VI

PREFACE TO 'LECTURES ON QUATERNIONS'*

[Dublin 1853, pp. 1–64.]

[1.] The volume now offered to the public is designed as an assistance to those persons who may be disposed to study and to employ a certain new mathematical method, which has, for some years past, occupied much of my own attention, and for which I have ventured to propose the name of the Method or Calculus of Quaternions. Although a copious analytical index, under the form of a Table of Contents, will be found to have been prefixed to the work, yet it seems proper to offer here some general and preliminary remarks: especially as regards that conception from which the whole has been gradually evolved, and the motives for giving to the resulting method an appellation not previously in use.

[2.] The difficulties which so many have felt in the doctrine of Negative and Imaginary Quantities in Algebra forced themselves long ago on my attention; and although I early formed some acquaintance with various views or suggestions that had been proposed by eminent writers, for the purpose of removing or eluding those difficulties (such as the theory of direct and inverse quantities, and of indirectly correlative figures, the method of constructing imaginaries by lines drawn from one point with various directions in one plane, and the view which refers all to the mere play of algebraical operations, and to the properties of symbolical language), yet the whole subject still appeared to me to deserve additional inquiry, and to be susceptible of a more complete elucidation. And while agreeing with those who had contended that negatives and imaginaries were not properly *quantities* at all, I still felt dissatisfied with any view which should not give to them, from the outset, a clear interpretation and *meaning*; and wished that this should be done, for the square roots of negatives, without introducing considerations *so expressly geometrical*, as those which involve the conception of an *angle*.

[3.] It early appeared to me that these ends might be attained by our consenting to regard ALGEBRA as being no mere Art, nor Language, nor *primarily* a Science of Quantity; but rather as the Science of Order in Progression. It was, however, a part of this conception, that the *progression* here spoken of was understood to be *continuous* and *unidimensional*: extending indefinitely *forward* and *backward*, but not in any *lateral* direction. And although the successive *states* of such a progression might (no doubt) be represented by *points upon a line*, yet I thought that their simple *successiveness* was better conceived by comparing them with *moments of time*, divested, however, of all reference to *cause* and *effect*; so that the 'time' here considered might be said to be abstract, ideal, or *pure*, like that 'space' which is the object of geometry. In this manner I was led, many years ago, to regard Algebra as the SCIENCE OF PURE TIME: and an Essay,† containing my views respecting it as such, was published‡ in 1835. If I now reproduce

* [Hereafter referred to as *Lectures*.]

† [See I.]

‡ I was encouraged to entertain and publish this view, by remembering some passages in Kant's Criticism of the Pure Reason, which appeared to justify the expectation that it should be *possible* to construct, *à priori*, a Science of Time, as well as a Science of Space. For example, in his Transcendental Æsthetic,

a few of the opinions put forward in that early Essay, it will be simply because they may assist the reader to place himself in that *point of view*, as regards the first elements of *algebra*, from which a passage was gradually made by me to that comparatively *geometrical* conception which it is the aim of this volume to unfold. And with respect to anything unusual in the *interpretations* thus proposed, for some simple and elementary notations, it is my wish to be understood as not at all insisting on them as *necessary*,* but merely proposing them as consistent among themselves, and preparatory to the study of the quaternions, in at least one aspect of the latter.

[4.] In the view thus recently referred to, if the letters A and B were employed as *dates*, to denote any two *moments* of time, which might or might not be distinct, the case of the coincidence or *identity* of these two moments, or of *equivalence* of these two dates, was denoted by the equation,

$$\mathrm{B}=\mathrm{A};$$

which symbolic assertion was thus interpreted as not involving any *original* reference to *quantity*, nor as expressing the result of any comparison between two *durations* as *measured*. It corresponded to the conception of simultaneity or *synchronism*; or, in simpler words, it represented the thought of the *present* in time. Of all possible answers to the general question, '*When*,' the *simplest* is the answer, 'Now:' and it was the *attitude of mind*, assumed in the making of this answer, which (in the system here described) might be said to be originally symbolized by the *equation* above written. And, in like manner, the two formulae of *non*-equivalence,

$$\mathrm{B}>\mathrm{A}, \quad \mathrm{B}<\mathrm{A},$$

were interpreted, without any *primary* reference to quantity, as denoting the two contrasted relations of *subsequence* and of *precedence*, which answer to the thoughts of the *future* and the *past* in time; or as expressing, simply, the one that the moment B is conceived to be *later* than A, and the other that B is *earlier* than A: without *yet* introducing even the *conception* of a *measure*, to determine *how much later*, or how much earlier, one moment is than the other.

[5.] Such having been proposed as the *first* meanings to be assigned to the three elementary marks $=\ >\ <$, it was next suggested that the *first* use of the mark $-$, in constructing a *science of pure time*, might be conceived to be the forming of a complex symbol $\mathrm{B}-\mathrm{A}$, to denote the *difference between two moments*, or the *ordinal relation* of the moment B to the moment A, whether that relation were one of identity or of diversity; and if the latter, then whether it were one of subsequence or of precedence, and in whatever degree. And *here*, no doubt, in

Kant observes: 'Zeit und Raum sind demnach zwey Erkenntnissquellen, aus denen *à priori* verschiedene synthetische Erkenntnisse geschöpft werden können, wie vornehmlich die reine Mathematik in Ansehung der Erkenntnisse vom Raume und dessen Verhältnissen ein glänzendes Beyspiel gibt. Sie sind nämlich beide zusammengenommen reine Formen aller sinnlichen Anschauung, und machen dadurch synthetische Sätze *à priori* möglich.' Which may be rudely rendered thus: 'Time and Space are therefore two knowledge-sources, from which different synthetic knowledges can be *à priori* derived, as eminently in reference to the knowledge of space and of its relations a brilliant example is given by the pure mathematics. For they are, both together [space and time], pure forms of all sensuous intuition, and make thereby synthetic positions *à priori* possible.' (*Critik der reinen Vernunft*, p. 41. Seventh Edition. Leipzig: 1828.)

* For example, the usual identity $(\mathrm{B}-\mathrm{A})+\mathrm{A}=\mathrm{B}$, which in the older Essay was interpreted with reference to *time*, as in paragraph [8] of this Preface, the letters A and B denoting *moments*, is in the present work (Lecture I, article 25) interpreted, on an analogous plan indeed, but with a reference to *space*, the letters denoting *points*. Still it will be perceived that there exists a close connexion between the two views; a *step*, in each, being conceived to be applied to a *state* of a progression, so as to generate (or conduct to) another state. And generally I think that it may be found useful to compare the interpretations of which a sketch is given in the present Preface, with those proposed in the body of the work.

attending to the *degree* of such diversity between two moments, the conception of *duration*, as *quantity* in time, was introduced: the *full* meaning of the symbol $B-A$, in any particular application, being (on this plan) not known, until we know *how long after*, or how long before, if at all, B is than A. But it is evident that the notion of a certain *quality* (or *kind*) of this diversity, or interval, enters into this conception of a *difference* between moments, at least as fully and as soon as the notion of *quantity*, amount, or duration. The contrast between the Future and the Past appears to be even earlier and more fundamental, in human thought, than that between the Great and the Little.

[6.] After *comparing moments*, it was easy to proceed to *compare relations*; and in this view, by an *extension* of the recent signification [4] of the sign $=$, it was used to denote *analogy* in time; or, more precisely, to express the *equivalence of two marks of one common ordinal relation*, between *two pairs* of moments. Thus the formula,

$$D-C=B-A,$$

came to be interpreted as denoting an *equality between two intervals in time*; or to express that the moment D is *related* to the moment C, *exactly as* B *is to* A, with respect to identity or diversity: the *quantity and quality* of such diversity (when it exists) being here *both* taken into account. A formula of this sort was shewn to admit of *inversion* and *alternation* ($C-D=A-B$, $D-B=C-A$); and generally there could be performed a number of *transformations* and *combinations* of equations such as these, which all admitted of being *interpreted* and *justified* by this mode of viewing the subject, but which *agreed* in all respects with the received *rules* of algebra. On the same plan, the two contrasted formulae of inequalities of differences,

$$D-C>B-A, \quad D-C<B-A,$$

were interpreted as signifying, the one that D was *later*, *relatively* to C, than B to A; and the other that D was *relatively earlier*.

[7.] Proceeding to the mark $+$, I used this sign *primarily* as a mark of combination between a symbol, such as the smaller Roman letter a, of a *step in time*, and the symbol, such as A, of the moment *from* which this *step* was conceived to be made, in order to form a complex symbol, $a+A$, *recording this conception of transition*, and denoting the moment (suppose B) *to* which the step was supposed to conduct. The step or transition here spoken of was regarded as a *mental act*, which might as easily be supposed to conduct *backwards* as *forwards* in the progression of time; or even to be a *null step*, denoted by 0, and producing *no effect* ($0+A=A$). Thus, with these meanings of the signs, the notation

$$B=a+A,$$

denoted the conception that the moment B might be *attained*, or mentally *generated*, by making (in thought) the step a from the moment A. And it appeared to me that without ceasing to regard the symbol $B-A$ as denoting, in one view [5], an *ordinal relation* between two moments, we might *also* use it in the *connected sense* of denoting this *step from one to another*: which would allow us (as in ordinary algebra) to write, with the recent suppositions,

$$B-A=a;$$

the two members of this new equation being here symbols for one common step.

[8.] The usual identity,

$$(B-A)+A=B,$$

came thus to be interpreted as signifying *primarily* (in the Science of Pure Time) a certain conceived *connexion* between the operations, of *determining* the difference between two moments as a *relation*, and of *applying* that difference as a *step*. And the two other familiar and connected identities,

$$\mathrm{C}-\mathrm{A}=(\mathrm{C}-\mathrm{B})+(\mathrm{B}-\mathrm{A}), \quad \mathrm{C}-\mathrm{B}=(\mathrm{C}-\mathrm{A})-(\mathrm{B}-\mathrm{A}),$$

were treated, on the same plan, as originally signifying certain *compositions* and *decompositions* of ordinal relations or of steps in time. A special symbol for *opposition* between any two such relations or steps was proposed; but it was remarked that the more usual notations, $+\mathrm{a}$ and $-\mathrm{a}$, for the step (a) itself, and for the opposite of that step, might, in full consistency with the same general view, be employed, if treated as abridgments for the more complex symbols $0+\mathrm{a}$, $0-\mathrm{a}$: the latter notation presenting *here* no difficulty of interpretation, nor requiring any attempt to conceive the *subtraction* of a *quantity* from *nothing*, but merely the *decomposition* of a *null step* into *two opposite steps*. But *operations on steps*, conducted on this plan, were shewn to agree in all respects with the usual *rules* of algebra, as regarded Addition and Subtraction.

[9.] One *time-step* (b) was next compared with another (a), in the way of algebraic *ratio*, so as to conduct to the conception of a certain complex *relation* (or *quotient*), determined partly by their *relative largeness*, but partly also by their *relative direction*, as similar or opposite; and to the closely connected conception of an algebraic *number* (or *multiplier*), which *operates* at once on the quantity and on the direction of the one step (a), so as to *produce* (or mentally *generate*) the quantity and direction of the other step (b). By a combination of these two conceptions, the usual identity,

$$\frac{\mathrm{b}}{\mathrm{a}}\times\mathrm{a}=\mathrm{b}, \quad \text{or} \quad \mathrm{b}=a\times\mathrm{a}, \quad \text{if} \quad \frac{\mathrm{b}}{\mathrm{a}}=a,$$

received an interpretation; the factor a being a *positive* or a *contra-positive* (more commonly called *negative*) *number*, according as it *preserved* or *reversed* the *direction* of the step on which it operated. The four primary operations, for combining any two such ratios or numbers or factors, a and b, among themselves, were *defined* by four equations which may be written thus, and which were indeed *selected* from the usual formulae of algebra, but were employed with new *interpretations*:

$$(b+a)\times\mathrm{a}=(b\times\mathrm{a})+(a\times\mathrm{a}); \quad (b-a)\times\mathrm{a}=(b\times\mathrm{a})-(a\times\mathrm{a});$$

$$(b\times a)\times\mathrm{a}=b\times(a\times\mathrm{a}); \quad b\div a=(b\times\mathrm{a})\div(a+\mathrm{a}).$$

[10.] *Operations* on algebraic *numbers* (positive or contra-positive) were thus made to depend (in thought) on operations of the same names on *steps*; which were again conceived to involve, in their ultimate analysis, a reference to comparison of *moments*. These conceptions were found to conduct to results agreeing with those usually received in algebra; at least when 0 was treated as a symbol of a *null number*, as well as of a null step [7], and when the symbols, $0+a$, $0-a$, were abridged to $+a$ and $-a$. In this view, there was no difficulty whatever, in interpreting the *product* of *two negative numbers*, as being equal to a *positive* number: the result expressing simply, in this view of it, that *two successive reversals restore* the direction of a step. And other difficulties respecting the *rule of the signs* appeared in like manner to fall away, more perfectly than had seemed to me to take place in any view of algebra, which made the thought of quantity (or of magnitude) the *primary* or *fundamental* conception.

[11.] This theory of algebraic numbers, as ratios of steps in time, was applied so as to include results respecting powers and roots and logarithms: but what it is at present chiefly important to observe is, that because, for the reason just assigned, the *square* of *every* number is *positive*, therefore *no number*, whether positive or negative, could be a *square root of a negative* number, in *this* any more than in *other* views of algebra. At least it was certain that no *single* number, of the kinds above considered, could possibly be such a root: but I thought that without going out of the same *general class* of interpretations, and especially without ceasing to refer all to the notion of *time*, explained and guarded as above, we might conceive and compare *couples of moments*; and so derive a conception of *couples of steps* (in time), on which might be founded a theory of *couples of numbers*, wherein no such difficulty should present itself.

[12.] In this extended view, the symbols A_1 and A_2 being employed to denote the two moments of one such pair or couple, and B_1, B_2 the two moments of another pair, I was led to write the formula,

$$(B_1, B_2) - (A_1, A_2) = (B_1 - A_1, B_2 - A_2);$$

and to explain it as expressing that the *complex ordinal relation* of one *moment-couple* (B_1, B_2) to another moment-couple (A_1, A_2) might be regarded as a *relation-couple*; that is to say, as a *system of two ordinal relations*, $B_1 - A_1$ and $B_2 - A_2$, between the *corresponding moments* of those two moment-couples: the *primary moment* B_1 of the one pair being compared with the primary moment A_1 of the other; and, in like manner, the *secondary moment* B_2 being compared with the secondary moment A_2. But, instead of this (analytical) *comparison* of moments with moments, and thereby of *pair with pair*, I thought that we might also conceive a (synthetical) *generation* [7] of one pair of moments from another, by the *application* of a *pair of steps* [11], or by what might be called the *addition* (see again [7]), of a *step-couple* to a *moment-couple*; and that an *interpretation* might thus be given to the following *identity*, in the theory of couples here referred to:

$$(B_1, B_2) = \{(B_1, B_2) - (A_1, A_2)\} + (A_1, A_2).$$

And other results, respecting the compositions and decompositions of *single ordinal relations*, or of *single steps in time*, such as those referred to in paragraph [8] of this Preface, were easily extended, in like manner, to the corresponding treatment of *complex relations*, and of *complex steps*, of the kinds above described.

[13.] There was no difficulty in interpreting, on this plan, such formulae of *multiplication* and *division*, as

$$a \times (\mathrm{a}_1, \mathrm{a}_2) = (a\mathrm{a}_1, a\mathrm{a}_2); \quad (a\mathrm{a}_1, a\mathrm{a}_2) \div (\mathrm{a}_1, \mathrm{a}_2) = a;$$

where the symbols a_1, a_2 denote any two steps in time, and a any number, positive or negative. But the question became less easy, when it was required to interpret a symbol of the form

$$(\mathrm{b}_1, \mathrm{b}_2) \div (\mathrm{a}_1, \mathrm{a}_2),$$

where b_1, b_2 denoted two steps which could not be derived from the two steps a_1, a_2, through multiplication by *any single number*, such as a. To meet this case, which is indeed the general one in this theory, I was led to introduce the conception [11] of *number-couples*, or of *pairs of numbers*, such as (a_1, a_2); and to regard every *single* number (a) as being a *degenerate form* of such a number-couple, namely of $(a, 0)$; so that the recent formula, for the *multiplication of a step-couple by a number*, might be thus written:

$$(a_1, 0)\,(\mathrm{a}_1, \mathrm{a}_2) = (a_1\mathrm{a}_1, a_1\mathrm{a}_2).$$

It appeared proper to establish also the following formula, for the *multiplication of a primary step*, by an arbitrary number-couple:

$$(a_1, a_2)(\mathrm{a}_1, 0) = (a_1 \mathrm{a}_1, a_2 \mathrm{a}_1);$$

and to regard every such number-couple as being the *sum* of two others, namely, of a *pure primary* and a *pure secondary*, as follows:

$$(a_1, a_2) = (a_1, 0) + (0, a_2):$$

the analogous decomposition of a step-couple having been already established.

[14.] The difficulty of the *general* multiplication of a step-couple by a number-couple came thus to be reduced to that of assigning the product of one pure secondary by another: and the spirit of this whole theory of couples led me to conceive that, for such a product, we ought to have an expression of the form,

$$(0, a_2)(0, \mathrm{a}_2) = (\gamma_1 a_2 \mathrm{a}_2, \gamma_2 a_2 \mathrm{a}_2);$$

the coefficients γ_1 and γ_2 being some two constant numbers, independent of the *step* a_2, and of the *number* a_2: which two coefficients I proposed to call the *constants of multiplication.* These constants might be variously assumed: but reasons were given for adopting the following *selection** of values, as the basis of all subsequent operations:

$$\gamma_1 = -1; \quad \gamma_2 = 0.$$

In this way, the required *law of operation*, of a general number-couple on a general step-couple, as multiplier on multiplicand, was found, with this choice of the *constants*, to be expressed by the formula:

$$(a_1, a_2)(\mathrm{a}_1, \mathrm{a}_2) = (a_1 \mathrm{a}_1 - a_2 \mathrm{a}_2, a_2 \mathrm{a}_1 + a_1 \mathrm{a}_2).$$

And in fact it was easy, with the assistance of this formula, to *interpret the quotient* [13] *of two step-pairs*, as being always equal to a *number-pair*, which could be definitely assigned, when the ratios of the four single steps were given.

[15.] With these conceptions and notations, it was allowed to write the two following equations:

$$(1, 0)(\mathrm{a}, \mathrm{b}) = (\mathrm{a}, \mathrm{b}); \quad (0, 1)(\mathrm{a}, \mathrm{b}) = (-\mathrm{b}, \mathrm{a});$$

and I thought that these two factors, $(1, 0)$ and $(0, 1)$, thus used, might be called respectively the *primary unit*, and the *secondary unit*, of number. It was proposed to establish, by definition, for the chief *operations on number-pairs*, a few rules which seemed to be natural extensions of those already established for the corresponding operations [9] on *single numbers*: and it was seen that because

$$(0, 1)(-\mathrm{b}, \mathrm{a}) = (-\mathrm{a}, -\mathrm{b}) = (-1, 0)(\mathrm{a}, \mathrm{b}),$$

we were allowed, as a consequence of those rules, or of the conception which had suggested them, namely, (compare [33]), by a certain *abstraction* of operators from operand, to establish the formula,

$$(0, 1)^2 = (-1, 0) = -1.$$

A new and (as I thought) clear *interpretation* was thus assigned, for that well-known expression in algebra, *the square root of negative unity*: for it was found that we might consistently write, on the foregoing plan,

$$(0, 1) = (-1, 0)^{\frac{1}{2}} = (-1)^{\frac{1}{2}} = \sqrt{-1};$$

without anything obscure, impossible, or *imaginary*, being in any way involved in the conception.

* In some of my unprinted investigations, other selections of these constants were employed.

[16.] In words, if after *reversing* the direction of the *second* of any two steps, we then *transpose* them, as to order; thus making the old but reversed second step the *first* of the *new* arrangement, or of the new step-couple; and making, at the same time, the old and unreversed first step the *second* of the same new couple; and if we then *repeat* this complex process of reversal and transposition, we shall, upon the whole, have *restored* the *order* of the two steps, but shall have *reversed* the *direction* of *each*. Now, it is the *conceived operator*, in this process of *passing from one pair of steps to another*, which, in the system here under consideration, was denoted by the celebrated symbol $\sqrt{-1}$, so often called IMAGINARY. And it is evident that the process, thus described, has no special reference whatever to the notion of *space*, although it has a reference to the conception of PROGRESSION. The symbol -1 denoted that NEGATIVE UNIT of number, of which the effect, as a *factor*, was to change a *single step* $(+\mathrm{a})$ to its own *opposite step* $(-\mathrm{a})$; and because *two* such reversals *restore*, therefore (see [10]) the usual algebraic equation,

$$(-1)^2 = +1,$$

continued to subsist, in *this* as in other systems. But the symbol $\sqrt{-1}$ was regarded as *not at all less real* than those other symbols -1 or $+1$, although *operating on a different subject*, namely, *on a pair of steps* (a, b), and changing them to a *new pair*, namely, the pair $(-\mathrm{b}, +\mathrm{a})$. And the *form* of this well-known symbol, $\sqrt{-1}$, as an *expression* (in the system here described) for what I had previously written as $(0, 1)$, and had called (see [15]) the SECONDARY UNIT of number, was justified by shewing that the effect of its *operation*, when *twice* performed, *reversed each step* of the pair.

[17.] The more general expression of algebra, $a_1 + \sqrt{-1}\, a_2$, for any (so called) *imaginary root* of a quadratic or other equation, was, on this plan, interpreted as being a symbol of the *number-couple* which I had otherwise denoted by (a_1, a_2); and of which the law of *operation on a step-couple* had already [14] been assigned: as also the analogous law, thence derived,* of its *multiplication by another number-couple*, namely, that which is expressed by the formula,

$$(b_1, b_2)(a_1, a_2) = (b_1 a_1 - b_2 a_2, b_2 a_1 + b_1 a_2).$$

In this view, instead of saying that the usual quadratic equation,

$$x^2 + ax + b = 0,$$

where a and b are supposed to denote two positive or negative numbers, has generally two roots, *real or imaginary*, it would be said that this *other form* of the same equation,

$$(x, y)^2 + (a, 0)(x, y) + (b, 0) = (0, 0),$$

is generally satisfied by *two* (real) *number-couples*; in which, according to the values of a and b, the *secondary number* (y) might or might not be zero. An equation of this sort was called a *couple-equation*, and was regarded as equivalent to a *system of two equations*† *between numbers*:

* The principles of such derivation were only hinted at in the Essay of 1835 (see page 403 of the Volume above cited): but it was perhaps sufficiently obvious that they depended on the 'separation of symbols,' or on the abstraction of a common operand. (Compare paragraphs [15], [33], of the present Preface.)

† M. Cauchy, in his *Cours d'Analyse* (Paris, 1821, page 176), has the remark: 'Toute équation imaginaire n'est que la représentation symbolique de deux équations entre quantités réelles.' That valuable work of M. Cauchy was early known to me: but it will have been perceived that I was induced to look at the whole subject of algebra from a somewhat different point of view, at least on the metaphysical side. As to the word 'numbers,' see a note to [33].

for example, the recent *quadratic couple-equation* breaks itself up into the two following separate equations,

$$x^2 - y^2 + ax + b = 0, \quad 2xy + ay = 0,$$

which always admit of real and numerical solutions, whether $\frac{1}{4}a^2 - b$ be a positive or a negative number; the difference being only that in the former case we are to take the factor $y = 0$, of the second equation of the pair, whereas in the latter case we are to take the *other factor* of that equation, and to suppose $2x + a = 0$. And similar remarks might be made on equations of higher orders: all notion of anything *imaginary*, *unreal*, or *impossible*, being quite excluded from the view.

[18.] The same view was extended, so as to include a theory of powers, roots, and logarithms of number-couples; and especially to confirm a remarkable conclusion which my friend John T. Graves, Esq., had communicated to me (and I believe to others) in 1826, and had published in the Philosophical Transactions* for the year 1829: namely, that *the general symbolical expression for a logarithm is to be considered as involving two arbitrary and independent integers*;† the *general logarithm of unity*, to the Napierian base, being, for example, susceptible of the form,

$$\log 1 = \frac{2\omega'\pi}{2\omega\pi - \sqrt{-1}},$$

where ω, ω' denote *any two whole numbers*, positive or negative or null. In fact, I arrived at an equivalent expression, in my own theory of number-couples, under the form,

$$\underset{\omega\ (e,\,0)}{\overset{\omega'}{\log}}\,(1,0) = \frac{(0, 2\omega'\pi)}{(1, 2\omega\pi)};$$

and generally an expression for the *logarithm-couple*, with the *order* ω, and *rank* ω', of any proposed *number-couple* (y_1, y_2), to any proposed *base-couple* (b_1, b_2), was investigated in such a way as to confirm‡ the results of Mr Graves.

[19.] After remarking that it was he who had proposed those names, of *orders and ranks of logarithms*, that early Essay of my own, of which a very abridged (although perhaps tedious) account has thus been given, continued and concluded as follows: 'But because Mr Graves employed, in his reasoning, the usual principles respecting *Imaginary Quantities*, and was content to prove the symbolical necessity without shewing the interpretation, or inner meaning, of his formulae, the present *Theory of Couples* is published to make manifest that hidden meaning: and to shew, by this remarkable instance, that expressions which seem, according to common views, to be merely symbolical, and quite incapable of being interpreted,

* [See I, p. 96, footnote.]

† It is proper to mention, that results substantially the same, respecting the entrance of two arbitrary whole numbers into the general form of a logarithm, are given by Ohm, in the second volume of his valuable work, entitled: *Versuch eines vollkommen consequenten Systems der Mathematik, vom Professor Dr Martin Ohm* (Berlin, 1829, Second Edition, page 440. I have not seen the first Edition). For other particulars respecting the history of such investigations, on the subject of *general logarithms*, I must here be content to refer to Mr Graves's subsequent Paper, printed in the *Proceedings* of the Sections of the British Association for the year 1834 (Fourth Report, pp. 523 to 531. London, 1835).

‡ Another confirmation of the same results, derived from a peculiar theory of conjugate functions, had been communicated by me to the British Association at Edinburgh in 1834, and may be found reported among the *Proceedings* of the Sections for that year, at pp. 519 to 523 of the Volume lately cited. [See II.] The partial differential 'equations of conjugation,' there given, had, as I afterwards learned, presented themselves to other writers: and the Essay on 'Conjugate Functions, or Algebraic Couples,' there mentioned, was considerably modified, in many respects, before its publication in 1835, in the *Transactions of the Royal Irish Academy*. [See I.]

may pass into the world of thoughts, and acquire reality and significance, if Algebra be viewed as not a mere Art or Language, but as the Science of Pure Time.* The author hopes to publish

* Perhaps I ought to apologize for having thus ventured here to reproduce (although only historically, and as marking the progress of my own thoughts) a view so little supported by scientific authority. I am very willing to believe that (though not unused to calculation) I may have habitually attended too little to the *symbolical* character of Algebra, as a Language, or organized system of *signs*: and too much (in proportion) to what I have been accustomed to consider its *scientific* character, as a Doctrine analogous to Geometry, through the Kantian parallelism between the *intuitions* of Time and Space. This is not a proper opportunity for seeking to do justice to the views of others, or to my own, on a subject of so great subtlety: especially since, in the *present* work, I have thought it convenient to adopt throughout a *geometrical basis*, for the exposition of the theory and calculus of the Quaternions. Yet I wish to state, that I do not despair of being able hereafter to shew that my own old views respecting Algebra, perhaps modified in some respects by subsequent thought and reading, are not fundamentally and irreconcilably opposed to the teaching of writers whom I so much respect as Drs Ohm and Peacock. The *Versuch*, &c., of the former I have cited (the date of the first Volume of the Second Edition is Berlin, 1828): and it need scarcely be said (at least to readers in these countries) that my other reference is to the *Algebra* (Cambridge, 1830); the *Report on Certain Branches of Analysis*, printed in the Third Report of the British Association for the Advancement of Science (London, 1834); the *Arithmetical Algebra* (Cambridge, 1842); and the *Symbolical Algebra* (Cambridge, 1845): all by the Rev. George Peacock. I by no means dispute the possibility of constructing a consistent and useful system of algebraical calculations, by starting with the notion of *integer number*; unfolding that notion into its necessary consequences; expressing those consequences with the help of *symbols*, which are already general in *form*, although supposed at first to be limited in their signification, or *value*: and then, by *definition*, for the sake of *symbolic generality*, *removing* the *restrictions* which the original notion had imposed; and so resolving to *adopt*, as perfectly *general* in *calculation*, what had been only *proved* to be *true* for a certain subordinate and limited extent of *meaning*. Such seems to be, at least in part, the view taken by each of the two original and thoughtful writers who have been referred to in the present Note: although Ohm appears to dwell more on the study of the *relations* between the fundamental *operations*, and Peacock more on the *permanence* of equivalent *forms*. But I confess that I do not find myself able to frame a distinct *conception* of *number*, without *some* reference to the thought of *time*, although this reference may be of a somewhat abstract and transcendental kind. I cannot fancy myself as *counting* any set of things, without first *ordering* them, and treating them as *successive*: however *arbitrary* and *mental* (or *subjective*) this assumed succession may be. And by consenting to *begin* with the abstract notion (or pure intuition) of TIME, as the *basis* of the exposition of those axioms and inferences which are to be expressed by the symbols of algebra, (although I grant that the commencing with the more familiar conception of *whole number* may be more convenient for purposes of elementary instruction), it still appears to me that an advantage would be gained: because the necessity for any merely *symbolical extension* of formulae would be at least considerably *postponed* thereby. In fact (as has been partly shewn above), *negatives* would then present themselves as easily and naturally as positives, through the fundamental contrast between the thoughts of *past* and *future*, used *here* as no mere *illustration* of a result otherwise and symbolically deduced, without any clear comprehension of its meaning, but as the very *ground* of the reasoning. The ordinary *imaginaries* of algebra could be *explained* (as above) by *couples*; but might *then*, for convenience of calculation, be *denoted* by *single letters*, subject to all the ordinary *rules*, which rules would *follow* (on this plan) from the combination of *distinct conceptions* with *definitions*, and would offer no result which was not perfectly and easily *intelligible*, in strict consistency with that *original* thought (or intuition) of time, from which the whole theory should (on this supposition) be evolved. The doctrine of the n roots of an equation of the n^{th} degree (for example) would thus suffer no attaint as to *form*, but would acquire (I think) new clearness as to *meaning*, without any assistance from geometry. The *quaternions*, as I have elsewhere shewn (*Trans. Roy. Irish Acad.* vol. XXI (1848), pp. 199–296; [see VII]), and even the *biquaternions* (as I hope to shew hereafter), [*Lectures*, articles 637, 669 *et seq.*] might have their laws explained, and their symbolical results interpreted, by comparisons of *sets of moments*, and by operations on *sets of steps* in time. Thus, in the phraseology of Dr Peacock, we should have a very wide 'science of suggestion' (or rather, suggestive science) as our *basis*, on which to build up afterwards a new structure of purely *symbolical generalization*, if the *science of time* were adopted, instead of merely Arithmetic, or (primarily) the doctrine of *integer number*. Still I admit fully that the actual *calculations* suggested by this, or by any other view, must be performed according to some fixed *laws of combination of symbols*, such as Professor De Morgan has sought to reduce, for ordinary algebra, to the smallest possible compass, in his Second Paper on the Foundation of Algebra (*Trans. Camb. Phil. Soc.* vol. VII (1839), pp. 173–187, 287–300), and in his work entitled *Trigonometry and Double Algebra* (London,

hereafter many other applications of this view; especially to Equations and Integrals, and to a Theory of Triplets and Sets of Moments, Steps, and Numbers, which includes this Theory of Couples.'*

[20.] The theory of *triplets* and *sets*, thus spoken of at the close of the Essay of 1835, had in fact formed the subject of various unpublished investigations, of which some have been preserved: and a brief notice of them here (especially as relates to triplets†) may perhaps be useful, by assisting to throw light on the nature of the passage, which I gradually came to make, from *couples* to *quaternions*.

Without departing from the same general view of algebra, as the science of pure time, it was obvious that no necessity existed for any limitation to *pairs*, of moments, steps, and numbers. Thus, instead of comparing, as in [12], *two moments*, B_1 and B_2, with two other moments, A_1 and A_2, it was possible to compare *three* moments, B_1, B_2, B_3, with three *other* moments, A_1, A_2, A_3; that is, more fully, to compare (or to conceive as compared) the *homologous* moments of these two *triads*, primary with primary, secondary with secondary, and tertiary with tertiary; and so to obtain a certain system or *triad of ordinal relations*, or a *triad of steps* in time, which might be denoted (compare [5], [7], [12]) by either member of the following equation:

$$(B_1, B_2, B_3) - (A_1, A_2, A_3) = (B_1 - A_1, B_2 - A_2, B_3 - A_3).$$

And on the same plan (compare [7], [8], [12]), if we denote the three *constituent steps* of such a triad as follows,

$$B_1 - A_1 = a_1, \quad B_2 - A_2 = a_2, \quad B_3 - A_3 = a_3,$$

it was allowed to write,

$$(B_1, B_2, B_3) = (a_1, a_2, a_3) + (A_1, A_2, A_3);$$

a triad of steps being thus (symbolically) *added* (or applied) to a triad of moments, so as to conduct (in thought) to another triad of moments. It appeared also convenient to establish the following formula, for the *addition of step-triads*,

$$(b_1, b_2, b_3) + (a_1, a_2, a_3) = (b_1 + a_1, b_2 + a_2, b_3 + a_3),$$

as denoting a certain *composition* of two such triads of steps, answering to that *successive application* of them to any given triad of moments (A_1, A_2, A_3), which conducts ultimately to a *third triad* of moments, namely, to the triad (C_1, C_2, C_3), if

$$C_1 - B_1 = b_1, \quad C_2 - B_2 = b_2, \quad C_3 - B_3 = b_3.$$

Subtraction of one step-triad from another was explained (see again [8]) as answering to the analogous decomposition of a given step-triad into others; or to a system of *three distinct*

1849): and that in following out such *laws* to their symbolical consequences, uninterpretable (or at least uninterpreted) *results* may be expected to arise. In the present Volume (as has been already observed), I have thought it expedient to present the quaternions under a *geometrical aspect*, as one which it may be perhaps more easy and interesting to contemplate, and more immediately adapted to the subsequent applications, of geometrical and physical kinds. And in the passage which I have made (in the Seventh Lecture), from *quaternions* considered as *real* (or as geometrically *interpreted*), to *biquaternions* considered as *imaginary* (or as geometrically *uninterpreted*), but as symbolically *suggested* by the generalization of quaternion formulae, it will be perceived, by those who shall do me the honour to read this work with attention, that I have employed a *method of transition*, from *theorems proved* for the *particular* to *expressions assumed* for the *general*, which bears a very close *analogy* to the methods of Ohm and Peacock: although I have *since* thought of a way of *geometrically interpreting the biquaternions* also.

* *Trans. Roy. Irish Acad.* vol. XVII (1837), p. 422 [see I, p. 96].

† These remarks on *triplets* are now for the first time published.

decompositions of so many single steps, each into two others, of which one was given; and it was expressed by the formula,

$$(c_1, c_2, c_3) - (a_1, a_2, a_3) = (c_1 - a_1, c_2 - a_2, c_3 - a_3):$$

while the usual rules of algebra were found to hold good, respecting *such* additions and subtractions of triads.

[21.] *Multiplication* of a step-triad by a positive or negative *number* (a) was easy, consisting simply in the multiplication of *each constituent step* by that number; so that I had the equation,

$$a(a_1, a_2, a_3) = (aa_1, aa_2, aa_3):$$

and conversely it was natural (compare [13]) to establish the following formula for a certain *case of division of step-triads*,

$$(aa_1, aa_2, aa_3) \div (a_1, a_2, a_3) = a.$$

But in the more general case (compare again [13]), where the steps b_1, b_2, b_3 of one triad were *not proportional* to the steps a_1, a_2, a_3, it seemed to me that the *quotient* of these two step-triads was to be interpreted, on the same general plan, as being equal to a certain triad or *triplet of numbers*, a_1, a_2, a_3; so that there should be conceived to exist generally two equations of the forms,

$$(b_1, b_2, b_3) \div (a_1, a_2, a_3) = (a_1, a_2, a_3);$$

$$(b_1, b_2, b_3) = (a_1, a_2, a_3)\,(a_1, a_2, a_3):$$

the *three* (positive or negative) *constituents* of this *numerical triplet* (a_1, a_2, a_3) depending, according to some definite laws, on the *ratios* of the *six steps*, $a_1, a_2, a_3, b_1, b_2, b_3$.

[22.] In this way there came to be conceived *three distinct and independent unit-steps*, a primary, a secondary, and a tertiary, which I denoted by the symbols,

$$1_1, \quad 1_2, \quad 1_3;$$

and also *three unit-numbers*, primary, secondary, and tertiary, each of which might *operate*, as a species of *factor*, or multiplier, on each of these three steps, or on their system, and which I denoted by these other symbols,

$$x_1, \quad x_2, \quad x_3:$$

or sometimes more fully thus,

$$(1,0,0), \quad (0,1,0), \quad (0,0,1).$$

A *triad of steps* took thus the form,

$$r1_1 + s1_2 + t1_3,$$

where r, s, t were *three numerical coefficients* (positive or negative), although $1_1\,1_2\,1_3$ were still supposed to denote *three steps in time*; and any *triplet factor*, such as (m, n, p), by which this *step-triplet* was to be multiplied, or *operated* upon, might be put under the analogous form,

$$mx_1 + nx_2 + px_3.$$

Continuing then to admit the *distributive* property of multiplication, it was only necessary to fix the significations of the *nine products*, or combinations, obtained by operating separately with *each* of the three units of number on *each* of the three units of step: every such product, or result, being conceived, in this theory, to be *itself*, in general, a *step-triad*, of which, however, some of the component steps might vanish. Hence, after writing

$$x_1\,1_1 = 1_{1,1}; \quad x_1\,1_2 = 1_{2,1}; \quad \ldots \quad x_3\,1_2 = 1_{2,3}; \quad x_3\,1_3 = 1_{3,3},$$

I proceeded to develop these *nine step-triplets* into *nine trinomial expressions* of the forms,

$$1_{f,g} = 1_{f,g,1}\,1_1 + 1_{f,g,2}\,1_2 + 1_{f,g,3}\,1_3,$$

where the *twenty-seven* symbols of the form $1_{f,g,h}$ represented certain *fixed numerical coefficients*, or *constants of multiplication*, analogous to those denoted by γ_1 and γ_2 in [14], and like them requiring to have their values *previously assigned*, before proceeding to multiplication, if it were demanded that the operation of a given triplet of numbers on a given triplet of steps should produce a perfectly *definite step-triad* as its result.

[23.] Conversely, when once these numerical *constants* had been assigned, I saw that the equation of multiplication,

$$(m\mathrm{x}_1+n\mathrm{x}_2+p\mathrm{x}_3)\,(r1_1+s1_2+t1_3)=x1_1+y1_2+z1_3,$$

was to be regarded as breaking itself up, on account of the supposed *mutual independence* of the three unit-steps, into *three ordinary algebraical equations, between the nine numbers*, $m, n, p, r, s, t, x, y, z$; namely, between the *coefficients* of the multiplier, multiplicand, and product. These three equations were *linear*, relatively to m, n, p (as also with respect to r, s, t, and x, y, z); and therefore while they gave, *immediately*, expressions for the coefficients xyz of the *product*, and so resolved *expressly* the problem of *multiplication*, they enabled me, through a simple system of three linear and ordinary equations, to resolve also the *converse* problem [21] of the *division* of one triad of steps by another: or to determine the coefficients mnp of the following *quotient* of two such triads,

$$m\mathrm{x}_1+n\mathrm{x}_2+p\mathrm{x}_3=(x1_1+y1_2+z1_3)\div(r1_1+s1_2+t1_3).$$

[24.] Such were the most essential elements of that *general* theory of triplets, which occurred to me in 1834 and 1835: but it is clear that, in its *applications*, everything depended on the *choice of the twenty-seven constants of multiplication*, which might *all be arbitrarily assumed, before* proceeding to *operate*, but were *then* to be regarded as *fixed*. It was *natural*, indeed, to consider the *primary number-unit* x_1 as producing *no change* in the step or triad on which it operates; and it was *desirable* to determine the constants so as to satisfy the condition,

$$\mathrm{x}_3\,\mathrm{x}_2=\mathrm{x}_2\,\mathrm{x}_3,$$

for the sake of conforming to analogies of algebra. Accordingly, in one of several triplet-systems which I tried, the constants were so chosen as to satisfy these conditions, by the assumptions,

$$\mathrm{x}_1 1_1=1_1,\quad \mathrm{x}_1 1_2=1_2,\quad \mathrm{x}_1 1_3=1_3,$$
$$\mathrm{x}_2 1_1=1_2,\quad \mathrm{x}_2 1_2=1_1+(b-b^{-1})\,1_2,\quad \mathrm{x}_2 1_3=b1_3,$$
$$\mathrm{x}_3 1_1=1_3,\quad \mathrm{x}_3 1_2=b1_3,\quad \mathrm{x}_3 1_3=1_1+b1_2+c1_3;$$

which still involved two arbitrary numerical constants, b and c, and gave, by a combination of *successive operations*, on any arbitrary *step-triad* (such as $r1_1+s1_2+t1_3$, whatever the *coefficients* r, s, t of this *operand triad* might be), the following *symbolic equations*,* expressing the *properties of the assumed operators*, $\mathrm{x}_2, \mathrm{x}_3$, and the laws of their mutual combinations:

$$\mathrm{x}_2^2=(b-b^{-1})\,\mathrm{x}_2+1;\qquad \mathrm{x}_2\mathrm{x}_3=\mathrm{x}_3\mathrm{x}_2=b\mathrm{x}_3;\qquad \mathrm{x}_3^2=c\mathrm{x}_3+b\mathrm{x}_2+1;$$

while the factor x_1 was suppressed, as being simply equivalent, in this system, to the factor 1, or to the ordinary unit of number. But although the symbol x_2 appeared thus to be given by a *quadratic* equation, with the *two real roots* b and $-b^{-1}$, I saw that it would be improper to *confound* the *operation* of this *peculiar* symbol x_2 with that of *either* of these two *numerical roots*, of that quadratic but *symbolical equation*, regarded as an *ordinary* multiplier. It was not

* These symbolic equations are copied from a manuscript of February 1835.

either, *separately*, of the two operations $x_2 - b$ and $x_2 + b^{-1}$, which, when performed on a *general step-triad*, reduced that triad to another with every step a *null* one: but the *combination* of these two operations, successively (and in either order) performed.

[25.] In the same particular triplet system, the three general equations [23] between the nine numerical coefficients, of multiplier, multiplicand, and product, became the following:

$$x = mr + ns + pt; \qquad y = ms + nr + (b - b^{-1})\,ns + bpt; \qquad z = mt + pr + b(nt + ps) + cpt;$$

whence it was possible, *in general*, to determine the coefficients m, n, p, of the quotient of any two proposed step-triads. The same three equations were found to hold good also, when the *number-triplet* (x, y, z) was considered as the *symbolical product of the two number-triplets*, (m, n, p) and (r, s, t); *this* product being obtained by a certain *detachment* (or separation) of the symbols of the *operators* from that of a common *operand*, namely here an arbitrary *step-triad*. In other words, the *same algebraical equations* between the nine numerical coefficients, x, y, z, m, n, p, r, s, t, expressed *also* the conditions involved in the formula of symbolical multiplication,

$$(x, y, z) = (m, n, p)\,(r, s, t),$$

regarded as an *abridgment* of the following *fuller* formula:

$$(x, y, z)\,(a_1, a_2, a_3) = (m, n, p)\,(r, s, t)\,(a_1, a_2, a_3);$$

where a_1, a_2, a_3 might denote *any three steps* in time. Or they might be said to be the conditions for the correctness of this other *symbolical equation*,

$$x x_1 + y x_2 + z x_3 = (m x_1 + n x_2 + p x_3)\,(r x_1 + s x_2 + t x_3),$$

interpreted on the same plan as the symbols x_2^2, $x_2 x_3$, $x_3 x_2$, x_3^2, in [24].

[26.] All the peculiar properties of the lately mentioned triplet system might be considered to be contained in the three ordinary and algebraical equations, [25], which connected the nine coefficients with each other (and in this case with two arbitrary constants). And I saw that these equations admitted of the three following combinations, by the ordinary processes of algebra:

$$x - b^{-1}y = (m - b^{-1}n)\,(r - b^{-1}s);$$

$$x + by + az = (m + bn + ap)\,\,(r + bs + at);$$

$$x + by + a'z = (m + bn + a'p)\,(r + bs + a't);$$

where a, a' were the two real and unequal roots of the ordinary quadratic equation,

$$a^2 = ca + b^2 + 1.$$

Here, then, was an *instance* of what occurred in *every other triplet system* that I tried, and seemed indeed to be a general and necessary consequence of the *cubic form* of a certain function, obtained by elimination between the three equations mentioned in [23], at least if we still (as is natural) suppose that $x_1 = 1$: namely, that *the product of two triplets may vanish*, *without either factor vanishing*. For if (as *one* of the ways of exhibiting this result), we assume

$$n = bm, \quad r = -bs, \quad t = 0,$$

the recent relations will then give

$$x = 0, \quad y = 0, \quad z = 0;$$

so that, whatever values may be assigned to m, p, s, we have, in this system, the formula:

$$(m, bm, p)\,(-bs, s, 0) = (0, 0, 0).$$

For the same reason, there were *indeterminate cases*, in the operation of *division of triplets*: for example, if it were required to find the coefficients mnp of a quotient, from the equation

$$(m,n,p)(-bs,s,0)=(x,y,z),$$

we should only be able to determine the function $m-b^{-1}n$, but not the numbers m and n themselves; while p would be entirely undetermined: at least if $x+by$ and z were each $=0$, for otherwise there might come *infinite* values into play.

[27.] The foregoing reasonings respecting triplet systems were quite independent of any sort of *geometrical interpretation*. Yet it was natural to interpret the results, and I did so, by conceiving the three sets of coefficients, (m,n,p), (r,s,t), (x,y,z), which belonged to the three triplets in the multiplication, to be the *co-ordinate projections*, on three rectangular axes, of *three right lines* drawn from a common origin; which *lines* might (I thought) be said to be, respectively, in this system of interpretation, the multiplier line, the multiplicand line, and the product line. And then, in the particular triplet system recently described, the formulae of [26] gave easily a simple rule, for *constructing* (on this plan) the *product of two lines in space*. For I saw that if *three fixed and rectangular lines*, A, B, C, distinct from the original axes, were determined by the three following pairs of ordinary equations in co-ordinates:

$$x+by=0,\quad z=0,\quad \text{for line } A;$$

$$y-bx=0,\quad z-ax=0,\ \ldots\ B;$$

$$y-bx=0,\quad z-a'x=0,\ldots\ C;$$

we might then enunciate this *theorem*:*

'If a line L'' be the product of two other lines L, L', then on whichever of the three rectangular lines A, B, C we project the two factors L, L', the product (in the ordinary meaning) of their two projections is equal to the product of the projections (on the same) of L'' and U, U being the primary unit-line $(1,0,0)$.'

[28.] I saw also that it followed from this theorem, or more immediately from the equations lately cited [26], from which the theorem itself had been obtained, that if we considered *three rectangular planes*, A', B', C', perpendicular respectively to the three lines A, B, C, or having for their equations,

$$y-bx=0,(A');\quad x+by+az=0,(B');\quad x+by+a'z=0,(C');$$

then *every line* in *any one* of these three fixed planes gave a *null product line*, when it was multiplied by a line *perpendicular* to that fixed plane: the line A, for example, as a factor, giving a null line as the product, when combined with any factor line in the plane A'. For the same reason (compare [26]), although the *division* of one line by another gave *generally* a determinate *quotient-line*, yet if the *divisor-line* were situated in any one of the three planes A', B', C', this quotient-line became then *infinite*, or *indeterminate*. And results of the same general character, although not all so simple as the foregoing, presented themselves in my examinations of various *other* triplet systems: there being, in all those which I tried, at least *one* system of line and plane, analogous to (A) and (A'), but not always *three* such (real) systems, not always at *right angles* to each other.

* This theorem is here copied, without any modification, from the manuscript investigation of February 1835, which was mentioned in a former note.

[29.] These speculations interested me at the time, and some of the results appeared to be not altogether inelegant. But I was dissatisfied with the departure from ordinary analogies of algebra, contained in the *evanescence* [26] [28] of a *product* of two triplets (or of two lines), in certain cases when neither *factor* was null; and in the connected *indeterminateness* (in the same cases) of a *quotient*, while the *divisor* was different from zero. There seemed also to be too much room for *arbitrary choice of constants*, and not any sufficiently decided reasons for finally preferring *one* triplet system to another. Indeed the assumption of the symbolic equation [24], $\mathrm{x}_1 = 1$, which it appeared to be convenient and *natural* to make, although *not essential* to the theory, determined immediately the values of *nine* out of the twenty-seven constants of multiplication; and *six* others were obtained from the assumptions, which also seemed to be *convenient* (although in *some* of my investigations the latter was not made),

$$\mathrm{x}_2\,\mathrm{l}_1 = \mathrm{l}_2, \quad \mathrm{x}_3\,\mathrm{l}_1 = \mathrm{l}_3.$$

The supposed *convertibility* (see again [24]), of the *order* of the two operations x_2 and x_3, gave then the three following conditions,

$$\mathrm{x}_3\,\mathrm{x}_2\,\mathrm{l}_1 = \mathrm{x}_2\,\mathrm{x}_3\,\mathrm{l}_1, \quad \mathrm{x}_3\,\mathrm{x}_2\,\mathrm{l}_2 = \mathrm{x}_2\,\mathrm{x}_3\,\mathrm{l}_2, \quad \mathrm{x}_3\,\mathrm{x}_2\,\mathrm{l}_3 = \mathrm{x}_2\,\mathrm{x}_3\,\mathrm{l}_3,$$

of which the first was seen at once to establish *three* relations between six of the twelve remaining coefficients of multiplication, namely (if the subscript commas be here for conciseness omitted),

$$\mathrm{l}_{231} = \mathrm{l}_{321}, \quad \mathrm{l}_{232} = \mathrm{l}_{322}, \quad \mathrm{l}_{233} = \mathrm{l}_{323}.$$

The two other equations between step-triads, given by the recent conditions of convertibility, resolved themselves into six equations between coefficients, which were, however, perceived to be not all independent of each other, being in fact all satisfied by satisfying the *three* following:

$$\mathrm{l}_{321} = \mathrm{l}_{223}\,\mathrm{l}_{332} - \mathrm{l}_{233}\,\mathrm{l}_{322};$$

$$\mathrm{l}_{221} = \mathrm{l}_{233}(\mathrm{l}_{233} - \mathrm{l}_{222}) + \mathrm{l}_{223}(\mathrm{l}_{322} - \mathrm{l}_{333});$$

$$\mathrm{l}_{331} = \mathrm{l}_{332}(\mathrm{l}_{233} - \mathrm{l}_{222}) + \mathrm{l}_{322}(\mathrm{l}_{322} - \mathrm{l}_{333});$$

of which the two former presented themselves to me under forms a little simpler, because, for the sake of preserving a *gradual ascent* from couples to triplets, or for preventing a *tertiary term* from appearing in the product, when no such term occurred in either factor, I assumed the value,

$$\mathrm{l}_{223} = 0.$$

There still remained *five* arbitrary coefficients,

$$\mathrm{l}_{222}, \quad \mathrm{l}_{322}, \quad \mathrm{l}_{323}, \quad \mathrm{l}_{332}, \quad \mathrm{l}_{333},$$

which it seemed to be permitted to choose at pleasure: but the decomposition of a certain *cubic function* [26] of r, s, t into *factors*, combined with *geometrical considerations*, led me, for the sake of securing the *reality* and *rectangularity* of a certain system of *lines* and *planes*, to assume the three following relations between those coefficients:

$$\mathrm{l}_{222} = \mathrm{l}_{323} - \mathrm{l}_{323}^{-1}, \quad \mathrm{l}_{322} = 0, \quad \mathrm{l}_{332} = \mathrm{l}_{323};$$

which gave also the values, $\mathrm{l}_{221} = 1, \quad \mathrm{l}_{321} = 0, \quad \mathrm{l}_{331} = 1.$

But the two constant coefficients l_{323} and l_{333} still seemed to remain wholly arbitrary,* and

* The system of constants $b = 1$, $c = 1$, might have deserved attention, but I do not find that it occurred to me to consider it. In some of those old investigations respecting triplets, the symbol $\sqrt{-1}$ presented itself as a coefficient: but this at the time appeared to me unsatisfactory, nor did I see how to interpret it in such a connexion.

were those undetermined elements, denoted by b and c, which entered into the formulae of triplet multiplication [25], already cited in this Preface.

[30.] I saw, however, as has been already hinted [19] [20], that the same general *view* of algebra, as the science of pure time, admitted easily, at least in thought, of an *extension* of this whole theory, not only from couples to triplets, but also from triplets to *sets*, of moments, steps, and numbers. Instead of *two* or even *three* moments (as in [12] or [20]), there was no difficulty in conceiving a system or *set* of n such moments, $A_1, A_2, \ldots, A_n$, and in supposing it to be compared with another *equinumerous momental set*, $B_1, B_2, \ldots, B_n$, in such a manner as to conduct to a new complex ordinal relation, or *step-set*, denoted by the formula,

$$(B_1, B_2, \ldots, B_n) - (A_1, A_2, \ldots, A_n) = (B_1 - A_1, B_2 - A_2, \ldots, B_n - A_n).$$

Such step-sets could be *added* or *subtracted* (compare [20]), by adding or subtracting their *component steps*, each to or from its own corresponding step, as indicated by the double formula,

$$(b_1, b_2, \ldots, b_n) \pm (a_1, a_2, \ldots, a_n) = (b_1 \pm a_1, b_2 \pm a_2, \ldots, b_n \pm a_n);$$

and a step-set could be *multiplied* by a *number* (a), or *divided* by *another step-set*, provided that the component steps of the one were *proportional* to those of the other (compare [13] [21]), by the formulae:

$$a(a_1, a_2, \ldots, a_n) = (aa_1, aa_2, \ldots, aa_n); \quad (aa_1, aa_2, \ldots, aa_n) \div (a_1, a_2, \ldots, a_n) = a.$$

[31.] But when it was required to divide one step-set by another, in the more general case (compare [13] [14] [21]), where the components or *constituent steps* $a_1, a_2, \ldots, a_n$ of the one set were *not* proportional to the corresponding components $b_1, b_2, \ldots, b_n$ of the other set, a difficulty again arose, which I proposed still to meet on the same general plan as before, by conceiving that a *numeral set*, or set or *system of numbers*, $(a_1, a_2, \ldots, a_n)$, might *operate* on the *one* set of steps, $(a_1, a_2, \ldots, a_n)$, in a way *analogous to multiplication*, so as to *produce* or generate the *other* given step-set, as a result which should be *analogous to a product*. Instead of *three* distinct and independent unit-steps, as in [22], I now conceived the existence of n such *unit-steps*, which might be denoted by the symbols,

$$1_1, 1_2, \ldots, 1_n;$$

and instead of *three unit-numbers* (see again [22]), I conceived n such *unit-operators*, which in those early investigations I denoted

$$x_1, x_2, \ldots, x_n,$$

and of which I conceived that *each* might operate on *each* unit-step, as a species of *multiplier*, or *factor*, so as to produce (generally) a *new step-set* as the result. There came thus to be conceived a number, $=n^2$, of such resultant step-sets, denoted, on the plan of [22], by symbols of the forms:

$$x_g 1_f = 1_{f,g,1} 1_1 + 1_{f,g,2} 1_2 + \ldots + 1_{f,g,n} 1_n;$$

where the n^3 symbols of the form $1_{f,g,h}$ denoted so many *numerical coefficients*, or *constants of multiplication*, of the kind previously considered in the theories of couples [14], and of triplets [22], which *all* required to have their values *previously assumed*, or assigned, *before* proceeding to *multiply* a step-set by a number-set, in order that this operation might give generally a *definite step-set* as the result.

[32.] Conversely, on the plan of [23], when the n^3 numerical *values* of these coefficients or constants $1_{f,g,h}$ had been once fixed, I saw that we could then definitely interpret a *product* of the form,

$$(mx_1 + \ldots + m_g x_g + \ldots + m_n x_n)(r_1 1_1 + \ldots + r_f 1_f + \ldots + r_n 1_n),$$

where $m_1, \ldots m_g, \ldots m_n$ and $r_1, \ldots r_f, \ldots r_n$ were any $2n$ given numbers, as being equivalent to a certain new or *derived* step-set of the form,

$$x_1 \mathrm{l}_1 + \ldots + x_h \mathrm{l}_h + \ldots + x_n \mathrm{l}_n;$$

where $x_1, \ldots x_h, \ldots x_n$ were n new or *derived numbers*, determined by n expressions such as the following:

$$x_h = \Sigma m_g r_f \mathrm{l}_{f,g,h};$$

the summation extending to all the n^2 combinations of values of the indices f and g. And because these expressions might in general be treated as a system of *n linear equations* between the n coefficients m_g of the multiplier set, I thought that the *division of one step-set by another* (compare [14] [23]), might thus in general be accomplished, or at least conceived and interpreted, as being the process of *returning to that multiplier*, or of *determining the numeral set* which would *produce the dividend step-set*, by *operating on the divisor step-set*, and which might therefore be denoted as follows:

$$m_1 \mathrm{x}_1 + \ldots + m_g \mathrm{x}_g + \ldots + m_n \mathrm{x}_n = (x_1 \mathrm{l}_1 + \ldots + x_h \mathrm{l}_h + \ldots + x_n \mathrm{l}_n) \div (r_1 \mathrm{l}_1 + \ldots + r_f \mathrm{l}_f + \ldots + r_n \mathrm{l}_n);$$

or more concisely thus, $\Sigma m_g \mathrm{x}_g = \Sigma x_h \mathrm{l}_h \div \Sigma r_f \mathrm{l}_f$:

while the numeral set thus found might be called the *quotient* of the two step-sets.

[33.] It may be remembered that even at so early a stage as the interpretation of the symbol $b \times a$, for the algebraic product of two positive or negative *numbers*,* it had been proposed to conceive a reference to a *step* (a), which should be first *operated on* by those two numbers *successively*, and then *abstracted from*, as was expressed by the elementary formula [9],

$$(b \times a) \times \mathrm{a} = b \times (a \times \mathrm{a}).$$

Thus to interpret the product -2×-3 as $= +6$, I conceived that some time-step (a) was first tripled in length and reversed in direction; then that the new step (-3a) was doubled and reversed; and finally that the last resultant step ($+6$a) was *compared* with the original step (a), in the way of algebraic *ratio* [9], thereby conducting to a result which was *independent* of that original step. All this, so far, was no doubt extremely easy; nor was it difficult to extend the same mode of interpretation to the case [17] of the multiplication of two *number couples*, and to interpret the product of two such couples as satisfying the condition,

$$(b_1, b_2)(a_1, a_2) \times (\mathrm{a}_1, \mathrm{a}_2) = (b_1, b_2) \times (a_1, a_2)(\mathrm{a}_1, \mathrm{a}_2);$$

the arbitrary *step-couple* $(\mathrm{a}_1, \mathrm{a}_2)$ being first operated on, and afterwards abstracted from. In like manner, in the theory of *triplets*, it was found possible [24] [25] to *abstract from an operand step-triad*, and thereby to obtain formulae for the symbolic *multiplication* of the *secondary* and *tertiary number-units*, $\mathrm{x}_2, \mathrm{x}_3$, and more generally of any two *numerical* triplets among themselves. But when it was sought to extend the same view to the still more general *multiplication of numeral sets*, new difficulties were introduced by the essential complexity of the subject, on which I can only touch in the briefest manner here.†

* This word 'number', whether with perfect propriety or not, is used throughout the present Preface and work, not as contrasted with *fractions* (except when accompanied by the word *whole* or *integer*), nor with incommensurables, but rather with those *steps* (in time, or on one axis), of some *two* of which it represents or denotes the *ratio*. In short, the *numbers* here spoken of, and elsewhere denominated '*scalars*' in this work, are simply those *positives* or *negatives*, on the *scale* of progression from $-\infty$ to $+\infty$, which are commonly called *reals* (or real quantities) in algebra.

† A fuller account of this theory of *sets*, with a somewhat different notation (the symbols $c_{r,s,t}$ and $n_{r,r',r''}$ being employed, for example, to denote the coefficients which would here be written as $\mathrm{l}_{t,r,s}$ and $\mathrm{l}'_{r,r',r''}$), and with a special application to the theory of *quaternions*, will be found in an Essay entitled:

[34.] After operating on an arbitrary step-set $\Sigma r_f 1_f$ by a number-set $\Sigma m_g x_g$, and so obtaining [32] another step-set, $\Sigma x_h 1_h$, we may conceive ourselves to operate on the same general plan, and with the same particular constants of multiplication, on this new step-set, by a *new number-set*, such as $\Sigma m'_{g'} x_{g'}$, and so to obtain a *third step-set*, such as $\Sigma x'_{h'} 1_{h'}$: which may then be supposed to be *divided* (see again [32]) *by the original step-set* $\Sigma r_f 1_f$, so as to conduct to a *quotient*, which shall be *another numeral set*, of the form $\Sigma m''_{g''} x_{g''}$. Under these conditions, we may certainly write,

$$\Sigma m'_{g'} x_{g'} (\Sigma m_g x_g . \Sigma r_f 1_f) = \Sigma m''_{g''} x_{g''} . \Sigma r_f 1_f;$$

but in order to justify the subsequent *abstraction of the operand step-set*, or the *abridgment* (compare [25]) of this formula of *successive operation* to the following,

$$\Sigma m'_{g'} x_{g'} . \Sigma m_g x_g = \Sigma m''_{g''} x_{g''},$$

which may be called a formula for the (symbolic) *multiplication of two number-sets*, certain *conditions of detachment* are to be satisfied, which may be investigated as follows.

[35.] Conceive that the required *separation of symbols* has been found possible, and that it has given, by a generalization of the process for triplets in [24], a system of n^2 symbolic equations of the form,

$$x_{g'} x_g = \Sigma 1'_{g,g',g''} x_{g''};$$

where $1'_{g,g',g''}$ is one of a *new system of* n^3 *numerical coefficients*, and the sum involves n terms, answering to n different values of the index g''. Under the same conditions, the recent formula for the multiplication of numeral sets breaks itself up into n equations, of the form,

$$m''_{g''} = \Sigma m_g m'_{g'} 1'_{g,g',g''};$$

the summation here extending to n^2 terms arising from the combinations of the values of the indices g and g'. For all such combinations, and for each of the n values of f, we are to have (if the required detachment be possible) the following equation between step-sets:

$$x_{g'} . x_g 1_f = x_{g'} x_g . 1_f;$$

and conversely, if we can satisfy these n^3 equations between step-sets, we shall thereby satisfy the *conditions of detachment* [34], which we have at present in view. But *each* of these n^3 equations between *sets* resolves itself generally into n equations between *numbers*: and thus there arise in general no fewer than n^4 *numerical equations*, as expressive of the conditions in question, which may all be represented by the formula,*

$$\Sigma 1_{f,g,h} 1_{h,g',h'} = \Sigma 1'_{g,g',h} 1_{f,h,h'};$$

all combinations of values of the indices f, g, g', h' (from 1 to n for each) being permitted, and the summation in each member being performed with respect to h. Now to satisfy these n^4 equations of condition, there were only $2n^3$ coefficients, or rather their ratios, disposable: and although the theories of couples and triplets already served to exemplify the *possibility* of effecting the desired *detachment*, at least in certain *cases*, yet it was by no means *obvious* that any *such extensive reductions*† were likely to present themselves, as were required for the

'Researches respecting Quaternions. First Series.' *Trans. Roy. Irish Acad.* vol. XXI (1848), pp. 199–296. [See VII.] This Essay was not fully printed till 1847, but several copies of it were distributed in that year, especially during the second Oxford Meeting of the British Association. The discussion of that portion of the subject which is here considered is contained chiefly in pages 225 to 231 of the volume above cited.

* A formula equivalent to this, but with a somewhat different notation, will be found at page 231 of the Essay and Volume referred to in a recent Note. [See VII, eqn (160).]

† On the subject of such general reductions, some remarks will be found at page 251 of the Essay and Volume lately cited. [See VII, section 29.]

accomplishment of the same object, in the more general theory of *sets*. And I believe that the compass and difficulty, which I thus perceived to exist, in that very *general* theory, deterred me from pursuing it farther at the time above referred to.

[36.] There was, however, a motive which induced me then to attach a special importance to the consideration of *triplets*, as distinguished from those more general *sets*, of which some account has been given. This was the desire to connect, in some new and useful (or at least interesting) way, *calculation* with *geometry*, through some undiscovered *extension*, to *space of three dimensions*, of a method of *construction* or representation [2], which had been employed with success by Mr Warren* (and indeed also by other authors,† of whose writings I had not

* *Treatise on the Geometrical Representation of the Square Roots of Negative Quantities. By the Rev. John Warren, A.M., Fellow and Tutor of Jesus College, Cambridge* (Cambridge, 1828). To suggestions from that Treatise I gladly acknowledge myself to have been indebted, although the interpretation of the symbol $\sqrt{-1}$, employed in it, is entirely distinct from that which I have since come to adopt in the geometrical applications of the quaternions.

† Several important particulars respecting such authors have been collected in the already cited *Report on certain Branches of Analysis* (see especially pp. 228 to 235), by Dr Peacock, whose remarks upon their writings, and whose own investigations on the subject, are well entitled to attention. As relates to the method described above (in paragraph [36] of this Preface), if *multiplication* (as well as *addition*) *of directed lines* in one plane be regarded (as I think it ought to be) as an *essential element* thereof, I venture here to state the impression on my own mind, that the true inventor, or at least the *first definite promulgator* of that method, will be found to have been Argand, in 1806: although his *Essai sur une Manière de représenter les Quantités Imaginaires*, which was published at Paris in that year, is known to me only by Dr Peacock's mention of it in his Report, and by the account of the same Essay given in the course of a subsequent correspondence, or series of communications (which also has been noticed in that Report, and was in consequence consulted a few years ago by me), carried on between Français, Servois, Gergonne, and Argand himself; which series of papers was published in Gergonne's *Annales des Mathématiques*, in or about the year 1813. My recollection of that correspondence is, that it was admitted to establish fully the priority of Argand to Français, as regarded the method [36] of (not merely *adding*, but) *multiplying* together directed lines in one plane, which is briefly described above: and which was afterwards independently reproduced, by Warren in 1828, and in the same year by Mourey, in a work entitled: *La Vraie Théorie des Quantités Négatives, et des Quantités prétendues Imaginaires* (Paris, 1828). If the list of such independent re-inventors of this important and modern method of constructing by a *line* the *product of two directed lines in one fixed plane* (from which it is to be remarked, in passing, that my own mode of representing by a *quaternion* the product of two directed lines *in space* is altogether different) were to be continued, it would include, as I have lately learned, the illustrious name of Gauss, in connexion with his *Theory of Biquadratic Residues* (Göttingen, 1832). ['Theoria residuorum biquadraticorum commentatio secunda', *Comm. Göttingen Recent.* VII (1832); Werke II, pp. 95–148.] On the other hand, I cannot perceive that *any distinct anticipation* of this method of *multiplication of directed lines* is contained in Buée's vague but original and often cited Paper, entitled 'Mémoire sur les Quantités Imaginaires', which appeared in the *Philosophical Transactions* (of London) for 1806, having been read in June 1805. The ingenious author of that Paper had undoubtedly formed the notion of *representing the directions of lines* by algebraical symbols: he even uses (in No. 35 of his Memoir) such expressions as $\sqrt{2}\,(\cos 45° \pm \sin 45°\sqrt{-1})$ to denote two different and *directed diagonals* of a square: and there is the high authority of Peacock ('Report on certain branches of analysis', *Brit. Assoc. Report*, 1833, p. 228), for considering that the geometrical interpretation of the symbol $\sqrt{-1}$, as denoting *perpendicularity*, was 'first formally maintained by Buée, though more than once suggested by other authors' ['Mémoire sur les Quantités Imaginaires', *Phil. Trans. Roy. Soc.* vol. 96 (1806), pp. 23–88]. In No. 43 of the Paper referred to, Buée constructs with much elegance, by a *bent line* AKE, or by an *inclined* line AE (where KE is a perpendicular, $=\frac{1}{2}a$, erected at the middle point K of a given line AB, or a), an *imaginary root* (x) of the quadratic equation, $x(a-x)=\frac{1}{2}a^2$, which had been proposed by Carnot (in p. 54 of the Géometrie de Position, Paris, 1804). But when he proceeds to explain (in No. 46 of his Paper) *in what sense* he regards the *two lines* AE and EB (or the two constructed *roots* of the quadratic) as having their *product* equal to the given value $\frac{1}{2}a^2$ or $\frac{1}{2}\overline{\mathrm{AB}}^2$, Buée *expressly limits* the signification of *such* a product of the result obtained by *multiplying the arithmetical values*, and *expressly excludes* the consideration of the *positions of the factor-lines* from his conception of their *multiplication*: whereas it seems to me to belong to the very *essence* of the method [36] of Argand and

then heard), for *operations on right lines in one plane*: which method had given a species of *geometrical interpretation* to the usual and well-known *imaginary symbol* of algebra. In the method thus referred to, *addition of lines* was performed according to the same rules as *composition of motions*, or of forces, by drawing the diagonal of a parallelogram; and the *multiplication* of two lines, in a given plane, corresponded to the construction of a species of *fourth proportional*, to an assumed line in the same plane, selected as the representative of *positive unity*, and to the two proposed *factor-lines*: such fourth proportional, or *product-line*, being *inclined* to *one* factor-line at the *same angle*, measured in the *same sense*, as that at which the *other* factor-line was inclined to the assumed *unit-line*; while its *length* was, in the old and usual signification of the words, a fourth proportional to the lengths of the unit-line and the two factor-lines. Subtraction, division, elevation to powers, and extraction of roots, were explained and constructed on the same general principles, and by processes of the same general character, which may easily be conceived from the slight sketch just given, and indeed are by this time known to a pretty wide circle of readers: and thus, no doubt, by operations on right lines *in one plane*, the symbol $\sqrt{-1}$ received a perfectly clear interpretation, as denoting a *second unit-line, at right angles** to that line which had been selected to represent positive

others, and generally to that system of geometrical interpretation whereon is based what Professor De Morgan has happily named *Double Algebra*, to *take account of those positions* (or directions), when *lines* are to be *multiplied* together. My *own* conception (as has been already hinted, and as will appear fully in the course of this work), of the *product of two directed lines in space* as a QUATERNION, is *altogether distinct*, both from the purely *arithmetical product* of numerical values of Buée, and from the *linear product* (or third coplanar line), in the method of Argand: yet I have thought it proper to submit the foregoing remarks, on the invention of this latter method, to the judgment of persons better versed than myself in scientific history. A few additional remarks and references on the subject will be found in a subsequent Note. [See VI, p. 150.]

* Besides what has been already referred to, as having been done on this subject of the interpretation of the symbol $\sqrt{-1}$ by the Abbé Buée, it has been well remarked by Mr Benjamin Gompertz, at page vi of his very ingenious Tract on *The Principles and Applications of Imaginary Quantities*, Book II, *derived from a particular case of Functional Projections* (London, 1818), that the celebrated Dr Wallis of Oxford, in his *Treatise of Algebra* (London, 1685), proposed to interpret the imaginary roots of a quadratic equation, by going *out of the line*, on which if real they should be measured. Thus Wallis (in his chapter lxvii) observes: 'So that whereas in case of Negative Roots we are to say, the point B cannot be found, so as is supposed in AC Forward, but Backward it may in the same Line: we must here say, in case of a Negative Square, the point B cannot be found so as was supposed, in the Line AC; but Above that Line it may in the same Plane. This I have the more largely insisted on, because the Notion (I think) is new; and this, the plainest Declaration that at present I can think of, to explicate what we commonly call the *Imaginary Roots* of Quadratick Equations. For such are these.' And again (in his following chapter lxviii, at page 269), Wallis proposes to construct thus the roots of the equation $aa \mp ba + æ = 0$: 'On AC$a = b$, bisected in C, erect a perpendicular CP $= \sqrt{æ}$. And taking PB $= \frac{1}{2}b$, make (on whether side you please of CP), PBC, a rectangled triangle. Whose right angle will therefore be at C or B, according as PB or PC is bigger; and accordingly, BC a sine or a tangent, (to the radius PB,) terminated in PC. The straight lines AB, Ba, are the two values of a. Both affirmative if (in the equation,) it be $-ba$. Both negative, if $+ba$. Which values be (what we call) *Real*, if the right angle be at C. But *Imaginary* if at B.' These passages must always (I think) possess an historical interest, as exemplifying the manner in which, in the seventeenth century, one so eminent for his powers of *interpretation* of analytical expressions, as Dr Wallis was, sought to apply those powers to the *geometrical construction* of the *imaginary roots* of an equation: and for the decision with which he held that such roots were quite *as clearly interpretable*, as '*what we call real*' values. His particular interpretation of those imaginary roots of a quadratic appears indeed to me to be inferior in elegance to that which was long afterwards proposed by Buée. But it may be noticed that, whether his point B were *on* or *off* the line ACa, Wallis seems (like Buée, and many other and more modern writers) to have regarded *that right line*, as being *in some sense a sum*, or at least *analogous to a sum*, of the *two successive lines* AB, Ba; which latter lines conduct, upon the whole, from the initial point A to the final point a; and construct according to him the two roots of the quadratic, whose algebraic sum is $=b$. Indeed, Wallis remarks (in the same page 269) that when those two roots are algebraically *imaginary*, or are geometrically

unity. But when it was proposed to *leave the plane*, and to construct a system which should have *some general analogy* to the known system thus described, but should *extend to space*,* then difficulties of a new character arose, in the endeavour at surmounting which I was encouraged by the friend already mentioned (Mr John T. Graves), who felt the wish, and formed the project, to surmount them in *some* way or other, as early, or perhaps earlier than myself.

[37.] A conjecture respecting such extension of the rule of multiplication of lines, from the plane to space, which long ago occurred to me (in 1831), may be stated briefly here, as an illustration of the general character of those old speculations. Let A denote a point assumed on the surface of a fixed sphere, described about the origin O of co-ordinates, with a radius equal to the unit of length; and let this point A be called the *unit-point*. Let also B and C be supposed to be two *factor-points*, on the same surface, representing the *directions* OA, OB, of the two *factor-lines* in space, of which lines it is required to perform, or to interpret, the multiplication; and so to determine, by some fixed rule to be assigned, the *product-point* D, or the direction of the *product-line*, OD. Then it appeared that the analogy to operations in the plane might be not ill observed, by conceiving D to be taken on the *circle* ABC; the *arcs*, AB, CD, of that (generally) *small circle* of the sphere being *equally long*, and *similarly measured*; so that the two *chords* AD, BC should be *parallel*: while the old rule of *multiplication of lengths* should be retained: and *addition of lines* be still interpreted as before. But in this system there were found to enter *radicals* and *fractions* into the expressions for the co-ordinates† of a product; and although the case of *squares of lines*, or products of equal factors, might be rendered *determinate* by agreeing to take the *great* circle AB, when the point C coincided with B, yet there seemed to be an essential *indetermination* in the construction of the *reciprocal* of a line: it being sufficient, according to the definition here considered, to take the chord BC parallel to the tangent plane to the sphere at the unit-point, in order to make the product point D

constructed (according to him) by the help of a point B which is *above the line* ACa, then that straight line is *not equal to the aggregate* of AB + Ba; but this seems to be no more than guarding himself against being supposed to assert, that two sides of a triangle can be equal *in length* to the third. In chap. lxix, p. 272, he thus sums up: 'We find therefore, that in Equations, whether Lateral or Quadratick, which in the strict Sense, and first Prospect, appear Impossible; some mitigation may be allowed to make them Possible; and in such a mitigated interpretation they may yet be useful.' For *lateral* equations (equations of the first degree), the *mitigation* here spoken of consists simply in the usual representation of *negative roots*, by lines drawn *backward* from a point, whereas they had been at first supposed to be drawn *forward*. For quadratic equations with *imaginary roots*, Wallis *mitigates* the problem, by substituting a *bent line* ABa for that *straight line* ACa, which *constructs the given algebraical sum* (b) of the two roots of the equation, or *parts of the bent line*, AB, Ba. It is also to be noticed that he appears to have regarded the *algebraical semi-difference* of those two roots, AB, Ba, as being in all cases constructed by the *line* BC, drawn to the middle point C of the line Aa: which would again agree with many modern systems. Thus Wallis seems to have possessed, in 1685, *at least in germ* (for I do not pretend that he fully and consciously possessed them), some elements of the modern methods of *Addition* and *Subtraction* of directed lines. But on the equally essential point of *Multiplication* of directed lines in one plane, it does not appear that Wallis, any more that Buée (see the foregoing Note), had anticipated the method of Argand.

* At a much later period I learned that others had sought to accomplish some such extension to space, but in ways different from mine.

† The rectangular co-ordinates (or projections) of the two factor-lines and of the product-line being denoted by xyz, $x'y'z'$, $x''y''z''$, if we also write, for conciseness,

$$r=\sqrt{(x^2+y^2+z^2)}, \quad r'=\sqrt{(x'^2+y'^2+z'^2)}, \quad p=xx'+yy'+zz',$$

then the expressions which I found for $x''y''z''$ may be included briefly in the equations:

$$\frac{x''-rr'}{rx'-r'x}=\frac{y''}{ry'-r'y}=\frac{z''}{rz'-r'z}=\frac{rx'-r'x}{p-rr'}.$$

coincide with that point A. There was also the great and (as I thought) fatal objection to this method of construction, that it did not preserve the *distributive principle* of multiplication; a *product of sums* not being equal, in it, to the *sum of the products*: and on the whole, I abandoned the conjecture.

[38.] Another construction, of a somewhat similar character, and liable to similar objections, for the product of two lines in space, occurred to me in 1835, and also independently to Mr J. T. Graves in 1836, in which year he wrote to me on the subject. It may be briefly stated, by saying that instead of considering, as in the last-mentioned system, the *small* circle ABC, and drawing the *chord* AD, from unit-point to product-point, so as to be *parallel* to the chord BC from one factor-point to the other, it was now the *arc* AD of a *great* circle on the sphere, which was to be drawn so as to *bisect the arc* BC, of another great circle, and *be bisected* thereby. Or as Mr Graves afterwards expressed to me the rule in question: 'Bisect the inclination of the factor-lines, and then double forward the angle between the linear unit and the bisecting line:' the rule of multiplying *lengths* being understood to be still observed. Mr Graves made several acute remarks on the consequences of this construction, and proposed a few supplementary *rules* to remove the *porismatic* character of some of them: but observed that, with these interpretations, the *square-root of the negative unit-line*, or the triplet $(-1, 0, 0)^{\frac{1}{2}}$, would still be indeterminate, and of the form $(0, \cos\theta, \sin\theta)$, where θ remained arbitrary: while cases might arise, in which the 'minutest alteration' of a factor-line would make a 'considerable change' in the position of the product-line: and this result he conceived to be, or to lead to, 'a breach of the grand property of multiplication,' respecting its operation on a *sum*. He left to me the investigation of the general expressions for the 'constituent co-ordinates' of the resultant 'triplet,' or product-line, in terms of the constituents of the factors: and in fact I had already obtained such expressions, and had found them to involve radicals and fractions, and to violate the distributive principle, as in the system recently described [37]; with which indeed the one here mentioned had been perceived by me to have a very close analytical connexion.*

[39.] Mr J. T. Graves, however, communicated to me at the same time another method, which he said that he *preferred*, among all the modes that he had tried, 'of representing lines in space, and of multiplying such lines together.' This method consisted in considering such a line as a species of 'compound couple,' or as determined by *two couples*, one in the plane of *xy*, and the other perpendicular to that plane: it having been easily perceived that the rules proposed by me for the addition and multiplication [17] of *couples*, agreed in all respects with the previously known method [36], of representing the operations of the same names on *lines in one plane*. From this conception of *compound couples* Mr Graves derived a 'general rule for

* With the notations recently employed, the expressions which I had found for the co-ordinates of the product, in the case or system [38], are included in the equations,

$$\frac{x''+rr'}{rx'+r'x}=\frac{y''}{ry'+r'y}=\frac{z''}{rz'+r'z}=\frac{rx'+r'x}{p+rr'};$$

which only differ from those for the former case [37], by a change of sign in the radical r' (or r), which represents the length of a factor-line. The conditions for both systems are contained in these other equations,

$$xx''+yy''+zz''=r^2x',\quad x'x''+y'y''+z'z''=r'^2x,\quad x''^2+y''^2+z''^2=r^2r'^2;$$

and the quadratic equation in x'', obtained by elimination of y'' and z'', resolves itself into two separate factors, each linear relatively to x'', namely,

$$(p-rr')\,(x''-rr')-(rx'-r'x)^2=0,\quad (p+rr')\,(x''+rr')-(rx'+r'x)^2=0.$$

The first corresponds to the system [37]; the second to the system [38].

the multiplication of triplets,' which I shall here transcribe,* only abridging the notation by writing ρ and ρ_1 to represent the radicals $\sqrt{(x^2+y^2)}$ and $\sqrt{(x_1^2+y_1^2)}$, or the projections of the factor-lines on the plane of xy: $(x, y, z)(x_1, y_1, z_1) = (x_2, y_2, z_2)$, where

$$x_2 = (\rho\rho_1 - zz_1)\left(\frac{xx_1 - yy_1}{\rho\rho_1}\right), \quad y_2 = (\rho\rho_1 - zz_1)\frac{xy_1 + yx_1}{\rho\rho_1}, \quad z_2 = z_1\rho + z\rho_1.$$

This particular system of expressions he does not seem to have developed farther, nor did it at the time attract much of my own attention: but I have thought it deserving of being put on record here, especially as, by a remarkable coincidence, it came to be independently and otherwise arrived at by another member of the same family, at a date later by ten years, and to be again communicated to me.† And perhaps I may be excused if I here leave the order of time, to give some short account of the train of thought by which his brother, the Rev. Charles Graves, appears to have been conducted, in 1846, to precisely the *same relations* between the constituents of three triplets.

[40.] Professor Graves employed a system of *two new imaginaries*, i and j, of which he conceived that i had the effect of causing a rotation (generally conical) through 90° round the axis of z, while j caused a line to revolve through an equal angle in its own vertical plane (that is, in the plane of the line and of z); and then he proceeded to *multiply* together the two triplets $x+iy+jz$, $x'+iy'+jz'$, by a peculiar process, and so to obtain a third triplet $x''+iy''+jz''$: the relations thus resulting, between the co-ordinates or constituents, being (as it turned out) identical with those which his brother had formerly found. These symbols i and j were *each a sort of fourth root of unity*: and the first, but *not* the second, had the property of operating on a *sum* by operating on each of its *parts* separately. Thus, as Professor Graves remarked, multiplication of triplets, on this plan, would *not* be a *distributive* operation, although it would be a *commutative* one. The method conducted him to an elegant exponential expression for a line in space, namely, $r\epsilon^{il}\epsilon^{j\lambda}$; where r was the *radius vector*, and l, λ might be called the *longitude* and *latitude* of the line, so that the co-ordinate projections were (some peculiar considerations being employed in order to justify these expressions of them, as connected with that of the line):

$$x = r\cos l\cos\lambda, \quad y = r\sin l\cos\lambda, \quad z = r\sin\lambda.$$

And then the rule for the *multiplication of two lines* came to be expressed by the very simple formula:

$$r\epsilon^{il}\epsilon^{j\lambda}.\, r'\epsilon^{il'}\epsilon^{j\lambda'} = rr'\epsilon^{i(l+l')}\epsilon^{j(\lambda+\lambda')};$$

the *lengths* being thus *multiplied* (as in the other systems above mentioned), but the *longitudes* and *latitudes* of the one line being respectively *added* to those of the other: which was in fact the rule expressed by Mr J. T. Graves's co-ordinate formulae [39].

[41.] It will not (I hope) be considered as claiming any merit to myself in this matter, but merely as recording an unpursued *guess*, which may assist to *illustrate* this whole inquiry, if I venture to mention here that the *first conjecture* respecting *geometrical triplets*, which I find noted among my papers (so long ago as 1830), was, that while *lines in space* might be *added* according to the same rule as in the plane, they might be *multiplied* by multiplying their lengths, and *adding* their polar angles. In the method [36], known to me then as that of Mr Warren, if we write $x = r\cos\theta$, $y = r\sin\theta$, we have, for multiplication *within* the plane,

* From Mr Graves's Letter of 8 August 1836.

† By the Rev. Charles Graves, Professor of Mathematics in the University of Dublin, in a letter of 14 November 1846.

equations which may be written thus, $r''=rr'$, $\theta''=\theta+\theta'$. It hence occurred to me, that if we employed for space these other known transformations of rectangular to polar co-ordinates,

$$x=r\cos\theta,\quad y=r\sin\theta\cos\phi,\quad z=r\sin\theta\sin\phi,$$

it might be natural to *define* multiplication of lines in space by the slightly extended but analogous formulae,

$$r''=rr',\quad \theta''=\theta+\theta',\quad \phi''=\phi+\phi':$$

which, however, conducted to *radicals*, as in the expression,

$$x''=xx'-(y^2+z^2)^{\frac{1}{2}}(y'^2+z'^2)^{\frac{1}{2}},$$

whereas within the plane there were *rational* values for the rectangular co-ordinates of the product, namely (compare [17]),

$$x''=xx'-yy',\quad y''=xy'+yx'.$$

But this old (and uncommunicated) conjecture of mine, which was inconsistent with the distributive principle, though possessing some general *resemblance* to the lately mentioned results [39] [40] of Messrs John and Charles Graves, cannot be considered to have been an *anticipation* of them. For while we all *agreed* in *adding* the *longitudes* of the two factors (in the sense lately mentioned), *they added latitudes* also; while I, less happily, had thought of *adding the colatitudes*, or the angular distances from a *line* (x), instead of those from a *plane* (xy). And this difference of plan produced a very important difference of results. Indeed the two systems are totally distinct, although there exists some sort of analogy between them.

[42.] I shall here mention one more system, which was communicated to me* in 1840, by the elder of those two brothers, and which involved a method of representing the usual imaginary quantities of algebra, *each by a corresponding unique point on the surface of a sphere*, described (as in [37]) about the origin with a radius = 1: whence it appeared that the ordinary imaginary expression $r(\cos\theta+\sqrt{-1}\sin\theta)$ might be denoted by a *triplet* (x,y,z), under the *condition*, $x^2+y^2+z^2=1$: and that the *rules* thus obtained, for the multiplication of *such* triplets, might perhaps afford some *analogy*, suggesting rules† for the more *general* case, where the constituents x,y,z are wholly *independent* of each other. Mr J. T. Graves's 'mode of representing quantity spherically' was stated by him to me as follows: 'All positive quantities r may be represented by points on an assumed semicircle, by taking the extremity of the arc $2\tan^{-1}r$ (counted from one end (A) of the semicircle) to represent r. Next let us consider our sphere as generated by the revolution of the semicircle‡ ABC round the axis AC (forwards or backwards, according to arbitrary convention). When the semicircle has moved through an angle θ, let the position of a point on its circumference denote $r(\cos\theta+\sqrt{-1}\sin\theta)$, if the same point in its original position denoted r.' I make a very easy transformation of this statement, when I present it thus: Construct all quantities (so called), real and imaginary, according to the known method already described in [36], by drawing right lines from the assumed point (A) of the unit-sphere, in the tangent plane at that point; double all the lines so drawn, and treat the ends of the doubled lines as the stereographic projections of points upon the sphere.

* In a letter of 17 October 1840, from J. T. Graves, Esq.

† Mr Graves appears not to have actually worked out such rules, at least I do not find that he communicated them to me. They would probably have been, on the plan described in [42], to have *multiplied* (as before) the *lengths*, and (as before) *added* the *longitudes*: but to have then *multiplied the tangents of the halves of the colatitudes* of the factors, in order to obtain the tangent of the half of the colatitude of the product.

‡ A figure, which it seems unnecessary here to reproduce, accompanied Mr Graves's Letter.

Infinity was thus represented, in the particular system of Mr Graves here described, by the point diametrically opposite to A. And in this endeavour of mine, to furnish faithfully a record of every circumstance, which, even as remotely *suggesting* to a *friend* a train of thought, may have *indirectly* stimulated *myself*, I must not suppress the following acknowledgment of Mr J. T. Graves: 'What led me to this was a passage in a letter from De Morgan,* in which he expressed a wish to be able to represent quantity *circularly*, in order to explain the passage from positive to negative through infinity.'

[43.] The foregoing specimens may suffice to exemplify the attempts which were made, a considerable number of years ago, by Mr Graves and by myself: on the one hand, to *extend* to *space* that geometrical construction for the multiplication of *lines*, which was known to us from the work of Mr Warren; and on the other hand, to render more entirely *definite* my conception of algebraical *triplets*. I will not here trouble my readers with any further account of the conjectures on those subjects which at various times occurred to him or me, before I was led to the quaternions, in a way which I shall presently explain. But I wish to mention first, that among the circumstances which assisted to prevent me from losing sight of the general subject, and from wholly abandoning the attempt to turn to some useful account those early speculations of mine, on triplets and on sets, was probably the publication of Professor De Morgan's first Paper on the Foundation of Algebra,† of which he sent me a copy in 1841. In that Paper, besides the discussion of other and more important topics, my Essay on Pure Time was noticed, in a free but friendly spirit; and the subject of triplets was alluded to, in such passages, for instance, as the following: 'But in this branch of logical algebra' (that referred to in paragraph [36] of the present Preface), 'the lines must be all in one plane, or at least affected by only one modification of direction: the branch which shall apply to a line drawn in any direction from a point, or modified by two distinct directions, is yet to be found.' ... 'An extension to geometry of three‡ dimensions is not practicable until we can assign two symbols, Ω and ω, such that $a+b\Omega+c\omega=a_1+b_1\Omega+c_1\omega$ gives $a=a_1$, $b=b_1$, and $c=c_1$: and no *definite* symbol of ordinary algebra will fulfil this condition.' My symbols x_2, x_3 (of 1834–5) had not then been published, nor otherwise exhibited to him; they were designed to fulfil precisely the foregoing conditions: but I was not myself satisfied with them, as not considering them '*definite*' enough (compare [29]).

[44.] In the early numbers of the *Cambridge Mathematical Journal*, there appeared some ingenious and original Papers, by the late Mr Gregory and by other able analysts, on the signs + and −, on the powers of +, on branches of curves in different planes, and on other connected subjects: but I hope that it will not be thought disrespectful if I confess that I do not remember their having had much influence on my own trains of thought. Perhaps I was not

* Augustus De Morgan, Esq., Professor of Mathematics in University College, London.

† *Trans. Camb. Phil. Soc.* vol. VII (1839), pp. 173–187.

‡ Professor De Morgan proposed at the same time a remarkable conjecture, which he may be considered to have afterwards illustrated and systematized, by his theory of *cube-roots* of negative unity, employed as *geometrical operators*, in his Paper on 'Triple Algebra' (*Camb. Phil. Trans.* vol. VIII (1844), pp. 241–254); namely, that 'an extension to three dimensions' might 'require a solution of the equation $\phi^2x=-x$.' I much regret that my plan will not allow me to attempt the giving any further account, in this Preface, of that very original Paper of Professor De Morgan, the first suggestion of which he was pleased to attribute to the publication of my own remarks on Quaternions, in the *Philosophical Magazine* for July 1844: and a similar expression of regret applies to the independent but somewhat later researches of Messrs John and Charles Graves, in the same year, respecting other Triplet Systems, which involved cube-roots of *positive* unity, and of which some account has been preserved in the *Proceedings of the Royal Irish Academy*. [See Authors' index, vol. III (1847).]

sufficiently prepared, or disposed, to look at algebra generally, and its applications to geometry, from the same point of view, and was thereby prevented from studying those Papers with the requisite attention. At least, if anything in my own views shall be found to be inconsistent with those put forward in the Papers thus alluded to, I wish it to be considered as offered with every deference, and not in a controversial spirit. And if for the present I omit all further mention of them, it is partly because, without a closer study, I should fear to do them injustice: and partly because I make no pretensions to be here an *historian of science*, even in *one* department of mathematical speculation, or to give anything more than an account of the *progress of* my *own thoughts*, upon one class of subjects. For the same reasons, I pass over some other investigations having reference to the imaginary* symbol of algebra, which were not used as suggestions by myself, and proceed at once to the quaternions.

[45.] With such preparations as I have described, I resumed (in 1843) the endeavour to adapt the general conception of triplets to the multiplication of lines in space, resolving to *retain* the *distributive* principle, with which some formerly conjectured systems had been inconsistent, and at first supposing that I *could* preserve the *commutative* principle *also*, or the convertibility [24] [29] of the factors as to their *order*. Instead of my old symbols $\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3$ (see [22]), I wrote more shortly $1, i, j$; so that a numerical triplet took the form $x+iy+jz$, where I proposed to interpret x, y, z as three rectangular co-ordinates, and the triplet itself as denoting a line in space. From the analogy of couples, I assumed $i^2=-1$; and tried the effect of assuming also $j^2=-1$, which I interpreted as answering to a rotation through two right angles in the plane of xz, as $i^2=-1$ had corresponded to such a rotation in the plane of xy. And because I at first supposed that ij and ji were to be *equal*, as in the ordinary calculations of algebra, the product of two triplets appeared to take the form,

$$(a+ib+jc)(x+iy+jz)=(ax-by-cz)+i(ay+bx)+j(az+cx)+ij(bz+cy):$$

but I did not at once see what to do with the *product* ij. The theory of triplets seemed to require that it should be *itself* a triplet, of the form,

$$ij=\alpha+i\beta+j\gamma,$$

* I am unwilling, however, to leave unmentioned here (although it did not happen to supply me with any suggestion), a remarkable use of the symbol $\sqrt{-1}$, which was made by the late Professor MacCullagh, of Dublin, whose great and original powers in mathematical and physical science must ever be remembered with admiration, and which he seems to have connected (in 1843) with investigations respecting the total reflexion of light. (See *Proc. Roy. Irish Acad.* vol. III (1847), pp. 49–51.) This use of imaginaries was founded on a theorem relative to the ellipse, which was expressed by him as follows, in a question proposed at the Examination for the Election of Junior Fellows in 1842 (see Dublin University Examination Papers for that year, published in 1843, p. lxxxiv): 'Detur in spatio ellipsis, cujus centrum est origo co-ordinatarum. Puncta xyz, $x'y'z'$ in ellipsi sint termini diametrorum conjugatarum. Ostendendum est quantitates imaginarias

$$\frac{y+y'\sqrt{-1}}{x+x'\sqrt{-1}}, \quad \frac{z+z'\sqrt{-1}}{x+x'\sqrt{-1}},$$

constantes esse pro quolibet systemate diametrorum conjugatarum.' This elegant theorem of Professor MacCullagh may easily be proved, without employing any but the usual principles respecting the symbol $\sqrt{-1}$, by observing that the following expressions, for the six co-ordinates in question,

$$x=a\cos v+a'\sin v, \quad y=b\cos v+b'\sin v, \quad z=c\cos v+c'\sin v,$$

$$x'=a'\cos v-a\sin v, \quad y'=b'\cos v-b\sin v, \quad z'=c'\cos v-c\sin v,$$

give

$$\frac{x+x'\sqrt{-1}}{a+a'\sqrt{-1}}=\frac{y+y'\sqrt{-1}}{b+b'\sqrt{-1}}=\frac{z+z'\sqrt{-1}}{c+c'\sqrt{-1}}=\cos v-\sin v\sqrt{-1}.$$

the coefficients α, β, γ being some three constant numbers: but the question arose, how were those numbers to be determined, so as to adapt in the best way the resulting formula of multiplication to some *guiding geometrical analogies*.

[46.] To assist myself in applying such analogies, I considered the case where the co-ordinates b, c were *proportional* to y, z, so that the two factor-lines were in one common *plane*, containing the unit-line, or the axis of x. In that particular *case*, there was ready a *known* signification [36] for the product line, considered as the fourth proportional to the unit-line (assumed here on the last-mentioned axis), and to the two coplanar factor-lines. And I found, without difficulty, that the co-ordinate projections of such a fourth proportional were here,

$$ax-by-cz, \quad ay+bx, \quad az+cx,$$

that is to say, the coefficients of $1, i, j$, in the recently written expression for the product of the two triplets, which had been supposed to represent the factor-lines. In fact, if we assume $y=\lambda b$, $z=\lambda c$, where λ is any coefficient, we have the two identical equations,

$$(ax-\lambda b^2-\lambda c^2)^2+(\lambda a+x)^2(b^2+c^2)=(a^2+b^2+c^2)(x^2+\lambda^2 b^2+\lambda^2 c^2),$$

$$\tan^{-1}\frac{(\lambda a+x)(b^2+c^2)^{\frac{1}{2}}}{ax-\lambda(b^2+c^2)}=\tan^{-1}\frac{(b^2+c^2)^{\frac{1}{2}}}{a}+\tan^{-1}\frac{\lambda(b^2+c^2)^{\frac{1}{2}}}{x},$$

which express that the required geometrical conditions are satisfied. It was allowed then, in this *case of coplanarity*, or under the particular *condition*,

$$bz-cy=0,$$

to treat the triplet, $(ax-by-cz)+i(ay+bx)+j(az+cx)$,

as denoting a *line* which might, consistently with known analogies, be regarded as the *product* of the two lines denoted by the two proposed triplets,

$$a+ib+jc, \quad \text{and} \quad x+iy+jz.$$

And here the *fourth term*, $ij(bz+cy)$,

appeared to be simply *superfluous*: which induced me for a moment to fancy that perhaps the *product* ij was to be regarded as $=0$. But I saw that this fourth term (or part) of the product was more immediately given, in the calculation, as the sum of the two following,

$$ib.jz, \quad jc.iy;$$

and that this *sum* would vanish, under the present *condition* $bz=cy$, if we made what appeared to me a *less harsh* supposition, namely, the supposition (for which my old speculations on *sets* had prepared me) that

$$ij=-ji:$$

or that $ij=+k, \quad ji=-k,$

the value of the product k being still left undetermined.

[47.] In this manner, *without* now assuming $bz-cy=0$, I had generally for the *product of two triplets*, the expression of *quadrinomial form*,

$$(a+ib+jc)(x+iy+jz)=(ax-by-cz)+i(ay+bx)+j(az+cx)+k(bz-cy);$$

and I saw that although the product of the sums of squares of the constituents of the two factors could not in general be decomposed into *three* squares of rational functions of them, yet it *could* be generally presented as the sum of *four* such squares, namely, the squares of the four coefficients of 1, i, j, k, in the expression just deduced: for, without any relation being assumed between a, b, c, x, y, z, there was the identity,

$$(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax-by-cz)^2+(ay+bx)^2+(az+cx)^2+(bz-cy)^2.$$

This led me to conceive that perhaps instead of seeking to *confine* ourselves to *triplets*, such as $a+ib+jc$ or (a,b,c), we ought to regard these as only *imperfect forms of* QUATERNIONS, such as $a+ib+jc+kd$, or (a,b,c,d), the symbol k denoting *some new sort of unit operator*: and that thus my old conception of *sets* [30] might receive a new and useful application. But it was necessary, for operating *definitely* with such quaternions, to fix the value of the *square* k^2, of this new symbol k, and also the values of the *products*, ik, jk, ki, kj. It seemed natural, after assuming as above that $i^2=j^2=-1$, and that $ij=k$, $ji=-k$, to assume also that $ki=-ik=-i^2j=+j$, and $kj=-jk=j^2i=-i$. The assumption to be made respecting k^2 was less obvious; and I was for a while disposed to consider this square as equal to *positive* unity, because $i^2j^2=+1$: but it appeared more convenient to suppose, in consistency with the foregoing expressions for the products of i, j, k, that

$$k^2=ijij=-iijj=-i^2j^2=-(-1)(-1)=-1.$$

[48.] Thus all the fundamental assumptions for the *multiplication of two quaternions* were completed, and were included in the formulae,

$$i^2=j^2=k^2=-1; \quad ij=-ji=k; \quad jk=-kj=i; \quad ki=-ik=j:$$

which gave me the equation,

$$(a,b,c,d)(a',b',c',d')=(a'',b'',c'',d''),$$

or

$$(a+ib+jc+kd)(a'+ib'+jc'+kd')=a''+ib''+jc''+kd'',$$

when and only when the following *four separate equations* were satisfied by the *constituents* of these three quaternions:

$$a''=aa'-bb'-cc'-dd', \qquad c''=(ac'+ca')+(db'-bd'),$$
$$b''=(ab'+ba')+(cd'-dc'), \qquad d''=(ad'+da')+(bc'-cb').$$

And I perceived on trial, for I was not acquainted with a theorem of Euler* respecting *sums of four squares*, which might have enabled me to anticipate the result, that these expressions for a'', b'', c'', d'' had the following *modular property*:

$$a''^2+b''^2+c''^2+d''^2=(a^2+b^2+c^2+d^2)(a'^2+b'^2+c'^2+d'^2).$$

I saw also that if, instead of representing a line by a triplet of the form $x+iy+jz$, we should agree to represent it by this *other trinomial form*,

$$ix+jy+kz,$$

we should then be able to express the desired *product of two lines in space* by a QUATERNION, of which the constituents have very *simple geometrical significations*, namely, by the following,

$$(ix+jy+kz)(ix'+jy'+kz')=w''+ix''+jy''+kz'',$$

where

$$w''=-xx'-yy'-zz', \quad x''=yz'-zy', \quad y''=zx'-xz', \quad z''=xy'-yx';$$

so that the part w'', independent of ijk, in this expression for the product, represents the *product of the lengths of the two factor-lines, multiplied by the cosine of the supplement of their inclination* to each other; and the remaining part $ix''+jy''+kz''$ of the same product of the two trinomials represents a *line*, which is in *length* the *product of the same two lengths*, *multiplied by*

* [See IV, p. 108, footnote 2.]

the sine of the same inclination, while in *direction* it is *perpendicular to the plane of the factor-lines*, and is such that the *rotation round the multiplier-line*, from the multiplicand-line towards the product-line (or towards the *line-part* of the whole quaternion product), has the *same right-handed* (or left-handed) *character*, as the rotation round the positive semiaxis of k (or of z), from the positive semiaxis of i (or of x), towards that of j (or of y).

[49.] When the conception, above described, had been so far unfolded and fixed in my mind, I felt that the *new instrument* for applying *calculation to geometry*, for which I had so long sought, was now, at least in part, attained. And although I had left several former conjectures respecting *triplets* for many years uncommunicated, except by name, even to friends, yet I at once proceeded to lay these results respecting *quaternions* before the Royal Irish Academy (at a Meeting of Council* in October 1843, and at a General Meeting† shortly subsequent): introducing also a theory of their connexion with spherical trigonometry, some sketch of which appeared a few months later in London (in the Philosophical Magazine for July 1844). On that *connexion of quaternions with spherical trigonometry*, and generally with *spherical geometry*, I need not at present dwell, since it is sufficiently explained in the concluding Lectures of this Volume: but it may be not improper that a brief account should here be given, of a not much later but hitherto unpublished speculation, of a character partly geometrical, but partly also metaphysical (or *à priori*), by which I sought to explain and confirm some results that might at first seem strange, among those to which my analysis had conducted me, respecting the *quadrinomial form*, and *non-commutative property*, of the *product* of two directed lines in space.

[50.] Let, then, the PRODUCT of two co-initial lines, or of two vectors from a common origin, be conceived to be *something* which has QUANTITY, in the sense that it is doubled, tripled, &c., by doubling, tripling, &c., either factor; let it also be conceived to have in some sense, QUALITY, *analogous to direction*, which is in some way *definitely connected* with the directions of the two factor lines. In particular let us conceive, in order to preserve so far an analogy to *algebraic multiplication*, that its direction is in all respects *reversed*, when *either* of those directions is reversed; and therefore that it is *restored*, when *both* of them are reversed. On the other hand, for the sake of recognizing what may be called the *symmetry of space*, let this *direction of the product*, so far as it can be constructed or represented by that of any *line in space*, be conceived as *not changing its relation to the system of those two factor directions*, when that system is in any manner *turned in space*: its own direction, *as a line*, being at the same time *turned with them*, as if it formed a part of one common and rigid system; and the *numerical element* of the same product (if it have any such) undergoing *no change* by such rotation. Let the product in question be conceived to be entirely *determined*, when the factors are determined; let it be

* The Minutes of Council of the R.I.A., for 16 October 1843, record 'Leave given to the President to read a paper on a new species of imaginary quantities, connected with a theory of quaternions.' It may be necessary to state, in explanation, that the Chair of the Academy, which has since been so well filled by my friends, Drs Lloyd and Robinson, was at that time occupied by me.

† At the Meeting of 13 November 1843, as recorded in the *Proceedings* of that date, in which the fundamental formulae and interpretations respecting the symbols ijk are given. [See V.] Two letters on the subject, which have since been printed, were also written in October 1843, to the friend so often mentioned in this Preface, Mr J. T. Graves: and the chief results were also exhibited to his brother, the Rev. C. Graves, before the public communication of November 1843. These circumstances (or some of them) have been stated elsewhere: but it seemed proper not to pass them over without some short notice here, as connected with the date of the invention and publication of the quaternions. [See IV, and VII, note A.]

made, if other conditions will allow, for the sake of general analogies, a *distributive function* of those two factors, summation of lines being performed by the same rules as composition of motions; and finally, if these various conditions can all be satisfied, and still leave anything undetermined, in the rules for *multiplication of lines*, let the indeterminateness be removed in such a way as to make these rules approach as much as possible to the other usual rules for the *multiplication of numbers* in algebra.

[51.] The *square of a given line* must *not* be *any line* inclined to that given line; for, even if we chose any particular *angle* of inclination, there would be nothing to determine the *plane*, and thus the square would be *indeterminate*, unless we selected some one direction in space as *eminent*, which selection we are endeavouring to *avoid*. Nor can the square of a given line be a line in the *same* direction, nor in the direction *opposite*; for if *either* of these directions were selected, by a definition, then this definition would oblige us to consider the square as *reversed* in direction, when the line of which it is the square is reversed; whereas if the two factors of a product *both* change sign, the direction of the product is always (by what has been above agreed on) preserved, or rather *restored*. We must, therefore, consider the SQUARE OF A LINE as having *no direction in space*, and therefore as being *not* (properly) *itself a line*; but nothing hitherto prevents us from regarding the *square* as a NUMBER, which has always one determined *sign* (as yet unknown), and varies in the duplicate ratio of the length of the line to be squared. If, then, the length of a line α contain a times the unit of length, we are led to consider $\alpha\alpha$ or α^2 as a symbol equivalent to la^2, in which l is some numerical coefficient, positive or negative, as yet unknown, but constant for all lines in space, or having *one common value* for all. And, consequently, if α, β be *any two lines* in any *one common direction*, and having their *lengths* denoted by the *numbers* a and b, we are led to regard the product $\alpha\beta$ as equal to the number lab, l being the same *coefficient* as before. But if the direction of β be exactly *opposite* to that of α, their lengths being still a and b, their product is then equal to the opposite number, $-lab$. The same general conclusions might perhaps have been more easily arrived at, if we had *begun* by considering the product of two equally long but *opposite* lines; for it might perhaps then have been even easier to see that, consistently with the *symmetry of space*, no *one* line rather than another could represent, even in part, the direction of the product.

[52.] Next, let us consider the product $\alpha\beta$ of *two mutually perpendicular lines*, α and β, of which each has its length equal to 1. Let α', β' be lines respectively equal in length to these, but respectively opposite in direction. Then $\alpha'\beta = -\alpha\beta = \alpha\beta'$; $\alpha'\beta' = \alpha\beta$. If the sought product $\alpha\beta$ were equal to any *number*, or even if it contained a number as a *part* of its expression, then, on our changing the multiplier α to its own opposite line α', this product or part ought *for one reason* (the symmetry of space) to remain *constant* (because the system of the factors would have been merely *turned in space*); and for another reason ($\alpha'\beta = -\alpha\beta$) the same product or part ought to *change sign* (because *one* factor would have been *reversed*): but this co-existence of opposite results would be absurd. We are led therefore to try whether the present condition (of *rectangularity of the two factors*) allows us to suppose the product $\alpha\beta$ to be a LINE.

[53.] Let γ be a third line, of which the length is unity, and which is at the positive side of β, with reference to α as an axis of rotation; right-handed (or left-handed) rotation having been previously selected as *positive*; let also γ' be the line opposite to γ. Then *any line* in space may be denoted by $m\alpha + n\beta + p\gamma$; we are therefore to try whether we can consistently suppose $\alpha\beta = m\alpha + n\beta + p\gamma$, m, n, p being some three numerical constants. If so, we should have (by the principle of the symmetry of space) $\alpha'\beta = m\alpha' + n\beta + p\gamma'$; and therefore (by a change of all the signs) $\alpha\beta = m\alpha + n\beta' + p\gamma$; therefore $n\beta' = n\beta$, and consequently $-n = n$, or finally $n = 0$. In like manner, since $\alpha\beta = -\alpha\beta' = -(m\alpha + n\beta' + p\gamma') = m\alpha' + n\beta + p\gamma$, we should have $m\alpha' = m\alpha$, and therefore $m = 0$. But there is no objection of *this* kind against supposing $\alpha\beta = p\gamma$, p being some numerical coefficient, constant for all pairs of rectangular lines in space: for the reversal of the direction of a factor has the effect of turning the system through two right angles round the other factor as an axis, and so reverses the direction of the product. And then if the lengths of these two lines α, β, instead of being each $= 1$, are respectively a and b, their product $\alpha\beta$ will be $= pab\gamma$; that is, it will be a line perpendicular to both factors, with a length denoted by pab, and situated always to the positive or always to the negative side of the multiplicand line β, with respect to the multiplier line α as an axis of rotation, according as the constant number p is positive or negative.

[54.] So far, then, without having yet used any property of multiplication, algebraical or geometrical, beyond the three principles: 1st, that *no one direction in space is to be regarded as eminent above another*; 2nd, that *to multiply either factor by any number, positive or negative, multiplies the product by the same*; and 3rd, that *the product of two determined factors is itself determined*; we are led to assign interpretations: 1st, to the product of two *co-axial* vectors, or of two lines parallel to each other, or to one common axis; and 2nd, to the product of two *rectangular* vectors; which interpretations introduce only *two constant*, but as yet unknown, *numerical coefficients*, l and p, depending, however, partly on the assumed unit of length. And we see that for any two co-axial vectors, α, β, the equation $\alpha\beta - \beta\alpha = 0$ holds good; but that for any two rectangular vectors, $\alpha\beta + \beta\alpha = 0$. A *product of two rectangular lines* is, therefore, so far as the foregoing investigation leads us to conclude, *not a commutative function of them*.

[55.] Since then we are compelled, by considerations which appear more primary, to *give up the commutative property* of multiplication, as not holding *generally* for *lines*, let us at least try (as was proposed) whether we can retain the *distributive* property. If so, and if the multiplicand line β be the sum of two others, β_1 and β_2, of which one (β_1) is co-axial with the multiplier line α, while the other (β_2) is perpendicular thereto, we must interpret the product $\alpha\beta$ as equal to the *sum of the two partial products*, $\alpha\beta_1$ and $\alpha\beta_2$. But one of these is a number, and the other is a line; we are, therefore, led to consider a number as being under these circumstances *added* to a line, and as forming with it a certain *sum*, or *system*, denoted by $\alpha\beta_1 + \alpha\beta_2$, or more shortly by $\alpha\beta$. And such a *sum of line and number* may perhaps be called a GRAMMARITHM,* from the two Greek words, γραμμή, a line, and ἀριθμός, a number. A grammarithm is thus to be conceived as being entirely *determined*, when its *two parts* or elements are so; that is, when its *grammic* part is a known line, and its *arithmic* part is a known number. A change in *either* part is to be conceived as changing the grammarithm: thus, *an equation between two grammarithms includes generally two other equations*, one between two numbers, and another between two lines. Adopting this view of a grammarithm, and *defining* that $\alpha\beta = \alpha\beta_1 + \alpha\beta_2$, when $\beta = \beta_1 + \beta_2$, $\beta_1 \| \alpha$, $\beta_2 \perp \alpha$, the product of any determined multiplier line and any determined multiplicand line will be itself entirely determined, as soon as the unit of length and the numbers l and p shall have been chosen; and it remains to consider whether those numbers

* The word 'grammarithm' was subsequently proposed in a communication to the Royal Irish Academy (see *Proc. Roy. Irish Acad.* vol. III (1847), p. 273), as one which *might* replace the word 'quaternion,' at least in the geometrical view of the subject: but it did not appear that there would be anything gained by the systematic adoption of this change of expression, although the mere *suggestion* of another *name*, as not inapplicable, seemed to throw a little additional light on the whole theory. [See XIX.]

can now be so selected, as to make the rules of multiplication of *lines* approach more closely still to the rules of multiplication of *numbers*.

[56.] The *general distributive* principle will be found to give *no new condition*; and we have seen cause to *reject* the *commutative* principle or property, as *not generally* holding good in the present inquiry. It remains, then, to try whether we can determine or *connect* the two coefficients, l and p, so as to satisfy the *associative* principle, or to verify the formula,

$$\alpha.\beta\gamma=\alpha\beta.\gamma$$

For this purpose we may first *distribute* the factors β, γ into others, $\beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3$, which shall be parallel or perpendicular to it and to each other; and then shall have to satisfy, if possible, *six* conditions, which may be reduced to the six following:

$$\alpha.\alpha\alpha=\alpha\alpha.\alpha; \quad \alpha.\alpha\alpha'=\alpha\alpha.\alpha'; \quad \alpha.\alpha\alpha''=\alpha\alpha.\alpha'';$$
$$\alpha.\alpha'\alpha=\alpha\alpha'.\alpha; \quad \alpha.\alpha'\alpha'=\alpha\alpha'.\alpha'; \quad \alpha.\alpha'\alpha''=\alpha\alpha'.\alpha'';$$

$\alpha, \alpha', \alpha''$ being three rectangular unit-lines, so placed that the rotation round α from α' to α'' is positive. Then, by what has been already found, the following relations will hold good:

$$\alpha\alpha=\alpha'\alpha'=\alpha''\alpha''=l; \quad \alpha\alpha'=-\alpha'\alpha=p\alpha''; \quad \alpha\alpha''=-\alpha''\alpha=-p\alpha'; \quad \alpha'\alpha''=-\alpha''\alpha'=+p\alpha;$$

and the six conditions to be satisfied become,

$$\alpha.l=l.\alpha; \quad \alpha.p\alpha''=l.\alpha'; \quad \alpha.-p\alpha'=l.\alpha''; \quad \alpha.-p\alpha''=p\alpha''.\alpha; \quad \alpha.l=p\alpha''.\alpha'; \quad \alpha.p\alpha=p\alpha''.\alpha''.$$

Of these the first suggests to us to treat an arithmic factor as *commutative* (as regards *order*) with a grammic one, or to treat the product 'line into number' as equivalent to 'number into line;' the fourth and sixth conditions afford no new information; and the second, third, and fifth become,

$$-p^2\alpha'=l\alpha'; \quad -p^2\alpha''=l\alpha''; \quad -p^2\alpha=l\alpha.$$

The *conditions of association* are therefore all satisfied by our assuming, with the present signification of the symbols,

$$\alpha l=l\alpha, \quad \text{and} \quad l=-p^2;$$

and they cannot be satisfied otherwise. The constant l is, therefore, by those conditions, necessarily *negative*; and EVERY LINE in tridimensional space has its SQUARE (on this plan) equal to a NEGATIVE NUMBER: which is one of the most novel but essential elements of the whole quaternion theory. (Compare the recent paragraph [48]; also art. 85, pages 81, 82, of the *Lectures*.) And that a *grammarithm* [55] may properly be called a *quaternion*, appears from the consideration that the *line*, which in it is *added* to a *number*, depends itself upon a *system of three numbers*, or may be represented by a *trinomial expression*, because it is always the *sum of three lines* (actual or null), which are parallel to three fixed directions (compare Lecture III). The coefficient p remains still undetermined, and may be made equal to positive one, by a suitable choice of the unit of length, and the direction of positive rotation. In this way we shall have finally the very simple values,

$$p=+1, \quad l=-1;$$

and the *rules* for the *multiplication of lines in space* will then become entirely *definite*, and will *agree* in all respects with the relations [48], between the symbols ijk.

[57.] Another train of *à priori* reasoning, by which I early sought to confirm, or (if it had been necessary) to correct, the results expressed by those new symbols, was stated to the R.I. Academy* in (substantially) the following way. Admitting, for directed and *coplanar* lines,

* See *Proc. Roy. Irish Acad.* vol. III (1847), pp. 1–16. [See XVI.]

the conception [36] of *proportion*; and retaining the symbols ijk, or more fully, $+i$, $+j$, $+k$, to denote three rectangular unit-lines as above, while the three respectively opposite lines may be denoted by $-i$, $-j$, $-k$; but *not assuming* the knowledge of any laws respecting their *multiplication*, I sought to determine *what ought to be considered as the* FOURTH PROPORTIONAL, u, *to the three rectangular directions** j, i, k, *consistently with that known conception* [36] *for directions within the plane*, and with some *general and guiding principles*, respecting *ratios* and *proportions*. These latter assumed principles (of a *regulative* rather than a *constitutive* kind) were simply the following: 1st, that ratios similar to the *same* ratio must be regarded as similar to *each other*; 2nd, that the respectively *inverse* ratios are also mutually similar; and 3rd, that ratios are similar, if they be *similarly compounded* of similar ratios: this similarity of *composition* being understood to include generally a sameness of *order*. It seemed to me that any proposed definitional† use of the word RATIO, which should be inconsistent with these principles, would depart thereby *too widely* from known *analogies*, mathematical and metaphysical, and would involve an impropriety of *language*: while, on the other hand, it appeared that if these principles were attended to, and other analogies observed, it was permitted to extend the use of that word *ratio*, and the connected phrase *proportion*, not only from *quantity* to *direction*, *within one*

* In the abstract published in the Proceedings, the words 'South, West, Up' were used at first instead of the symbols i, j, k; and the sought fourth proportional to jik, which is here denoted by u, was called, provisionally, 'Forward.'

† As an example of the use of the first of these very simple principles, in serving to *exclude a definition* which might for a moment appear plausible, let us take the construction [38], and inquire whether (as that construction would suggest) we can *properly say* that *four directions* (or four diverging unit-lines), $\alpha, \beta, \gamma, \delta$, form generally a *proportion in space*, when the angles $\widehat{\alpha\delta}$, $\widehat{\beta\gamma}$, between the extremes and means have one *common bisector* (ϵ). If so, when the three directions α, β, γ became *rectangular*, we should have $\alpha:\beta::\gamma:-\alpha$, and $\gamma:-\alpha::\beta:-\gamma$; but we should have also, $\alpha:\beta::\beta:-\alpha$, and *not* $\alpha:\beta::\beta:-\gamma$; so that the two ratios, $\alpha:\beta$ and $\beta:-\gamma$, would be said to be similar to one *common* ratio $(\gamma:-\alpha)$, without being similar to *each other*, if the foregoing construction for a *fourth proportional* were to be, by definition, adopted: and this objection *alone* would be held by me to be *decisive* against the introduction of such a *definition*; and therefore also against the adoption of the connected *rule* mentioned in [38], as having at one time occurred to a friend (J. T. G.) and to myself, for the multiplication of lines in space, even if there were *no other reasons* (as in fact there are), for the rejection of that rule. A similar objection applies, with equal decisiveness, against the rule mentioned in [37], as an earlier conjecture of my own. On the other hand, an analogous and equally simple argument may serve to *justify* the notation $\mathrm{D}-\mathrm{C}=\mathrm{B}-\mathrm{A}$, employed by me in the following *Lectures*, and elsewhere, to express that the two right lines AB and CD are *equally long* and *similarly directed*, against an objection made some years ago, in a perfectly candid spirit, by an able writer in the *Philosophical Magazine* (for June 1849, p. 410) [J. Cockle, 'On the symbols of Algebra and on the Theory of Tessarines', *Phil. Mag.* vol. XXIV (1849), pp. 406–410]; who thought that interpretation *more arbitrary* than it had appeared to me to be; and suggested that the *same notation* might as well have been employed to signify this *other conception*: that the two equally long lines AB, CD *met somewhere*, at a finite or infinite distance. I could not admit this extension; for it would lead to the conclusion that two lines AB, EF might be *equal* to the same *third* line CD, without being equal to *each other*: which would (in my opinion) be so great a violation of analogy, as to render the use of the *word* 'EQUAL', or of the *sign* $=$, with the interpretation referred to, an embarrassment instead of an assistance. But I do not feel that analogies are thus violated, by the simultaneous admission of the *two contrasted proportions* (see (3) (4) (5) of [57]),

$$u:i::j:k, \quad u:j::i:-k;$$

for the elementary theorem called often 'alternando,' (ἐναλλὰξ λόγος, Euc. V. Def. 13, and Prop. 16) is by its nature limited (in its original meaning) to the CASE where the *means* which change places are *homogeneous* with each other: whereas two *rectangular directions*, as here i and j, are in this whole theory regarded as being in some sense *heterogeneous*. They have at least no relation to each other, which can be represented by any *ratio*, such as Euclid considers, of *magnitude to magnitude*; and therefore we have no right to *expect*, from analogy to old results, that *alternation* shall *generally* be allowed in a *proportion* involving such directions: although, *within* the plane, alternation is *found* to be admissible.

plane, as had been done [36] by other writers,* but also from the *plane* to *space*.† The supposed proportion,

$$j : i :: k : u, \tag{1}$$

* Since the note to paragraph [36], was in type, I have had an opportunity of re-consulting the fourth volume of the *Annales de Mathématiques*, and have found my recollections (agreeing indeed in the main with the formerly cited page 228 of Dr Peacock's admirable *Report*), respecting the admitted priority of Argand, confirmed. Français, indeed (in 1813), published in those *Annales* (Tome IV, pp. 61–71) a paper which contained a theory of 'proportion de grandeur et de position,' with a connected theory of multiplication (and also of addition) of lines in a given plane; but he expressly and honourably stated at the same time (p. 70), that he owed the substance of those new ideas to another person ('le fond de ces idées nouvelles ne m'appartient pas'): and on being soon afterwards shewn, through Gergonne, whose conduct in the whole matter deserves praise, a copy of Argand's earlier and printed Essay (Paris, 1806), Français most fully and distinctly recognized (p. 225) that the true author of the method was Argand ('il n'y a pas le moindre doute qu'on ne doive à M. Argand la premiere idée de représenter géométriquement les quantités imaginaires'). Nothing more lucid than Argand's own statements (see the same volume, pp. 136, 137, 138), as regards the *fundamental principles* of the theory of the *addition* and *multiplication* of coplanar lines, has since (so far as I know) appeared; not even in the writings of Professor De Morgan on Double Algebra, referred to in former notes. But Argand had not anticipated De Morgan's theory of Logometers; and was on the contrary disposed (pp. 144–146) to regard the symbol $\sqrt{-1}^{\sqrt{-1}}$, notwithstanding Euler's well-known result, as denoting a *line* (KP), *perpendicular to the plane* of the lines 1 and $\sqrt{-1}$: and to consider it as offering an example of a quantity which was *irreducible to the form* $p+q\sqrt{-1}$, and was (according to him) *as heterogeneous* with respect to $\sqrt{-1}$, as the latter with respect to $+1$ ('aussi hétérogéne' &c.). The word modulus ('module'), so well known by the important writings of M. Cauchy, occurs in a later paper by Argand, in the following volume of the Annales, as denoting the real quantity $\sqrt{p^2+q^2}$. If I have seemed to dwell too much on the speculations of Argand (not all adopted by myself), it has been partly because (so far as I have observed) his merits as an original inventor have not yet been sufficiently recognized by mathematicians in these countries: and partly because *one of the two most essential links* (the other being *addition*) between Double Algebra and Quaternions, is Argand's main and *fundamental principle* respecting COPLANAR PROPORTION, expressed by him as follows (*Annales*, T. IV, pp. 136, 137): 'Si (fig. 2) *Ang.* AKB = *Ang.* A′K′B′, on a, abstraction faite des grandeurs absolues, KA : KB :: K′A′ : K′B′. C'est là le principe fondamental de la theorie dont nous avons essayé de poser les premières bases, dans l'écrit dont nous donnons ici un extrait' (namely, Argand's printed Essay of 1806, exhibited by Gergonne to Français, after the appearance of the first paper of the latter author on the subject in 1813). Argand continued thus (in p. 137): 'Ce principe n'a rien au fond de plus étrange que celui sur lequel est fondée la concéption du rapport géometrique entre deus lignes de signes differens, et il n'en est proprement qu'une généralisation:' a remark in which I perfectly concur.

† Although the observations in par. [57] relate rather to *proportions* than to *imaginaries*, yet the present may be a convenient occasion for remarking that Buée, and even Wallis, had speculated, before Argand and Français, on interpretations of the symbol $\sqrt{-1}$, which should extend to *space*: but that the *nearest approach* to an *anticipation of the quaternions*, or at least to an *anticipation of triplets*, seems to me to have been made by Servois, in a passage of the lately cited volume of Gergonne's *Annales*, which appears curious and appropriate enough to be extracted here. Servois had been following up a hint of Gergonne, respecting the representation of ordinary imaginaries of the form $x+y\sqrt{-1}$ (x and y being whole numbers), by a *table of double argument* (p. 71); and thought (p. 235) that *such* a table might be regarded as only a *slice* (une tranche) of a table of TRIPLE argument, for representing *points* (*or lines*) *in* SPACE. He thus continued: 'Vous donneriez sans doute à chacune terme la forme *trinomiale*; mais quel coefficient aurait le troisième terme? Je ne le vois pas trop. L'analogie semblerait exiger que le trinôme fût de la forme, $p\cos\alpha+q\cos\beta+r\cos\gamma$, α, β, γ étant les angles d'une droite avec trois axes rectangulaires; et qu'on eût

$$(p\cos\alpha+q\cos\beta+r\cos\gamma)\,(p'\cos\alpha+q'\cos\beta+r'\cos\gamma)=\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

Les valeurs de p, q, r, p', q', r' qui satisferaient à cette condition seraient *absurdes*' ('quantités non-réelles,' as he shortly afterwards calls them): 'mais seraient-elles *imaginaires* réductibles à la forme génerale $A+B\sqrt{-1}$? Voila une question d'analise fort singulière, que je soumets à vos lumières.' The six NON-REALS which Servois thus with remarkable sagacity *foresaw*, without being able to *determine* them, may now be identified with the then unknown symbols $+i$, $+j$, $+k$, $-i$, $-j$, $-k$, of the quaternion theory: at least, these latter symbols fulfil precisely the *condition* proposed by him, and furnish an *answer* to his 'singular question.' It may be proper to state that my own theory had been constructed and published for a long time, before the lately cited passage happened to meet my eye.

gave thus, by inversion, $$u:k::i:j; \tag{2}$$

but also, in the planes of ij, ik, there were the two proportions,

$$i:j::j:-i, \quad \text{and} \quad k:i::-i:k; \tag{3}$$

compounding therefore, on the one hand, the two ratios, $u:k$ and $k:i$, and, on the other hand, the two respectively similar ratios, $j:-i$, and $-i:k$, there resulted the new proportion,

$$u:i::j:k; \tag{4}$$

which differed from the proportion (2) only by a *cyclical transposition* of the three directions ijk. For the same reason, we may make another cyclical change of the same sort, and may write

$$u:j::k:i; \tag{5}$$

while, in this *cycle* of three rectangular directions, ijk, the *right-handed* (or left-handed) *character* of the *rotation*, round the first from the second to the third, is easily seen to be unaffected by such a transposition. Again compounding the two similar ratios (1) with these two others, which are evidently similar, whatever the unknown direction u may be,

$$i:-i::u:-u, \tag{6}$$

we find this other proportion, $$j:-i::k:-u; \tag{7}$$

and therefore, by (2) and (3), $$u:k::k:-u. \tag{8}$$

In like manner, $$u:i::i:-u, \quad \text{and} \quad u:j::j:-u; \tag{9}$$

and in any one of these proportions, any two terms, whether belonging to the same or to different ratios, may have their *signs* changed together. All these proportions, (2) ... (9), follow from the original supposition (1), by the general principles above stated, without the direction u being as yet any otherwise determined.

[58.] Suppose now that the two rectangular directions j and k are made to *turn together*, in their own plane, round i as an axis, till they take two new positions j_1 and k_1, which will therefore satisfy the proportion, $$j:k::j_1:k_1. \tag{10}$$

We shall then have, by (4), $$u:i::j_1:k_1; \tag{11}$$

and therefore, by a cyclical change of these three new rectangular directions,

$$u:j_1::k_1:i::l:i_1, \tag{12}$$

if l and i_1 be obtained from k_1 and i by any common rotation round j_1. Another cyclical change, combined with a rotation round the new line l, gives finally,

$$u:l::i_1:j_1::m:n; \tag{13}$$

where l, m, n may represent *any three rectangular directions whatever*, subject only to the condition that the *rotation* round l from m to n shall be of the *same character* as that round i from j to k. With this *condition*, therefore, the first assumed proportion (1) may be replaced by this *more general* one: $$n:m::l:u; \tag{14}$$

while for (8) and (9) may now be written, with the same signification of the symbols,

$$u:l::l:-u; \quad u:m::m:-u; \quad u:n::n:-u; \tag{15}$$

and because $n:m::m:-n$, we have these other and not less general proportions,

$$m:-n::l:u; \quad m:n::l:-u. \tag{16}$$

If, then, there be *any* such fourth proportional, u, as has been above supposed, to the three *given* rectangular directions j, i, k, the *same* direction u, or the *opposite* direction $-u$, will also be, in the same sense, the fourth proportional to *any other three* rectangular directions n, m, l, or m, n, l, according as the character of a certain rotation is *preserved* or *reversed*.

[59.] This remarkable result appeared to me to justify the regarding the directions here called $+u$ and $-u$ rather as *numerical* (or algebraical) than as *linear* (or geometrical) *units*; and to make it proper to denote them simply by the symbols $+1$ and -1; because their directions were seen to admit only of a certain *contrast* between themselves, but not of any *other* change: all that *geometrical variety*, which results from the conception of *tridimensional space*, having been found to *disappear*, as regarded them, in an investigation conducted as above. And in fact it is *not permitted*, on the foregoing principles, to *identify* the direction u with that of *any line* (l) *whatever*: for in that case the proportion (13) would give the result $l:l::m:n$, which must be regarded in this theory as an *absurd* one, the two terms of one ratio being *coincident* directions, while those of the other ratio are *rectangular*. But there is no objection of *this* sort against our supposing, as above, that

$$+u=+1, \quad -u=-1; \tag{17}$$

and *then* the *proportions*, derived from (13), (15),

$$1:l::m:n::n:-m; \quad 1:l::l:-1, \tag{18}$$

may be conveniently and concisely *expressed* by formulae of *multiplication*, as follows:

$$lm=n; \quad ln=-m; \quad l^2=-1. \tag{19}$$

[60.] In this way, then, or in one not essentially different, the fundamental formulae [48] of the calculus of quaternions, as first exhibited to the R.I.A. in 1843, namely, the equations,

$$i^2=-1, \quad j^2=-1, \quad k^2=-1, \tag{A}$$

$$ij=+k, \quad jk=+i, \quad ki=+j, \tag{B}$$

$$ji=-k, \quad kj=-i, \quad ik=-j, \tag{C}$$

were shewn (in 1844) to be consistent with *à priori* principles, and with considerations of a general nature; a *product* being *here* regarded as a FOURTH PROPORTIONAL, to a certain *extra-spatial* unit*, and to *two directed factor-lines* in space: whereas, in the investigation of paragraphs [50] to [56], it was viewed rather as a certain FUNCTION of those two factors, the *form* of which function was to be determined in the manner most consistent with some general and guiding analogies, and with the conception of the *symmetry of space*. But there was still *another view* of the whole subject, sketched not long afterwards in another communication to the R.I. Academy,† on which it is unnecessary to say more than a few words in this place,

* It seemed (and still seems) to me natural to connect this *extra-spatial unit* with the conception [3] of TIME, regarded here merely as an *axis of continuous and uni-dimensional progression*. But whether we thus *consider jointly time and space*, or conceive generally *any system of four independent axes*, or scales of progression (u, i, j, k), I am disposed to infer from the above investigation the following LAW OF THE FOUR SCALES, as one which is at least consistent with analogy, and admissible as a *definitional extension* of the fundamental equations of quaternions: 'A formula of *proportion between four independent and directed units* is to be considered as remaining true, when *any two* of them *change places* with each other (in the formula), provided that the *direction* (or *sign*) of *one* be *reversed*.' Whatever may be thought of these abstract and semi-metaphysical *views*, the *formulae* (A) (B) (C) of par. [60] are in any event a sufficient *basis* for the erection of a CALCULUS of quaternions.

† See *Proc. Roy. Irish Acad.* vol. III (1847), Appendix pp. xxxi–xxxvi. [See XVIII.]

because it is, in substance, the view adopted in the following Lectures, and developed with some fulness in them: namely, that view according to which a QUATERNION is considered as the QUOTIENT of two directed lines in tridimensional space.

[61.] Of such a *geometrical quotient*,* b ÷ a, the fundamental property is in this theory conceived to be, that by *operating*, as a *multiplier* (or at least in a way *analogous* to multiplication), on the *divisor-line*, a, it *produces* (or generates) the *dividend-line*, b; and that thus it may be interpreted as satisfying the general and identical formula (compare [9]):

$$(\mathrm{b} \div \mathrm{a}) \times \mathrm{a} = \mathrm{b}.$$

The *analogy to multiplication* consists partly in the operation being one which is performed at once on *length* and on *direction*, as in the ordinary multiplication of a line by a positive or negative number; or as is done in that known *generalization* [36] of such multiplication, for lines within one plane, which (for reasons assigned in notes for former paragraphs) ought (I think) to be called the *Method of Argand*: and partly in the circumstance that the new operation possesses, like that older one (from which, however, it is entirely *distinct*,† in many other and important respects), the *distributive* and *associative*,‡ though *not* like it (generally) the *commutative* properties, of what is called *multiplication in algebra*;§ at least when a few

* This view of a *geometrical quotient* was also developed to a certain extent, in an unfinished series of papers, which appeared a few years ago in the *Cambridge and Dublin Mathematical Journal*, under the head of *Symbolical Geometry*: a title adopted to mark that I had attempted, in the composition of that particular series, to allow a more prominent influence to the general *laws of symbolical language* than in some former papers of mine; and that to this extent I had on that occasion sought to imitate the *Symbolical Algebra* of Dr Peacock, and to profit also by some of the remarks of Gregory and Ohm. [*Cambridge and Dublin Math. J.* vols. I–IV (1846–1849). To be published in vol. IV of the *Mathematical Papers*.]

† Among these *distinctions* of method, it is important to bear in mind that *no one line* is taken, in my system, as representing the *direction of positive unity*: and that, on the contrary, *every vector-unit* is regarded as *one of the square roots of negative unity*. It is to be remarked, also, that the *product* of two inclined but non-rectangular vectors is considered in this theory as *not a line*, but a *quaternion*: all which will be found fully illustrated in the *Lectures*.

‡ To this *associative* principle, or property of multiplication, I attach much importance, and have taken pains to shew, in the Fifth and Sixth Lectures, that it can be *geometrically proved* for quaternions, *independently* of the *distributive* principle, which may, however, in a different arrangement of the subject, be made to *precede* and *assist* the proof of the associative property, as shewn in the Seventh Lecture, and elsewhere. The *absence* of the associative principle appears to me to be an *inconvenience* in the *octaves* or octonomials of Messrs J. T. Graves and Arthur Cayley (see *Lectures*, Appendix B, p. 730): thus in the notation of the former we should indeed have, as in quaternions, $ij = k$, but *not generally* $i.j\omega = k\omega$, if ω represent an octave; for $i.jl = in = -o = -kl = -ij.l$. [See A. Cayley, *Phil. Mag.* vol. XXVI (1845), pp. 210–211; also Appendix 3 for researches of John T. Graves.]

§ The expression 'algebra,' or 'ordinary algebra,' occurs several times in these *Lectures*, as denoting merely *that usual species of algebra*, in which the equation $ab = ba$ is treated as universally true, and not (of course) as implying any degree of disrespect to those many and eminent writers, who have not hitherto chosen to admit into their calculations such equations as $\alpha\beta = -\beta\alpha$, for the multiplication of two rectangular lines, or for other and more abstract purposes. It is proper to state here, that a species of *non-commutative multiplication* for inclined lines (äussere Multiplikation) occurs in a very original and remarkable work by Prof. H. Grassmann (*Ausdehnungslehre*, Leipzig, 1844), which I did not meet with till after years had elapsed from the invention and communication of the quaternions: in which work I have also noticed (when too late to acknowledge it elsewhere) an employment of the symbol $\beta - \alpha$, to denote the *directed line* (Strecke), drawn from the point α to the point β. Notwithstanding these, and perhaps some other coincidences of view, Prof. Grassmann's system and mine appear to be perfectly distinct and independent of each other, in their conceptions, methods, and results. At least, that the profound and philosophical author of the *Ausdehnungslehre* was not, at the time of its publication, in possession of the theory of the *quaternions*, which had in the preceding year (1843) been applied by me as a sort of organ or *calculus for spherical trigonometry*, seems clear from a passage of his Preface (Vorrede, p. xiv), in which he states (under date of 28 June 1844), that he had not then succeeded in *extending the use of imaginaries* definitional formulae (resembling those in par. [9]) are established. And the *motive* (in this view) for calling such a *quotient* a QUATERNION, or the ground for connecting its conception with the NUMBER FOUR, is derived from the consideration that while the RELATIVE LENGTH of the two lines compared depends only on *one number*, expressing their RATIO (of the ordinary kind), their RELATIVE DIRECTION depends on a *system of three numbers*: *one* denoting the ANGLE ($a \wedge b$) between the two lines, and the *two others* serving to determine the *aspect* of the PLANE of that angle, or the *direction* of the AXIS of the positive *rotation* in that plane, *from* the divisor-line (a) *to* the dividend-line (b).

[62.] For the unfolding of this general view,* and the deduction from it of many geometrical† and of some physical‡ consequences, I must refer to the following *Lectures*; of which a considerable part has been drawn up in a more popular§ style than this Preface: while the whole has been composed under the influence of a sincere desire to render the exposition of the subject as clear and elementary as possible. The prefixed *Table of Contents* (pp. ix to lxxii), though somewhat fuller than usual, will be found useful (it is hoped) not merely as an analytical *Index*, assisting a reader to *refer* easily to any part of the volume which he has once carefully read, but also as a general *abridgment* of the work, and in some places as a *commentary*.|| The *Diagrams* are numerous, and have been engraved¶ with care from my drawings: some of them may perhaps be thought to have been unnecessary, but it appeared better to err,

from the plane to space; and generally that unsurmounted difficulties had opposed themselves to his attempts to construct, on his principles, a theory of *angles in space* (hingegen ist es nicht mehr möglich, vermittelst des Imaginären auch die Gesetze für den Raum abzuleiten. Auch stellen sich überhaupt der Betrachtung der Winkel im Raume Schwierigkeiten entgegen, zu deren allseitiger Lösung mir noch nicht hinreichende Musse geworden ist). The earlier treatise by Prof. A. F. Möbius (*Der barycentrische Calcul*, Leipzig, 1827), referred to in the same Preface by Grassmann, appears to be a work which likewise will deserves attention, for its conceptions, notations, and results; as does also another work of Möbius (*Mechanik des Himmels*, Leipzig, 1843), elsewhere referred to in these *Lectures* (page 614).

* I may just hint here that the BIQUATERNIONS of Lect. VII admit of being *geometrically interpreted* (comp. note to [19]), by considering each as a *couple of quotients* $\left(\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}\right)$, *constructed* by a TRIRADIAL (α, β, γ), and *multiplied* by a *commutative* factor of the form $\sqrt{-1}$ (compare [16]), when the *line-couple* (β, γ) is changed to ($-\gamma, \beta$), or when the *angle* $\widehat{\beta\gamma}$ is changed to an *adjacent angle*.

† Notwithstanding some references to works of M. Chasles, and other eminent foreign geometers, my acquaintance with their writings is far too imperfect to give me any confidence in the *novelty* of various theorems in the VIIth Lecture and Appendix (such as those respecting generations of the ellipsoid, and inscriptions of gauche polygons in surfaces of the second order), beyond what is derived from the opinion of a few geometrical friends.

‡ Some such *physical* applications were early suggested by Sir J. Herschel. [See XLI.]

§ It had been designed that these Lectures should not go much more into detail than those which have been actually delivered on the subject by me, in successive years, in the Halls of this University; and the First Lecture, printed in 1848 (as the astronomical allusions at its commencement may indicate), was in fact delivered in that year, in very nearly the form in which it now appears. But it was soon found necessary to extend the plan of the composition: and it is evident that the subsequent Lectures, as printed, are too long, and that the last of them involves too much calculation, to have been delivered in their present form: though something of the style of actual lecturing has been here and there retained. The real *divisions* of the work are not so much the *Lectures* themselves, as the shorter and more numerous *Articles*, to which accordingly the *references* have been chiefly made. An intermediate form of subdivision into *Sections* has however been used in drawing up the *Contents*, which the reader may adopt or not at his discretion, marking or leaving unmarked the margin of the Lectures accordingly. Some new terms and symbols have been unavoidably introduced into the work, but it is hoped that they will not be found embarrassing, or difficult to remember and apply.

|| For instance, as regards the formation of the Adeuteric Function (p. xliii).

¶ By Mr W. Oldham, whose fidelity and diligence are hereby acknowledged.

if at all, on the side of clearness and fulness of illustration, especially in the early parts of a work based on a new mathematical conception, and designed to furnish, to those who may be disposed to employ it, a new mathematical organ. Whatever may be thought of the degree of success with which my exertions in this matter have been attended, it will be felt, at least, that they must have been arduous and persevering. My thanks are due, at this last stage, to the friends who have cheered me throughout by their continued sympathy; to the scientific contemporaries* who have at moments turned aside from their own original researches, to notice, and in some instances to extend, results or speculations of mine; to my academical superiors who have sanctioned, as a subject of public and repeated examination in this University, the theory to which this Volume relates, and have contributed to lighten, to an important extent, the pecuniary risk of its publication: but, above all, to that Great Being, who has graciously spared to me such a measure of health and energy as was required for bringing to a close this long and laborious undertaking.

WILLIAM ROWAN HAMILTON

Observatory of T. C. D., June 1853

* In these countries, Messrs Boole, Carmichael, Cayley, Cockle, De Morgan, Donkin, Charles and John Graves, Kirkman, O'Brien, Spottiswoode, Young, and perhaps others: some of whose researches or remarks on subjects connected with quaternions (such as the *triplets, tessarines, octaves*, and *pluquaternions*) have been elsewhere alluded to, but of which I much regret the impossibility of giving here a fuller account. As regards the theory of *algebraic keys* (clefs algébriques), lately proposed by one of the most eminent of continental analysts, as one that *includes* the quaternions (*Comptes Rendus* for 10 Jan. 1853, p. 75), it appears to me to be virtually *included* in that theory of SETS in algebra (explained in the present Preface), which was announced by me in 1835, and published in 1848 [see VII] (*Trans. Roy. Irish Acad.* vol. XXI, part II, p. 229, etc., the symbols x_r being in fact what M. Cauchy calls KEYS), as an *extension* of the theory of *couples* (and therefore also of imaginaries): of which SETS I have always considered the QUATERNIONS (in their *symbolical* aspect) to be merely a *particular* CASE. Before the publication of those *sets*, the closely connected conception of an '*algebra of the n^{th} character*' had occurred to Prof. De Morgan in 1844, avowedly as a suggestion from the quaternions (*Trans. Camb. Phil. Soc.* vol. VIII (1839)).