XXV

ON THEOREMS RELATING TO SURFACES, OBTAINED BY THE METHOD OF QUATERNIONS*

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The following letter from Sir William R. Hamilton was read, giving some general expressions of theorems relating to surfaces, obtained by his method of quaternions:

'The equation of a curved surface being put under the form

$$f(\rho) = \text{const.}:$$

while its *tangent plane* may be represented by the equation,

$$df(\rho) = 0,$$

or

$$\mathrm{S}.\nu d\rho = 0,$$

if $d\rho$ be the vector drawn to a point of that plane, from the point of contact; the equation of *an osculating surface of the second order* (having complete contact of the second order with the proposed surface at the proposed point) may be thus written:

$$0 = df(\rho) + \tfrac{1}{2}d^2f(\rho);$$

(by the extension of Taylor's series to quaternions); or thus,

$$0 = 2\mathrm{S}.\nu d\rho + \mathrm{S}.d\nu d\rho,$$

if

$$df(\rho) = 2\mathrm{S}.\nu d\rho.$$

'The *sphere, which osculates in a given direction*, may be represented by the equation

$$0 = 2\mathrm{S}\frac{\nu}{\Delta\rho} + \mathrm{S}\frac{d\nu}{d\rho};$$

where $\Delta\rho$ is a chord of the sphere, drawn from the point of osculation, and

$$\mathrm{S}\frac{d\nu}{d\rho} = \frac{\mathrm{S}.d\nu d\rho}{d\rho^2} = \frac{d^2f(\rho)}{2d\rho^2}$$

is a scalar function of the versor $\mathrm{U}d\rho$, which determines the direction of osculation. Hence the important formula:

$$\frac{\nu}{\rho - \sigma} = \mathrm{S}\frac{d\nu}{d\rho};$$

where σ is the vector of the centre of the sphere which osculates in the direction answering to $\mathrm{U}d\rho$.

'By combining this with the expression formerly given by me for a normal to the ellipsoid, namely

$$(\kappa^2 - \iota^2)^2\nu = (\iota^2 + \kappa^2)\rho + \iota\rho\kappa + \kappa\rho\iota,$$

* [See *Lectures*, Lecture VII.]

the known value of the curvature of a normal section of that surface may easily be obtained. And for *any* curved surface, the formula will be found to give easily this general theorem,* which was perceived by me in 1824; that if, on a normal plane OPP', which is drawn through a given normal PO, and through any linear element PP' of the surface, we project the infinitely near normal $P'O'$, which is erected to the same surface at the end of the element PP'; the projection of the near normal will cross the given normal in the centre O of the sphere which osculates to the given surface at the given point P, in the direction of the given element PP'.

'I am able to shew that the formula

$$0 = \delta \mathrm{S} \frac{d\nu}{d\rho},$$

which follows from the above, for determining the directions of osculation of the greatest and least osculating spheres, agrees with my formerly published formula,

$$0 = \mathrm{S} . \nu d\nu d\rho,$$

for the directions of the lines of curvature.

'And I can deduce Gauss's *general* properties of geodetic lines† by showing that if σ_1, σ_2 be the two extreme values of the vector σ, then

$$\frac{-1}{(\rho-\sigma_1)(\rho-\sigma_2)} = \text{measure of curvature of surface} = \frac{1}{R_1 R_2} = \frac{d^2 \mathrm{T} \delta\rho}{\mathrm{T}\delta\rho . d\rho^2};$$

where d answers to motion along a normal section, and δ to the passage from one near (normal) section to another; while S, T, and U, are the characteristics of the operations of taking the scalar, tensor, and versor of a quaternion: and the variation δv of the inclination v of a given geodetic line to a variable normal section, obtained by passing from one such section to a near one, without changing the geodetic line, is expressed by the analogous formula,

$$\delta v = -\frac{d \mathrm{T} \delta\rho}{\mathrm{T} d\rho}.$$

* [See *Lectures*, p. 600.]

† [See *Lectures*, article 617, for details and bibliography.]