^2+1)^2}dx=\frac{x}{2}\frac{1}{x^2+1}+\frac{1}{2}\int\frac{dx}{x^2+1}=\frac{x}{2(x^2+1)}+\frac{1}{2}\tan^{-1}x.$$

(ii) $I=\int\frac{2x^3-1}{(x^3+x+1)^2}dx$. Writing

$$(a_0+a_1x)(1+x+x^3)+(b_0+b_1x+b_2x^2)(1+3x^2)\equiv -1+2x^3,$$

we have $\quad a_1=b_0=b_2=0,\quad a_0=-1,\quad b_1=1\;;$

$$\therefore\; I=\int\frac{-(x^3+x+1)+x(3x^2+1)}{(x^3+x+1)^2}dx=-\frac{x}{x^3+x+1}.$$

Note E. Legendre's Substitution applied to Functions of Form $1/X\sqrt{Y}$.

1894. With regard to integrals of the form $I\equiv\int\frac{Mx+N}{X\sqrt{Y}}dx$, where $X=a_1x^2+2b_1x+c_1$, $Y=a_2x^2+2b_2x+c_2$ discussed in Art. 291 onwards, in which we have adopted the substitution $y=\frac{Y}{X}$, it should be mentioned that Greenhill in his "Chapter on the Integral Calculus" generally prefers to put $y^2=\frac{Y}{X}$. This of course alters the character of the substitution-graphs, making them symmetrical about the x-axis. (See Ex. 56, p. 323.

Vol. I.) An alternative substitution is mentioned by Mr. Hardy as being followed by Stolz (*Grundzüge der Diff. und Int.-rechnung*) and by Dr. I'A. Bromwich, viz. to use the same substitution as that of Legendre in the reduction of an Elliptic Integral to Standard form, viz. $x=\dfrac{\lambda\xi+\mu}{\xi+1}$, whereby X takes the form

$$\{(a_1\lambda^2+c_1)\xi^2+2(a_1\lambda\mu+b_1\overline{\lambda+\mu}+c_1)\xi+(a_1\mu^2+c_1)\}/(\xi+1)^2$$

and Y takes a similar form with suffixes 2. Then, if λ, μ be so chosen that

$$a_1\lambda\mu+b_1(\lambda+\mu)+c_1=0,\quad a_2\lambda\mu+b_2(\lambda+\mu)+c_2=0 \quad \text{(cf. Art. 1463)}$$

I is reduced to the form

$$A\int\frac{\xi\,d\xi}{(a\xi^2+b)\sqrt{a'\xi^2+b'}}+B\int\frac{d\xi}{(a\xi^2+b)\sqrt{a'\xi^2+b'}},$$

where A, B, a, b, a', b' are certain constants. And now we may proceed either as in Art. 310, or use the substitutions $u\sqrt{a'\xi^2+b'}=1$ in the first; $v\sqrt{a'\xi^2+b'}=\xi$ in the second, which reduce each integral to the form $\displaystyle\int\frac{dv}{Pv^2+Q}$. This method fails if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$. But we may then put $a_1x+b_1=\xi$ and proceed as in Art. 309.

Note F. Continuity, Double Limits, Differentiation of an Integral, etc.

1895. **Continuity of a Function of two real Independent Variables.**

Let $z\equiv f(x, y)$ be a single-valued function of two independent real variables x and y which may be regarded as fixing a definite point. Construct a small rectangle with centre at x, y and with corners $x\pm\xi$, $y\pm\eta$. Then if θ_1, θ_2 be positive proper fractions and finite values of ξ, η can be found for which the value of $f(x\pm\theta_1\xi, y\pm\theta_2\eta)-f(x, y)$ taken positively is determinate and less than any arbitrarily chosen positive quantity ϵ, however small, for all combinations of the quantities θ_1, θ_2, the function is said to be continuous at the point x, y and throughout any region of the x-y plane for each point of which the same test is satisfied.

1896. In the case of such a function as the above, viz. $z=f(x, y)$, it may happen that in evaluating the value of z for a point for which $x=x_0$ and $y=y_0$, the mode of approach of x, y to the limiting position x_0, y_0 is not immaterial. That is $Lt_{x\to x_0} Lt_{y\to y_0} f(x, y)$ may not be the same thing as

$$Lt_{y\to y_0} Lt_{x\to x_0} f(x, y).$$

Take for instance the case of Sir R. Ball's Cylindroid, viz. the surface $z=\dfrac{2axy}{x^2+y^2}$. At any point for which $x=x_0$, $y=y_0$ other than those which lie on the z-axis, the value of z is $\dfrac{2ax_0y_0}{x_0^{\,2}+y_0^{\,2}}$, and is not dependent upon the direction in which x, y approaches its limiting position. But for points on the z-axis putting $y=mx$ so that the direction of approach is defined as being in a definite direction, $z=\dfrac{2am}{1+m^2}$, and as m changes from 0 to 1, z changes from 0 to a, so that if the direction of approach to the point for which $x=0$, $y=0$ be unassigned, the value of z cannot be assigned, and there is discontinuity in that its value is not independent of the relative mode of approach of x and y to their ultimately zero values. As a matter of fact, the z-axis is a nodal line upon the cylindroid.

1897. In partial differentiation of a function of two independent variables, $z=f(x, y)$, which is itself single-valued, finite and continuous for all values of x and y which lie within specified limits, the value of the fraction $\dfrac{f(x, y+\delta y)-f(x, y)}{\delta y}$ will in general approach a definite limit when δy becomes indefinitely small for each value of x within the specified range. The limit is then denoted by $\dfrac{\partial}{\partial y}f(x, y)$. But it is possible that within this range of values of x there may be one or more values of x for which no such limit exists. In such case the operation of differentiation fails and is an illegitimate process. Take the case $f(x, y)=x\sin xy$. Here

$$\frac{f(x, y+\delta y)-f(x, y)}{\delta y}=\frac{x\sin x(y+\delta y)-x\sin xy}{\delta y},$$

and for all finite values of x and y this tends uniformly to the limit $x^2\cos xy$ when δy is indefinitely diminished.

But if x be increased indefinitely, the limit when $\delta y=0$ of

$$\frac{x\sin x(y+\delta y)-x\sin xy}{\delta y}-x^2\cos xy$$

does not vanish, but may assume any value we please, however great. Therefore, for instance, the second differentiation suggested in Ex. 37, p. 381, Vol. I., would be an illegitimate operation.

But in the case $u=\int_0^\infty x^r e^{-ax}dx$, where r is a positive integer and a is real and positive, $\frac{\delta u}{\delta a}=\int_0^\infty x^r e^{-ax}\frac{e^{-x\delta a}-1}{\delta a}dx$, and whether x be zero, finite or infinitely large, $x^r e^{-ax}\frac{e^{-x\delta a}-1}{\delta a}$ tends uniformly to the limiting form $-x^{r+1}e^{-ax}$, vanishing whether $x=0$ or $x=\infty$. Hence the differentiations employed in Ex. 3 p. 369, Vol. I., are legitimate although the range of x is infinite. Similar remarks apply to Arts. 1039, 1041, 1046, etc., as therein noted.

1898. If discontinuity in such a function as $z=f(x, y)$ exists for any values of x, y, the equation $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial y\,\partial x}$ is not necessarily true for such points. This equation holds for any point x, y if a small rectangle whose centre is x, y can be constructed in the plane of x-y within which each of the differentiations is a possible operation, *i.e.* provided there be no discontinuity in the function or in either of its differential coefficients.

The rule $\frac{\partial}{\partial c}\int\phi(x, c)dx=\int\frac{\partial}{\partial c}\phi(x, c)dx$ (Art. 354) ....(1)

is virtually a consequence of

$$\frac{\partial^2 z}{\partial x\,\partial c}=\frac{\partial^2 z}{\partial c\,\partial x}. \quad \text{..........................(2)}$$

For $\psi(x, c)=\int\phi(x, c)dx$ is only another way of writing $\phi(x, c)=\frac{\partial\psi(x, c)}{\partial x}$; whence $\frac{\partial\phi}{\partial c}=\frac{\partial^2\psi}{\partial c\,\partial x}$. And the assertion of rule (1) is that

$\frac{\partial}{\partial c}\psi(x, c)=\int\frac{\partial}{\partial c}\phi(x, c)dx$, which is the same as $\frac{\partial}{\partial x}\frac{\partial\psi}{\partial c}=\frac{\partial\phi}{\partial c}$.

Hence the assertion (1) is equivalent to the assertion (2); and therefore, where the one rule fails, the other breaks down also.

1899. In all multiple integral evaluations and theorems, such for instance as that of Art. 361, viz.

$$\int_{c_0}^{c}\int_{a}^{b}\phi(x, y)\,dx\,dy=\int_{a}^{b}\int_{c_0}^{c}\phi(x, y)\,dy\,dx,$$

it is assumed that the subject of integration remains finite and continuous for all points within and at the boundaries of the region over which the integration is conducted; and moreover that the differentials which we integrate do not become infinite or discontinuous at any point within the range of the integration at each step of the process. If this be not the case, anomalies and contradictions may arise such as that noted in Ex. 38, p. 381, Vol. I.

Note G. Uniform Convergence.

1900. After the investigations of Stokes (*Trans. Camb. Phil. Soc.*, viii. 1847) and Seidel (*Abh. d. Bayerischen Akad.*, 1848), some time elapsed before writers on the General Theory of Functions realised fully the importance of careful distinction between the uniform and non-uniform convergence of infinite series. The question of uniformity of convergence is a fundamental point in this General Theory, and it always arises when we have under consideration the limiting value of a function depending upon more than one independent variable. For a very useful discussion of the Convergence of Infinite Series and Products, we may refer to Chrystal's *Algebra*, vol. ii., pages 113-185. Reference may also be made to Dr. Hobson's *Trigonometry*, ch. xiv., or Harkness and Morley, *Th. of F.*, ch. iii.

1901. Consider any series $u_1+u_2+u_3+\dots+u_n+\dots$, in which each term is a single-valued finite and continuous function of a variable z, which may be complex, and lying within a given region Γ in the Argand diagram, and of the integral number n which signifies its position in the series; then, if for every positive value of ϵ, however small we can assign a positive integer ν independent of z, such that for all values of n

greater than ν, the modulus of the residue of the series beyond the term u_n is less than ϵ, the series is said to be uniformly convergent for all points within that region (Chrystal, *Alg.*, ii., p. 144). If Σu_n converges uniformly within the aforesaid region to a definite value $\phi(z)$, then $\phi(z)$ is itself a continuous function of z for all points within the region. That is at each point z_0 within the region Γ, writing $u_r \equiv f(z, r)$,

$$\phi(z_0) = Lt_{z \to z_0} \sum_1^\infty f(z, r) = \sum_1^\infty Lt_{z \to z_0} f(z, r) = \sum_1^\infty f(z_0, r).$$

(See references above.)

1902. With the definition of an integral as in Art. 1266, viz. $Lt_{n \to \infty} \sum_1^n (z_r - z_{r-1}) \omega_{r-1}$, and supposing that each of the ω's is a single-valued finite and continuous function of z and a complex constant a, which both lie in a definite region Γ of the Argand diagram, say $\omega_r = f_r(a, z)$, and that when a and z are made to approach indefinitely near definitely assigned points a_0 and z_0 lying within the region Γ, the function $f_r(a, z)$ tends uniformly to the value $f_r(a_0, z_0)$ and is continuous, then we shall have

$$Lt_{a \to a_0} Lt_{n \to \infty} \sum_1^n (z_r - z_{r-1}) \omega_{r-1} = Lt_{n \to \infty} \sum_1^n (z_r - z_{r-1}) Lt_{a \to a_0} \omega^{r-1},$$

i.e.

$$Lt_{a \to a_0} \int f(a, z)\, dz = \int Lt_{a \to a_0} f(a, z)\, dz = \int f(a_0, z)\, dz.$$

This result, for the case when z and a are real, has been assumed in Art. 354.

Note H. Unicursal Curves.

1903. In any case of a rational integral function of x and y, say $\phi(x, y)$, in which the real variables x, y are connected by a rational integral algebraic equation $F(x, y) = 0$ whose graph is a curve of deficiency zero, and therefore unicursal, both x and y are expressible as rational algebraic functions of a third variable t, as also $\frac{dx}{dt}$, and therefore in all such cases the integration $\int \phi(x, y)\, dx$ can be effected with the limitation mentioned in Note D, and the result is partly rational and partly a logarithmic transcendent of form $\Sigma A \log(x-a)$, where A and a are certain constants.

1904. The principal elementary cases of unicursal curves are (*a*) the conic, (*b*) the nodal cubic, (*c*) the three-node quartic.

(*a*) The equation of a conic may be written as $u_1 v_1 = w_1$, where u_1, v_1, w_1 are linear functions of x and y. Putting $u_1 = \lambda w_1$, $v_1 = \lambda^{-1}$ and solving, we may express both x and y as rational algebraic functions of λ.

(*b*) The equation of a nodal cubic may be written $u_1 v_1 = w_3$, where u_1, v_1 are linear homogeneous functions of x and y, and w_3 is homogeneous and of degree 3. Putting $y = \lambda x$, we can express both x and y as rational algebraic functions of λ.

(*c*) The general equation of a three-node quartic may be written in homogeneous coordinates (say areals) as

$$ax^{-2} + by^{-2} + cz^{-2} + 2fy^{-1}z^{-1} + 2gz^{-1}x^{-1} + 2hx^{-1}y^{-1} = 0,$$

and therefore, taking another point x', y', z' connected with x, y, z by the relations $x/x'^{-1} = y/y'^{-1} = z/z'^{-1}$, we have

$$ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' = 0,$$

i.e. the three-node quartic may be regarded as the "inverse" of a conic, using the term "inversion" in the sense in which it is employed by Dr. Salmon, *H. Pl. Curves*, p. 244.

Now x', y', z' being the coordinates of a point on a conic, which is a unicursal curve, may be expressed in terms of a fourth new variable t as rational functions of t, and therefore x, y, z, the coordinates of a point on the inverse three-node quartic, can also be expressed in the same manner. For writing

$$\frac{x'}{f_1(t)} = \frac{y'}{f_2(t)} = \frac{z'}{f_3(t)} = \frac{1}{F(t)},$$

where $F = f_1 + f_2 + f_3$ and $\phi = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$, we have

$$\frac{x}{1/f_1} = \frac{y}{1/f_2} = \frac{z}{1/f_3} = \frac{1}{\phi}.$$

So that if $x' = \frac{f_1}{F}$, etc., then $x = \frac{1}{\phi f_1}$, etc. Hence the "inverse" of any unicursal curve is itself unicursal.

In all such cases the integral $\int \phi(x, y)\,dx$ will only require for its expression, rational integral algebraic functions and simple logarithmic transcendents.

The general cubic may be written $uvw=z$, where u, v, w, z are linear functions of x and y. Any point upon it may be defined by the equations $vw=\lambda z$, $u=\frac{1}{\lambda}$. If there be no node, the deficiency is unity. The curve is not then unicursal. But if these equations be solved for x and y, we have λx and λy expressed in the form $P+\sqrt{Q}$, where P and Q are rational polynomials in λ of degrees not higher than 2 and 4 respectively. Hence in this case, for the integration of $\int\phi(x, y)dx$ elliptic integrals will in general be required. Similarly, if the deficiency of the connecting relation be of higher degree, transcendents of a higher complexity than the elliptic integrals would in general be required.

Note I. General Review.

1905. The functions of a single variable x, with which we have been more particularly concerned, may be classed as (I) Algebraic, (II) Transcendental.

(I) An Algebraic function is one which may be theoretically expressed as a root of the equation

$$f_0(x)y^n+f_1(x)y^{n-1}+\ldots+f_n(x)=0,$$

where n is a positive integer and $f_0, f_1, \ldots f_n$ are polynomials, rational as regards x, but in which the coefficients may be either commensurable or incommensurable, real or imaginary, but independent of x.

This will include as particular cases,

(*a*) The general rational integral polynomial.

(*b*) The rational algebraic function, which is the ratio of two rational polynomials.

(*c*) The general irrational species, in which commensurable fractional indices may occur as powers of rational polynomials.

(II) Of Transcendental functions we have such as involve an exponentiation of the variable or the taking of a logarithm. And as the variable may be a complex quantity, this will include, besides the elementary cases of e^x or $\log x$, the trigonometrical or hyperbolic functions and their inverses. For a single exponentiation or the taking of a logarithm, the function is said to be a transcendent of the first order, but if these operations be repeated the function is said to be a transcendent of the second or higher order. Thus e^{e^x}, $\log\log\log x$ are said to be respectively of the second and third orders of transcendents.

We may also have any arithmetical combination of the sum, difference, product or quotient of two or more of these groups. Such functions are said to be simple or elementary functions.

1906. We have, besides such functions as described above, transcendents of a higher degree of complexity, such as Soldner's function $\mathrm{li}(x)$, which is $\int^x \frac{dx}{\log x}$ or $\int^x \frac{e^x dx}{x}$; the Cosine and Sine integrals, viz. $\mathrm{Ci}(x) \equiv \int^x \frac{\cos x}{x} dx$; $\mathrm{Si}(x) \equiv \int^x \frac{\sin x}{x} dx$; Fresnel's Integrals; Kramp's Integral; Spence's Transcendents, defined as $L^n(1 \pm x) = \pm \frac{x^1}{1^n} - \frac{x^2}{2^n} \pm \frac{x^3}{3^n} - \frac{x^4}{4^n} \pm \text{etc.}$, the Elliptic Integrals, or others which have been computed and tabulated for special purposes.

1907. The problem of Integration with which we have been confronted is this: Supposing that we are given the differential equation $\frac{dy}{dx} = f(x)$, where $f(x)$ is one or other of the known classes of functions, or a combination of them, is it possible for us to solve this equation so that y can be recognised as itself one or other of these classes of functions or a combination of them? When no such solution exists y is a new transcendent.

1908. The general discussion as to how completely this question can be answered would occupy much more space than we have at disposal. The reader may be referred to Bertrand, *Calc. Int.*, ch. v., and to *Camb. Math. Tracts*, No. 2 (2nd ed.), by Mr. G. H. Hardy.

But we may remark that, in the first place, if $f(x)$ be a rational function of x, it appears from Chap. V. and the remarks in Note D that the integral y is in all cases partly rational, partly logarithmic; that when the denominator is factorisable into linear or quadratic factors, the complete integral can be found. But when the denominator is of the fifth or higher degree and unfactorisable, though the rational part can be found by Hermite's process, the transcendental logarithmic portion can only be obtained in certain cases. But the only barrier to complete integration in all such general cases is that of the impossibility of solving the general quintic or higher degree equation.

If $f(x)$ be an irrational algebraic function of the form $\frac{A+B\sqrt{Q}}{C+D\sqrt{Q}}$, where A, B, C, D are rational polynomials and Q is a polynomial of not more than the fourth degree, it has been seen that its integration can always be effected, and when the degree of Q is not above the second, only simple functions will be required; but when Q is of the third or fourth degree, the integration will usually call for the assistance of the Elliptic Integrals.

It has also been seen that in all cases in which $\phi(x, y)$ is a rational integral algebraic function of x and y, and y is connected with x by an equation whose graph is unicursal, the integration $\int \phi(x, y)\,dx$ can be effected in terms of the elementary rational algebraic and logarithmic functions.

1909. In addition to these facts, a theorem due to Abel states that if y be an algebraic function of x, defined as above in (I) by the equation $f_0(x)y^n+f_1(x)y^{n-1}+\ldots+f_n(x)=0$, then $\int y\,dx$ can always be expressed as $B_0+B_1y+\ldots+B_{n-1}y^{n-1}$, where $B_0, B_1, \ldots B_{n-1}$ are polynomials in x. And further, that in the case when $y^n=$ a rational function of x, the integral $\int y\,dy=y\times$ a rational function of x. The proof of the first of these theorems is somewhat difficult and long. Reference for them both may be made to the works already cited. Other forms for which $\int y\,dx$ is expressible by means of algebraic functions and logarithms will be found given by Bertrand.

1910. It may be noted that, since differentiation of a function involving irrational algebraic quantities or exponentials cannot destroy them, such quantities cannot appear upon the integration of a function that does not already contain them. Logarithms may appear upon the integration of an algebraic function, but always multiplied by mere constants and by no functions of x. For the operation of differentiation upon the result could not eliminate logarithmic terms otherwise involved.

If, therefore, the integral of an algebraic function be expressible by means of the simple functions at all, it cannot contain exponentials, and whatever logarithmic terms occur are such as to appear in the first degree as transcendents of the first order multiplied by constants.

Many cases have been discussed of the integration $\int f(x)\,dx$, in which $f(x)$ has involved exponential, logarithmic, trigonometric or hyperbolic functions, but there is no general rule which would indicate the nature of the result to be expected as there is in the case of rational algebraic functions, and the theory is far less complete. Reference may be made to Liouville's "Mémoire" (*Jour. f. Math.*, 1835).

PROBLEMS.

1. Integrate

$$(a)\ \frac{4x^5-1}{(x^5+x+1)^2},\quad (b)\ \frac{1-7x^8-8x^9}{(1+x^8+x^9)^2},\quad (c)\ \frac{x+6x^5+12x^6+6x^{11}}{(1+x+x^6)^2}.$$

2. Obtain the rational part of $\int \frac{1+2x+6x^5+13x^6+6x^{11}}{(1+x+x^6)^2}\,dx.$

3. Show that

$$\int_2^3 \frac{x^2(2x^3-1)(x^4-3x^2+2x+1)}{(x^3-x+1)^2(x^4-2x+1)}\,dx=\frac{1}{2}\log\frac{76}{13}-\frac{29}{175}.$$

4. Show that if $\int \frac{ax^2+2bx+c}{(a'x^2+2b'x+c')^2}\,dx$ be rational, $ac'+a'c=2bb'$, and find the integral. [Hardy, No. 2, *Camb. Math. Tracts*, p. 18.

5. Discuss the convergency of the integrals (*a*) $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$, (*b*) $\int_0^\infty x^{n-1} e^{-x} dx$, (*c*) $\int_0^1 \frac{\log x}{1+x} dx$, (*d*) $\int_0^\infty \frac{x^{n-1}}{1+x} dx$.

6. Show that $\int_0^\infty \frac{\sin x}{x} dx$, although convergent, is not *absolutely* convergent. [CARSLAW, *Fourier's Series*, p. 103.]

7. If the function $\phi(x)$ be positive in sign, but diminishing in value as x varies from a to ∞, then the series $\sum_0^\infty \phi(a+x)$ is convergent or divergent according as $\int_0^\infty \phi(x) \, dx$ is finite or infinite, and the series lies between $\int_a^\infty \phi(x) \, dx$ and $\int_{a-1}^\infty \phi(x) \, dx$.

[CAUCHY, BOOLE, *F. Diff.*, p. 126.]

8. If $a > 0$, discuss the convergency of the series

$$\text{(i)} \ \sum_0^\infty \frac{1}{(a+n)^m}; \quad \text{(ii)} \ \sum_0^\infty \frac{1}{(a+n)\{\log(a+n)\}^m};$$

$$\text{(iii)} \ \sum_0^\infty \frac{1}{(a+n)\log(a+n)\{\log\log(a+n)\}^m}.$$

[BOOLE, *l.c.*]

9. In the curve $x^3 + y^3 + b^3 = 3axy$, show that we may express x and y in the form $2x - c + a\lambda = \pm R$, $2y - c + a\lambda = \mp R$, where

$$3R^2 = 4\lambda^3 - 9a^2\lambda^2 + 6ac\lambda - c^2 \quad \text{and} \quad c = a^3 - b^3,$$

by putting $x + y + a = c\lambda^{-1}$.

Hence show that $\int F(x, \sqrt[3]{\alpha + \beta x + \gamma x^2 + \delta x^3}) \, dx$ can in all cases be reduced to an elliptic integral. [*See* HARDY, *l.c. sup.*, p. 50.]

10. Prove that

$$\int_0^\infty f\left(x + \frac{1}{x}\right) \log x \frac{dx}{x} = 0;$$

$$\int_0^\infty f\left(x + \frac{1}{x}\right) \tan^{-1} x \frac{dx}{x} = \frac{\pi}{4} \int_0^\infty f\left(x + \frac{1}{x}\right) \frac{dx}{x}.$$

[LIOUVILLE.]

11. If $f(x)$ be an even function of x, prove that

$$\text{(i)} \ \int_0^\infty f\left(x^2 + \frac{1}{x^2}\right) dx = \int_0^\infty f(x^2 + 2) \, dx;$$

$$\text{(ii)} \ \int_0^{\frac{\pi}{2}} f(\sin 2\theta) \sec \theta \, d\theta = 2 \int_0^{\frac{\pi}{2}} f(\cos^2 \theta) \sec \theta \, d\theta.$$

[GLAISHER.]

12. If $\phi(x)=\phi(2a-x)$, show that

$$\text{(i)}\quad \int_0^a \phi(x)\,F(x)\,dx=\frac{1}{2}\int_0^a \phi(x)\{F(x)+F(a-x)\}\,dx\,;$$

$$\text{(ii)}\quad \int_0^{\frac{\pi}{2}} f(\sin 2\theta)\sin^2\theta\,d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}} f(\sin 2\theta)\,d\theta\,;$$

$$\text{(iii)}\quad \int_0^{\frac{\pi}{2}} f(\sin 2\theta)\sec^2\theta\,d\theta=2\int_0^{\frac{\pi}{2}} f(\cos\theta)\sec^2\theta\,d\theta.$$

[GLAISHER.]

13. If $I_n=\int_0^\pi x^n f(\sin x)\,dx$, show that if n be an odd integer,

$$\text{(i)}\quad 2I_n-n\pi I_{n-1}+\frac{n(n-1)}{1\,.\,2}\pi^2 I_{n-2}-\ldots-\pi^n I_0=0\,;$$

$$\text{(ii)}\quad (n+1)\,I_n-\frac{(n+1)n}{1\,.\,2}\pi I_{n-1}+\frac{(n+1)n(n-1)}{1\,.\,2\,.\,3}\pi^2 I_{n-2}-\ldots-\pi^n I_0=0.$$

[GLAISHER.]

14. Prove that if $\phi(x)=\phi(1-x)$, then will

$$\text{(i)}\quad \int_0^1 \phi(x)\log\Gamma(x)\,dx=\frac{1}{2}\log\pi\int_0^1 \phi(x)\,dx-\frac{1}{2}\int_0^1 \phi(x)\log\sin\pi x\,dx\,;$$

$$\text{(ii)}\quad \int_0^1 \sin\pi x\log\Gamma(x)\,dx=\frac{1}{\pi}\log\pi-\frac{1}{\pi}(\log 2-1)\,;$$

$$\text{(iii)}\quad \int_0^1 \sin^2\pi x\log\Gamma(x)\,dx=\frac{1}{8}(2\log 2\pi-1).$$

[GLAISHER.]

15. By the transformation $x=\dfrac{1-y}{1+y}$, show that

$$\int_0^1 \tan^{-1}\frac{3(1+x)}{1-2x-x^2}\cdot\frac{dx}{1+x^2}=\frac{\pi^2}{8}.$$

[GLAISHER.]

16. Show that the curve $\theta=\phi$ on unit sphere consists of two loops each of area $\pi-2$; θ and ϕ being colatitude and azimuthal angle.

17. Show that the solid angle of the cone

$$z^2(x^2+y^2)^2=x^4(x^2+y^2+z^2)$$

is π.

18. Examine the nature of the curve on unit sphere defined by the equation $2\sin\frac{1}{2}\theta\cos\frac{1}{2}\phi=1$, and show that the solid angle of this cone is $2\sqrt{3}$.

19. Prove that

$$\iint \rho^{-2}\cos\theta\cos\theta'\,dS\,dS' = -\tfrac{1}{2}\iint \log\rho\cos\psi\,ds\,ds',$$

where dS, dS' are any elements of two unclosed surfaces over which the first integral is taken, and ρ the distance between them which makes angles θ and θ' with the normals at its extremities; also ds, ds' are any two elements of their bounding arcs over which the second integral is taken, the directions of these elements of arcs being inclined at an angle ψ. Give an optical interpretation of the result. [MATH. TRIP., 1886.]

[See Arts. 846, 1783, and Herman, *Optics*, Art. 157.]

20. If x, y, z be each real, finite and determinate functions of $\cos\alpha$, $\sin\alpha\cos\beta$ and $\sin\alpha\sin\beta$, the locus of the point x, y, z will be a closed surface containing a volume

$$\frac{1}{3}\int_0^{\pi}\int_0^{2\pi}\begin{vmatrix} x_\alpha, & y_\alpha, & z_\alpha \\ x_\beta, & y_\beta, & z_\beta \\ x, & y, & z \end{vmatrix} d\alpha\,d\beta, \quad \text{where } x_\alpha \equiv \frac{\partial x}{\partial \alpha}, \text{ etc.}$$

[MATH. TRIP., 1870.]

21. The volume enclosed by a closed oval (synclastic) surface is V; its area is S, and I denotes the integral $\iint\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)d\sigma$ extended over the surface, ρ_1, ρ_2 being the principal radii of curvature at the point where $d\sigma$ is the element of area. A sphere of any diameter rolls on the outside of the surface; and for the envelope of the sphere the corresponding integrals are constructed. Show that

$$V - \frac{1}{8\pi}I.S + \frac{1}{192\pi^2}I^3$$

is the same for the envelope as for the original surface.

22. Show that the length of an arc of a curve on the sphere $x^2+y^2+z^2=r^2$ may be expressed in terms of the coordinates u, v of a point on a plane curve by the transformation

$$\frac{x}{4r^2u} = \frac{y}{4r^2v} = \frac{z}{(u^2+v^2-4r^2)r} = \frac{1}{u^2+v^2+4r^2},$$

by the formula
$$s = \int \frac{\sqrt{du^2+dv^2}}{1+(u^2+v^2)/4r^2}.$$

[G. B. MATHEWS, *Nature*, Feb. 1921. Art. on "Einstein's Theory of Relativity".]