Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 2, pp. 365–376, Warszawa 1973

Acceleration waves in isotropic simple materials

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THE ACOUSTIC tensor for the propagation of acceleration waves in isotropic simple materials is calculated, when the simple material is undergoing an arbitrary large deformation. From this acoustic tensor, the speeds of propagation of principal acceleration waves into an isotropic solid at rest in a homogeneously deformed state are determined, along with the compatibility conditions. Next, the theory is applied to the propagation of waves in incompressible simple fluids and explicit speeds are obtained for a shear wave moving into a liquid undergoing simple shearing, and a discontituity moving axially in simple extension. For the viscoelastic solid, it is observed that the linear functionals occurring in the acoustic tensor are those existing in the theory of small deformations on large, while for the simple fluid in steady shearing, the functionals are those of the theory of nearly viscometric flows.

Obliczono tensor akustyczny rozprzestrzeniania się fal przyśpieszenia w izotropowych materiałach prostych przy założeniu, że materiał poddany jest dowolnie dużym deformacjom. Z tego tensora akustycznego określono zarówno prędkość rozprzestrzeniania się głównych fal przyśpieszenia w ośrodek izotropowy w stanie spoczynku w stan jednorodnej deformacji, jak również warunki zgodności. Następnie zastosowano tę teorię do rozprzestrzeniania się fal w nieściśliwych cieczach prostych i otrzymano wyraźne prędkości dla fali ścinania, poruszającej się w cieczy poddanej prostemu ścinaniu, oraz dla poruszającej się osiowo nieciągłości w prostym rozciąganiu. Zaobserwowano, że dla ciała sprężystolepkiego funkcjonały liniowe, występujące w tensorze akustycznym, są takie same jak w teorii małych deformacji nałożonych na duże, podczas gdy dla prostej cieczy przy ustalonym ścinaniu funkcjonały te są takie jak w teorii wiskometrycznego płynięcia.

Вычислен акустический тензор для распространения волн ускорения в изотропных простых средах. При этом предполагается, что простой материал подвергается произвольным конечным деформациям. С помощью акустического тензора определяются скорости распространения главных волн ускорения в неподвижной изотропной среде, находящейся в однородном деформированном состоянии, а также устанавливаются условия совместности. Затем данная теория применяется к исследованию распространения волн в несжимаемых простых жидкостях. Получены в явном виде формулы для скоростей распространения волн сдвига в жидкости, подверженной простому сдвигу, и для волны разрыва, движущейся по оси простого растяжения. Для вязкоупругого тела обнаружено, что линейные функционалы, входящие в акустический тензор, совпадают с функционалами, возникающими в теории малых деформаций, наложенных на конечные. В случае простых жидкостей в состоянии стационарного сдвига эти функционалы совпадают с теми, которые описывают течения, близкие вискозиметрическим.

1. Introduction

LET X denote the generic particle belonging to the body $\mathscr{B}$ and let t denote the time. Further, let $\mathbf{X}$, $\boldsymbol{\xi}$ and $\mathbf{x}$ be the respective positions of X in a fixed reference configuration, at time $(t-s)$, $0 \leqslant s < \infty$, and at time t. Of course,

$$\boldsymbol{\xi}(X, t-s)|_{s=0} = \mathbf{x}(X, t). \tag{1.1}$$

In a fixed Cartesian coordinate system, assuming the fixed reference configuration to be the natural state for a homogeneous isotropic simple solid the constitutive equation for the stress $\mathbf{T}(X, t)$ is given by [1, § 31]:

$$T_{ij}(X, t) = f_{ij}\big(B_{pq}(t)\big) + \mathop{\mathscr{F}_{ij}}_{s=0}^{\infty}\big(G_{kl}(s); B_{uv}(t)\big), \tag{1.2}$$

where δ_{kl} is the Kronecker delta; B_{pq} are the components of the strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, where $\mathbf{F}$ is deformation gradient of $\mathbf{x}(X, t)$ relative to $\mathbf{X}$, and $\mathbf{F}^T$ is the transpose of $\mathbf{F}$.

Also,

$$G_{kl}(s) = \big(\mathbf{C}_t(t-s)\big)_{kl} - \delta_{kl}, \quad 0 \leqslant s < \infty, \tag{1.3}$$

where $\big(\mathbf{C}_t(t-s)\big)_{kl}$ are the components of $\mathbf{C}_t(t-s) = \mathbf{F}_t^T(t-s)\mathbf{F}_t(t-s)$, so that $\mathbf{F}_t(t-s)$ is the gradient of $\boldsymbol{\xi}(X, t-s)$ with respect to $\mathbf{x}(X, t)$. The function $\mathbf{f}(\mathbf{B})$ in (1.2) determines the elastic response, while the functional $\mathop{\mathscr{F}}_{s=0}^{\infty}\big(\mathbf{G}(s); \mathbf{B}\big)$ yields the viscoelastic (or memory) part of the stress. This functional may be normalized, so that

$$\mathop{\mathscr{F}}_{s=0}^{\infty}(\mathbf{0}; \mathbf{B}) = 0, \tag{1.4}$$

without any loss of generality. The main purpose of this paper is to determine the acoustic tensor for the propagation of acceleration waves and to determine the speeds of such waves in isotropic simple solids and fluids.

Limiting the discussion to isotropic materials, one notes that an explicit formula for the acoustic tensor is not available in the paper by COLEMAN and GURTIN [2] or that of VARLEY [3], for these authors considered materials with arbitrary symmetry. No doubt such a formula may be found from their papers by suitable modifications. But this process would be cumbersome and thus one may follow ERICKSEN [4] and TRUESDELL [5] by starting with the constitutive equation for isotropic materials, and hence obtain the formula for the acoustic tensor (see § 2).

Next, in § 3, the speeds of propagation of principal waves in an isotropic solid at rest under a large homogeneous strain are examined and results generalizing those of TRUESDELL [5], who discussed elastic solids, and VARLEY [3], who examined the case of the homogeneous isotropic viscoelastic solid at rest in the undeformed state, are obtained. In other words, we find speeds of propagation when $\mathbf{C}_t(t-s) = \mathbf{1}$, $0 \leqslant s < \infty$, and

$$B_{ij} = \operatorname{diag}\{v_1^2, v_2^2, v_3^2\}. \tag{1.5}$$

In deriving these principal wave speeds, it is noted that the linear functionals occurring in the acoustic tensor are identical to those occurring in the theory of a small deformation superposed on a large homogeneous strain of an isotropic solid and use is made of the explicit formulae given by PIPKIN and RIVLIN [6] or PIPKIN [7].

In § 4, the results of § 2 are specialized to incompressible simple fluids and the acoustic tensor for an acceleration wave propagating into the simple fluid undergoing an arbitrary motion is determined. To arrive at this result, one uses ERICKSEN'S approach [4], and then shows that the linear functionals occurring in this acoustic tensor are identical to those of the theory of nearly viscometric flows [8], if the acceleration wave propagates into the simple fluid in steady simple shearing. Thus, a close relationship with the work of COLEMAN and GURTIN [9] is established. The speed of acceleration waves in a fluid undergoing simple extension is calculated as a second example.

Before proceeding to § 2, the attention of the reader is drawn to the recent article by HAYES and RIVLIN [10] who discussed the propagation of sinusoidal small-amplitude waves in a deformed, initially isotropic, viscoelastic solid. Our results are derived in a different manner and it is found that the speeds of propagation of principal waves reported here do not agree with those listed in [10]. An equivalence between the two sets of speeds need not exist always, though it does in finite elasticity [5, § 4; 11] and in some linear theories of continua [12, § 194A]. Of course, the present results in three dimensions confirm those obtained by COLEMAN and GURTIN [13, § 7], who proved that the speeds of acceleration waves and damped oscillatory waves are not equal, at least for one-dimensional motions; however, they established that the ultrasonic wave speed is equal to the acceleration wave speed. We examine this question briefly in § 3 as it applies to the results of this paper and those in [10].

In a future article, the growth and decay of acceleration waves will be studied.

2. The acoustic tensor

As is customary, one assumes that the motion, deformation gradient and the velocity field at time t are continuous across the wave, which at time t is to be found at X and occupies the spatial position $\mathbf{x}(X, t)$. Or, the jumps of x_i, $x_{i,\alpha}$ and v_i are zero; and using the notation $[f]$ to denote the jump of a quantity f across the wave, one has

$$[x_i] = [x_{i,\alpha}] = [v_i] = 0. \tag{2.1}$$

In (2.1), $\mathbf{v}$ is the velocity of X at time t and

$$F_{i\alpha} = x_{i,\alpha} = \frac{\partial x_i}{\partial X_\alpha} \tag{2.2}$$

is the deformation gradient at time t. Next, the compatibility conditions [12, § 190] for acceleration waves yield

$$[\ddot{x}_i] = U^2 a_i, \tag{2.3}$$

$$[F_{i\alpha,\beta}] = F_{j\alpha} F_{k\beta} n_j n_k a_i, \tag{2.4}$$

where $\mathbf{a}$, the jump of the acceleration across the wave, is the amplitude, U is the local speed of propagation and $\mathbf{n}$ is the unit normal to the wave surface at $\mathbf{x}(X, t)$.

Using the definitions of the strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and $\mathbf{C}_t(t-s) = \mathbf{F}_t^T(t-s)\,\mathbf{F}_t(t-s)$, $0 \leqslant s < \infty$, we have:

$$B_{pq}(t) = F_{p\alpha}(t) F_{q\alpha}(t), \tag{2.5}$$

$$(\mathbf{C}_t(t-s))_{kl} = X_{\alpha,k}(t) C_{\alpha\beta}(t-s) X_{\beta,l}(t). \tag{2.6}$$

In writing (2.6), the identity

$$\mathbf{C}_t(t-s) \equiv (\mathbf{F}(t)^{-1})^T \mathbf{C}(t-s) \mathbf{F}(t)^{-1}, \tag{2.7}$$

has been used, so that

$$C_{\alpha\beta}(t-s) = \xi_{i,\alpha}\xi_{i,\beta}. \tag{2.8}$$

From (2.4) and (2.5), it is quite easy to verify that

$$[B_{pq,j}] = F_{m\alpha}F_{q\alpha}n_m n_j a_p + F_{p\alpha}F_{m\alpha}n_m n_j a_q. \tag{2.9}$$

Next, from the identity

$$X_{\alpha,p}x_{q,\alpha} \equiv \delta_{pq}, \tag{2.10}$$

one obtains

$$X_{\alpha,pj} = -X_{\gamma,p}X_{\beta,j}X_{\alpha,q}x_{q,\gamma\beta}, \tag{2.11}$$

and thus

$$[X_{\alpha,pj}] = -X_{\alpha,q}n_p n_j a_q, \tag{2.12}$$

where (2.1) and (2.4) have been used. Next,

$$[X_{\alpha,kj}]C_{\alpha\beta}(t-s)X_{\beta,l} = -a_m n_k n_j \big(\mathbf{C}_t(t-s)\big)_{ml}, \tag{2.13}$$

$$X_{\alpha,k}C_{\alpha\beta}[X_{\beta,lj}] = -a_m n_l n_j \big(\mathbf{C}_t(t-s)\big)_{km}. \tag{2.14}$$

To calculate the jump of $C_{\alpha\beta,\gamma}$, one has that

$$C_{\alpha\beta,\gamma} = \xi_{i,\alpha\gamma}\xi_{i,\beta} + \xi_{i,\alpha}\xi_{i,\beta\gamma}. \tag{2.15}$$

Thus the jump of $C_{\alpha\beta,\gamma}$ depends on the jump of $\xi_{i,\alpha\gamma}$. However, this is the gradient of the deformation gradient at X at time $t-s$, $0 \leqslant s < \infty$, and this gradient of $\mathbf{F}(t-s)$ is *continuous* at X until the arrival of the wave at $s = 0$. In other words, $[C_{\alpha\beta,\gamma}]$ is to be calculated at the time $s = 0$ only, with no attention paid to other values of s. In fact, as long as the wave is found at X on a (time) set of measure zero, this argument implies that the jump is to be calculated on this set. For simplicity, the rest of the article assumes that the jump in $\xi_{i,\sigma\gamma}$ occurs at $s = 0$ only (1).

We now return to (1.2) and obtain

$$T_{ij,j} = \frac{\partial f_{ij}}{\partial B_{pq}} B_{pq,j} + \partial_{B_{uv}} \overset{\infty}{\underset{s=0}{\mathscr{F}}}_{ij}\big(G_{kl}(s);\, B_{mj}\big)\, B_{uv,j} + \delta \overset{\infty}{\underset{s=0}{\mathscr{F}}}_{ijkl}\Big(G_{pq}(s);\, B_{uv}\big|\big(\mathbf{C}_t(t-s)\big)_{kl,j}\Big), \tag{2.16}$$

where the "elasticities" $\partial\mathbf{f}/\partial\mathbf{B}$ and $\partial\mathscr{F}/\partial\mathbf{B}$, and the functional $\delta\mathscr{F}(\,\cdot\,;\,.\,|\operatorname{grad}\mathbf{C}_t(t-s))$, which is linear and continuous in $\operatorname{grad}\mathbf{C}_t(t-s)$, are all *assumed* to be continuous for all $X \in \mathscr{B}$ and $t \in (-\infty, \infty)$, including the wave surface. Then, the jump $[T_{ij,j}]$ is given by

$$[T_{ij,j}] = \frac{\partial f_{ij}}{\partial B_{pq}} [B_{pq,j}] + \partial_{B_{uv}} \overset{\infty}{\underset{s=0}{\mathscr{F}}}_{ij}(\,.\,;\,.)[B_{uv,j}] + \delta \overset{\infty}{\underset{s=0}{\mathscr{F}}}_{ijkl}\Big(\,.\,;\,.\big|\big[\big(\mathbf{C}_t(t-s)\big)_{kl,j}\big]\Big). \tag{2.17}$$

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(1) This is equivalent to the condition used by COLEMAN, GURTIN and HERRERA [14] in their study of acceleration waves.

Now, the linear functional $\delta\mathcal{F}$ is a sum of three linear functionals as below:

$$\delta \mathop{\mathcal{F}_{ijkl}}_{s=0}^{\infty}\Big(. ; .|-a_m n_k n_j \big(\mathbf{C}_t(t-s)\big)_{ml}\Big) + \delta \mathop{\mathcal{F}_{ijkl}}_{s=0}^{\infty}\Big(. ; .|-a_m n_l n_j \big(\mathbf{C}_t(t-s)\big)_{km}\Big) + \delta \mathop{\mathcal{F}_{ijkl}}_{s=0}^{\infty}(. ; .|X_{\alpha,k}[C_{\alpha\beta,j}]X_{\beta,l}). \tag{2.18}$$

In the above sum, the last functional is zero because the jump occurs at $s = 0$ (or on a set of measure zero). Thus, instead of (2.18), we have the following expression:

$$-2a_m \delta \mathop{\mathcal{F}_{ijkl}}_{s=0}^{\infty}\Big(. ; .|n_k n_j \big(\mathbf{C}_t(t-s)\big)_{ml}\Big), \tag{2.19}$$

where the fact that no loss of generality results by assuming $\delta \mathop{\mathcal{F}_{ijkl}}_{s=0}^{\infty} = \delta \mathop{\mathcal{F}_{ijlk}}_{s=0}^{\infty}$ has been used. The interesting part is that (2.19) is linear in $\mathbf{a}$, as are the jumps $[B_{pq,j}]$ and $[B_{uv,j}]$. Using these facts, one can write the jump of the equation of motion, i.e.,

$$[T_{ij,j}] = \varrho[\ddot{x}_i], \tag{2.20}$$

which is derived as a consequence of the assumed continuity of the body force field, as:

$$Q_{ij}(\mathbf{n})a_j = \tilde{\varrho}\, U^2 a_i, \tag{2.21}$$

where $\tilde{\varrho}$ is the density at X in the fixed reference configuration, and the *acoustic tensor* $\mathbf{Q}(\mathbf{n})$ is:

$$\frac{\varrho}{\tilde{\varrho}} Q_{ij}(\mathbf{n}) = 2\frac{\partial f_{ik}}{\partial B_{jl}} B_{lm} n_m n_k + 2\frac{\partial \mathcal{F}_{ik}}{\partial B_{js}} B_{rs} n_k n_s - 2\delta \mathop{\mathcal{F}_{imkl}}_{s=0}^{\infty}\Big(. ; .|n_m n_k \big(\mathbf{C}_t(t-s)\big)_{jl}\Big), \tag{2.22}$$

where a condensed notation is employed.

Of course, (2.22) can also be derived from the work of VARLEY [3] when it is assumed that the constitutive relation (1.2) is expressible in the form of multiple integrals, or from the work of COLEMAN and GURTIN [2]. However, as mentioned in § 1, such a procedure is not quite straightforward.

It should be emphasized that (2.21) and (2.22) yield the speeds of propagation of an arbitrary acceleration wave moving into an isotropic simple solid undergoing an arbitrary large deformation.

The reader will also note that this article does not discuss thermodynamic aspects of wave propagation. If it be assumed, as is natural, that the material is a definite conductor, then all acceleration waves are homothermal [2]. So the formulae for the acoustic tensor and speeds of propagation are not significantly altered by including temperature effects, and since the determination of the acoustic tensor and the wave speeds are the aims of this article, the omission of temperature effects is not a major limitation on the results quoted here.

3. Isotropic solid in finite homogeneous strain

In this section, it is assumed that the finite homogeneous strain is such that the strain tensor $\mathbf{B}(t)$ has the form

$$B_{ij} = \mathrm{diag}\{v_1^2, v_2^2, v_3^2\}; v_\Gamma^2 > 0, \quad \Gamma = 1, 2, 3; \tag{3.1}$$

further, after an initial deformation to yield the above value for **B**, the material has been held at rest for an infinitely long time, so that

$$\mathbf{C}_t(t-s) = \mathbf{1}, \quad 0 \leqslant s < \infty. \tag{3.2}$$

Then, from (2.22), one can obtain the acoustic tensor as follows. If (3.1) and (3.2) hold, then from (1.2) one has that

$$\mathbf{T} = \mathbf{f}(\mathbf{B}) + \underset{s=0}{\overset{\infty}{\mathscr{F}}}(\mathbf{0}; \mathbf{B}). \tag{3.3}$$

Now, as mentioned in § 1, no loss of generality occurs in *assuming* that

$$\underset{s=0}{\overset{\infty}{\mathscr{F}}}(\mathbf{0}; \mathbf{B}) = \mathbf{0}, \tag{3.4}$$

or that the equilibrium stress is elastic. Now, if (3.4) holds for all positive definite **B**, then

$$\partial_{B_{uv}} \underset{s=0}{\overset{\infty}{\mathscr{F}}}_{ij}(0; B_{rs}) = 0. \tag{3.5}$$

Under this assumption, $Q_{ij}(n)$ in (2.22) becomes

$$\frac{\varrho}{2\tilde{\varrho}} Q_{ij}(\mathbf{n}) = \frac{\partial f_{ik}}{\partial B_{jl}} B_{lm} n_m n_k - \delta \underset{s=0}{\overset{\infty}{\mathscr{F}}}_{imkl}(v_\Gamma^2 | n_m n_k \delta_{jl}) \tag{3.7}$$

$$= \frac{\partial f_{ik}}{\partial B_{jl}} B_{lm} n_m n_k - n_m n_k \delta \underset{s=0}{\overset{\infty}{\mathscr{F}}}_{imjk}(v_\Gamma^2 | 1), \quad \Gamma = 1, 2, 3. \tag{3.6}$$

Any time one desires, one can write the linear functional $\delta\mathscr{F}$ as an integral of course, since the domain of $\mathscr{F}$ is the Hilbert space of histories [1, § 38].

From (3.7), it is trivial to establish the following: *the speeds of propagation and the acoustic axes are determined by the strain* **B** *alone, the memory of the material appearing through the linear functional* $\delta\mathscr{F}(\mathbf{B}|1)$. The above result generalizes that of TRUESDELL [5, p. 274] for isotropic elastic solids to isotropic solids with memory, when these materials are under the deformations (3.1) and (3.2).

Now, let the elastic part $\mathbf{f}(\mathbf{B})$ of the stress be written as

$$\mathbf{f}(\mathbf{B}) = f_0 \mathbf{1} + f_1 \mathbf{B} + f_2 \mathbf{B}^2, \tag{3.8}$$

where the f_Γ are analytic functions of the principal invariants I, II and III of **B**. Thus, by Eq. (7.8) of [5], and (3.7) above, one can obtain the squared wave speed U_{11}^2 of a longitudinal wave travelling down the principal axis with the stretch v_1 as:

$$\frac{\varrho U_{11}^2}{2v_1^2} = f_1 + 2v_1^2 f_2 + \sum_{\Gamma=0}^{2} v_1^{2\Gamma}\left(\frac{\partial f_\Gamma}{\partial \mathrm{I}} + (v_2^2+v_3^2)\frac{\partial f_\Gamma}{\partial II} + v_2^2 v_3^2 \frac{\partial f_\Gamma}{\partial \mathrm{III}}\right) - \frac{1}{2v_1^2} n_i^1 n_j^1 n_k^1 n_l^1 \delta \underset{s=0}{\overset{\infty}{\mathscr{F}}}_{ilkj}(v_\Gamma^2|1), \tag{3.9}$$

where $\{n_i^1\}$ are the components of the unit vector $\mathbf{n}^1$ along the principal axis with the stretch v_1. Similarly, the squared wave speed U_{12}^2, of a transverse wave travelling down $\mathbf{n}^1$ with amplitude parallel to $\mathbf{n}^2$, is:

$$\text{(3.10)} \qquad \frac{\varrho U_{12}^2}{v_1^2} = f_1 + (v_1^2 + v_2^2) f_2 - \frac{1}{v_1^2} n_i^2 n_j^2 n_k^1 n_l^1 \, \delta \mathop{\mathscr{F}}_{s=0}^{\infty}{}_{ilkj}(v_\Gamma^2 | 1).$$

Thus it is obvious that more definite statements can be made only if one knows the explicit form of $\delta \mathop{\mathscr{F}}_{s=0}^{\infty}{}_{iklj}(v_\Gamma^2|1)$. This we shall derive next.

Consider two strain histories $\mathbf{C}_t^*(t-s)$ and $\mathbf{C}_t(t-s)$, $0 \leqslant s < \infty$, such that they are close to each other in the sense of the norm of the Hilbert space. Let $\mathbf{B}^*(t)$ and $\mathbf{B}(t)$ be the corresponding two strains, also close to one another, so that one is considering the situation of "small on large". Then, the corresponding stresses $\mathbf{T}^*$ and $\mathbf{T}$ are related through

$$\text{(3.11)} \qquad T_{ij}^* - T_{ij} \approx \frac{\partial f_{ij}}{\partial B_{pq}} (B_{dq}^* - B_{pq}) + \frac{\partial \mathscr{F}_{ij}}{\partial B_{uv}} (.\,;.)(B_{uv}^* - B_{uv})$$
$$+ \delta \mathop{\mathscr{F}}_{s=0}^{\infty}{}_{ijkl} \Big(.\,;.|(\mathbf{C}_t^*(t-s) - \mathbf{C}_t(t-s))_{kl}\Big).$$

Thus the operators appearing in the acoustic tensor in (2.22) are identical to the *incremental response operators* appearing in (3.11), because of the uniqueness of the derivatives. This result is of course well known.

Now, it is very difficult to obtain an explicit form for $\delta \mathop{\mathscr{F}}_{s=0}^{\infty}{}_{ijkl}(G_{pq}(s);\, B_{uv} \,|\, [(\mathbf{C}_t(t-s)_{kl,j})]$ unless $G_{pq} = 0$. Then, from PIPKIN and RIVLIN [6, Eq. (9.14)] or PIPKIN [7, Eq. (9.7) one has:

$$\text{(3.12)} \qquad \delta \mathop{\mathscr{F}}_{s=0}^{\infty}{}_{ijkl} \Big(0;\, B_{uv} | [(C_t(t-s))_{kl,j}]\Big) = \int_0^\infty \Big\{k_0(s)\,\delta_{ik}\,\delta_{jl} + k_1(s)(\delta_{ik} B_{jl} + \delta_{jl} B_{ik})$$
$$+ k_2(s)(\delta_{ik} B_{jn} B_{nl} + \delta_{jl} B_{in} B_{nk}) + \sum_{M,N=0}^{2} k_{MN}(s)(\mathbf{B}^M)_{ij}(\mathbf{B}^N)_{kl}\Big\} [(\mathbf{C}_t(t-s))_{kl,j}]\, ds,$$

where the scalar coefficients are functions of the invariants of $\mathbf{B}$ and for $[(\mathbf{C}_t(t-s))_{kl,j}]$, one puts $-2a_m n_k n_j \delta_{ml}$ because of (2.13), (2.14) and the fact that $\mathbf{C}_t(t-s) = \mathbf{1}$.

Now, return to (3.7) and note that if $\mathbf{B}$ is given by (3.1), an eigenvector $\mathbf{n}^1$ of $\mathbf{B}$ is also the eigenvector of $\mathbf{Q}(\mathbf{n}^1)$. Thus we have proved

the acoustic axes for principal waves coincide with the principal axes; further, all principal waves are longitudinal or transverse (2),

which extends to isotropic solids with memory, the theorem of TRUESDELL [5, p. 275]. Of course the above theorem holds if $\mathbf{B}$ and $\mathbf{C}_t(t-s)$ obey (3.1) and (3.2), respectively. Next, since

$$\text{(3.13)} \qquad \delta \mathop{\mathscr{F}}_{s=0}^{\infty}{}_{1111}(v_\Gamma^2|1) = \int_0^\infty \Big\{k_0(s) + 2k_1(s) v_1^2 + 2k_2(s) v_1^4 + \sum_{M,N=0}^{2} k_{MN}(s) v_1^{2(M+N)}\Big\}\, ds \equiv f(v_1^2),$$

(2) This was proved by HAYES and RIVLIN [10, § 5] also.

the longitudinal speed U_{11}^2 in (3.9) is given by

$$\frac{\varrho U_{11}^2}{2v_1^2} = f_1 + 2v_1^2 f_2 + \sum_{\Gamma=0}^{2} v_1^{2\Gamma}\left(\frac{\partial f_\Gamma}{\partial \mathrm{I}} + (v_2^2+v_3^2)\frac{\partial f_\Gamma}{\partial \mathrm{II}} + v_2^2 v_3^2 \frac{\partial f_\Gamma}{\partial \mathrm{III}}\right) - \frac{1}{2v_1^2} f(v_1^2), \tag{3.14}$$

where one has put $\{n_i^1\} = \{1, 0, 0\}$. Moreover, with $\{n_i^2\} = \{0, 1, 0\}$,

$$\delta \overset{\infty}{\underset{s=0}{\mathscr{F}}}{}_{2112}(v_\Gamma^2|1) = 0, \tag{3.15}$$

and thus from (3.10), one has

$$\frac{\varrho U_{12}^2}{v_1^2} = f_1 + (v_1^2+v_2^2) f_2. \tag{3.16}$$

It is trivial to verify, from (3.16), that the compatibility conditions (9.1)–(9.3) of TRUESDELL [5] hold here too. Next, if one puts

$$A_1 = \frac{1}{2}\varrho \frac{U_{11}^2}{v_1^2} + \frac{1}{2v_1^2} f(v_1^2) - f_1 - 2v_1^2 f_2 - \sum_{\Gamma=1}^{2} v_1^{2\Gamma}\left(\frac{\partial f_\Gamma}{\partial \mathrm{I}} + (v_2^2+v_3^2)\frac{\partial f_\Gamma}{\partial \mathrm{II}} + v_2^2 v_3^2 \frac{\partial f_\Gamma}{\partial \mathrm{III}}\right), \tag{3.17}$$

then the function f_0, occurring in (3.8) here, is again given by equation (9.8) of [5].

The above compatibility conditions may be used to determine the function $f(v_1^2)$ of (3.13), once a knowledge of f_0, f_1 and f_2 is available.

As mentioned in § 1, the wave speeds and compatibility conditions derived here are not in agreement with those in [10, § 5], though the speeds reduce to those found for the special cases treated in [3, 5, 13].

A brief examination of the differences will be made next. The principal transverse acceleration wave speed U_{21}^2 is (cf. (3.16)):

$$\frac{\varrho U_{21}^2}{v_2^2} = f_1 + (v_1^2+v_2^2) f_2, \tag{3.18}$$

while from HAYES and RIVLIN [10, Eq. (5.10)]

$$\frac{\varrho U_{21}^2}{v_1^2} = \frac{t_1 - t_2}{v_1^2 - v_2^2} + \frac{1}{v_2^2}\{\tilde{\alpha}_3 + (v_1^2+v_2^2)\tilde{\alpha}_4 + (v_1^4+v_2^4)\tilde{\alpha}_5\}, \tag{3.19}$$

where

$$\frac{t_1 - t_2}{v_1^2 - v_2^2} = f_1 + (v_1^2+v_2^2) f_2, \tag{3.20}$$

and $\tilde{\alpha}_j$ $(j = 3, 4, 5)$ are functions of the principal invariants of $\mathbf{B}$ and $i\omega$ $(i^2 = -1)$. As mentioned in § 1, COLEMAN and GURTIN [13, § 7] proved that the acceleration wave speed and the infinitesimal progressive wave speed are identical if the latter is the ultrasonic wave speed, i.e., the speed corresponding to $\omega = \infty$. If we conjecture this theorem of equivalence to hold here also, then

$$\lim_{\omega\to\infty} \tilde{\alpha}_j(\mathrm{I}, \mathrm{II}, \mathrm{III}, i\omega) = 0, \quad j = 3, 4, 5. \tag{3.21}$$

These and other matters related to growth and decay of acceleration waves are currently under study and the conclusions will be reported later on.

4. Acceleration waves in simple fluids

The constitutive equation for an incompressible simple fluid is [1, § 32]:

$$T^E_{ij} = T_{ij} + p\delta_{ij} = \mathop{\mathscr{H}_{ij}}_{s=0}^{\infty}\big(G_{kl}(s)\big), \tag{4.1}$$

where it is assumed that

$$\mathop{\mathscr{H}_{ii}}_{s=0}^{\infty}(.) = 0. \tag{4.2}$$

Now, assuming a continuous body force field, the jump of the equation of motion yields

$$[-p_{,i}] + [T^E_{ij,j}] = \varrho[\ddot{x}_i], \tag{4.3}$$

where [cf. (2.19)]:

$$[T^E_{ij,j}] = -2a_m\,\delta \mathop{\mathscr{H}_{ijkl}}_{s=0}^{\infty}\Big(G_{pq}(s)|\,n_k n_j\big(\mathbf{C}_t(t-s)\big)_{ml}\Big), \tag{4.4}$$

and $\delta\underset{\sim}{\mathscr{H}}$ is a linear, continuous functional of $n_k n_j\big(\mathbf{C}_t(t-s)\big)_{ml}$. In incompressible materials, all waves are transverse and thus $a_i n_i = 0$. If one substitutes (2.3), (4.4) and the compatibility condition [12, § 175]

$$[-p_{,i}] = \lambda n_i \tag{4.5}$$

into (4.3), one obtains (cf. [4]):

$$\lambda = 2n_i a_m\,\delta \mathop{\mathscr{H}_{ijkl}}_{s=0}^{\infty}\Big(G_{pq}(s)|\,n_k n_j\big(\mathbf{C}_t(t-s)\big)_{ml}\Big). \tag{4.6}$$

Hence (4.3) yields

$$Q_{ij}(\mathbf{n})\,a_j = \varrho U^2 a_i, \tag{4.7}$$

where

$$Q_{ij}(\mathbf{n}) = 2\Big\{n_i n_p\,\delta \mathop{\mathscr{H}_{pmkl}}_{s=0}^{\infty}\Big(.\,|\,n_k n_m\big(\mathbf{C}_t(t-s)\big)_{jl}\Big) - \delta \mathop{\mathscr{H}_{imkl}}_{s=0}^{\infty}\Big(.\,|\,n_k n_m\big(\mathbf{C}_t(t-s)\big)_{jl}\Big)\Big\}. \tag{4.8}$$

If needed, one may write the linear functionals in (4.8) as integrals of course. At any rate, (4.7) and (4.8) yield the squared speed of propagation of acceleration waves into an incompressible fluid undergoing an arbitrary motion.

Let the base motion, before arrival of the wave, be a steady simple shearing flow, i.e., the strain history has the form

$$\|\mathbf{C}_t(t-s)\| = \begin{bmatrix} 1+\varkappa^2 s^2 & -\varkappa s & 0 \\ . & 1 & 0 \\ . & . & 1 \end{bmatrix}, \tag{4.9}$$

where we have assumed that $\dot{x} = \dot{z} = 0$, $\dot{y} = \varkappa x$, $\varkappa = \text{const}$, is the velocity field. Then, using the argument of § 3, we can see that the linear functionals $\delta\mathscr{H}_{ijkl}$ are identical to those functionals $\delta\mathscr{S}_{ijkl}$ occurring in the theory of nearly viscometric flows [8], i.e.,

$$(4.10)\qquad \delta \overset{\infty}{\underset{s=0}{\mathscr{H}}}{}_{ijkl}\Big(G_{pq}(s)|\,n_k n_j\big(\mathbf{C}_t(t-s)\big)_{ml}\Big) = \delta \overset{\infty}{\underset{s=0}{\mathscr{S}}}{}_{ijkl}\Big(\varkappa, s|\,n_k n_j\big(C_t(t-s)\big)_{ml}\Big),$$

where $\delta\underset{\sim}{\mathscr{S}}$ is a linear functional of $n_k n_j\big(\mathbf{C}_t(t-s)\big)_{ml}$.

As an example, consider the problem solved by COLEMAN and GURTIN [9] regarding the wave speed of a shear wave in an incompressible simple fluid undergoing steady simple shearing motion. Let us choose $\mathbf{n} = \mathbf{n}^1$, the unit vector along the x-axis, $\mathbf{a} = a\mathbf{n}^2$, where $\mathbf{n}^2$ is the unit vector along the y-axis. Then

$$(4.11)\qquad Q_{ij}(\mathbf{n}^1) = 2\int_0^\infty (S_{111l}n_i^1 - S_{i11l})\big(\mathbf{C}_t(t-s)\big)_{jl}\,ds,$$

where the linear functionals in (4.10) have been written as integrals. Putting $a_2 = an_2^2$, one gets

$$(4.12)\qquad Q_{22}(\mathbf{n}^1)a_2 = \varrho U^2 a = 2a\int_0^\infty (S_{111l}n_2^1 - S_{211l})\big(\mathbf{C}_t(t-s)\big)_{2l}\,ds.$$

or

$$(4.13)\qquad \varrho U^2 = 2\int_0^\infty \{S_{2111}(\varkappa, s)\varkappa s - S_{2112}(\varkappa, s)\}\,ds,$$

where one uses the fact that the y-th component of $\mathbf{n}^1$ is zero, i.e., $n_2^1 = 0$. The integral (4.13) is exactly what was called by COLEMAN and GURTIN [9] as $E(\varkappa_0)$.

According to the notation of COLEMAN and GURTIN [9], if $\tau(\varkappa) = \overset{\infty}{\underset{s=0}{\mathscr{T}}}(-\varkappa s)$ is the shear stress in steady simple shearing, then

$$(4.14)\qquad E(\varkappa) = -\delta \overset{\infty}{\underset{s=0}{\mathscr{T}}}(-\varkappa s|1),$$

where $\delta\overset{\infty}{\underset{s=0}{\mathscr{T}}}(-\varkappa s|g)$ linear functional of g and depends non-linearly on $\varkappa$. Also, trivially,

$$(4.15)\qquad \frac{d\tau(\varkappa)}{d\varkappa} = -\delta \overset{\infty}{\underset{s=0}{\mathscr{T}}}(-\varkappa s|s).$$

Hence, if one were to represent the latter by an integral:

$$(4.16)\qquad -\delta \overset{\infty}{\underset{s=0}{\mathscr{T}}}(-\varkappa s|s) = \int_0^\infty f(\varkappa, s)s\,ds,$$

where $f(\varkappa, s)$ is the kernel, then by linearity

$$(4.17)\qquad -\delta \overset{\infty}{\underset{s=0}{\mathscr{T}}}(-\varkappa s|1) = \int_0^\infty f(\varkappa, s)\,ds.$$

Now, we note from Eq. (7.6) of [8] that

$$\frac{d\tau(\varkappa)}{d\varkappa} = 2\int_0^\infty \{S_{2111}(\varkappa, s)\varkappa s - S_{2112}(\varkappa, s)\} s\, ds. \tag{4.18}$$

Hence, Eqs. (4.18) and (4.13) are consistent with Eqs. (4.16) and (4.17) above as they should be. A relation of the form (4.13) has also been derived by SADD [15] for incompressible BKZ fluids [16].

As a second example, consider the propagation of an acceleration wave into an incompressible fluid undergoing steady simple extension. The velocity field for this flow [17] is

$$\dot{x}_i = a_i x_i, \quad i = 1, 2, 3; \text{ no sum}, \tag{4.14}$$

$$a_1 + a_2 + a_3 = 0, \tag{4.15}$$

and the tensor $\mathbf{C}_t(t-s)$ is given by

$$\mathbf{C}_t(t-s) = \exp(-2s\mathbf{L}), \quad 0 \leqslant s < \infty, \tag{4.16}$$

where $\mathbf{L}$ is the velocity gradient. Let x_3 be the axis along which the fluid is being pulled. Then, taking $\{n_i\} = \{0, 0, 1\}$ and $\{a_i\} = \{a_1, 0, 0\}$, i.e., that the acceleration wave travels in the x_3-direction with a jump in the x_1-direction, one obtains [cf. (4.7)]:

$$Q_{11}(\mathbf{n}) a_1 = \varrho U^2 a_1. \tag{4.17}$$

Using the value of Q_{11}, one has that

$$\frac{1}{2}\varrho U^2 = \delta \mathop{\mathscr{H}}_{s=0}^{\infty}{}_{3331}(a_1, a_2, a_3, s|\exp(-2sa_1)) - \delta \mathop{\mathscr{H}}_{s=0}^{\infty}{}_{1331}(a_1, a_2, a_3, s|\exp(-2sa_1)). \tag{4.18}$$

In writing (4.17), the dependence of $\delta\mathscr{H}$ on $\mathbf{G}(s)$ through a_i and s is used. One may also write (4.17) as

$$\frac{1}{2}\varrho U^2 = \int_0^\infty \{\mathscr{H}_{3331}(a_i, s) - \mathscr{H}_{1331}(a_i, s)\} \exp(-2sa_1)\, ds. \tag{4.19}$$

The reader must note that the velocity U, which appears in (4.17) and (4.18), is the velocity of the wave relative to the material, so that the speed of displacement [12, § 177] is:

$$u = U + a_3 z, \tag{4.20}$$

but for the shear wave considered earlier, $u = U$. Though it is obvious, it may be important to emphasize that the linear functionals appearing in (4.18) are not those which occur in nearly viscometric flows.

References

1. C. TRUESDELL, W. NOLL, *The non-linear field theories of mechanics*, Handbuch der Physik, Vol. III/3, Ed. S. FLÜGGE, Springer-Verlag, New York 1965.
2. B. D. COLEMAN, M. E. GURTIN, *Waves in materials with memory* IV. *Thermodynamics and the velocity of general acceleration waves*, Arch. Rat. Mech. Anal., **19**, 317, 1965.

3. E. VARLEY, *Acceleration fronts in viscoelastic materials*, Arch. Rat. Mech. Anal., **19**, 215, 1965.
4. J. L. ERICKSEN, *On the propagation of waves in isotropic incompressible perfectly elastic materials*, J. Rat. Mech. Anal., **2**, 329, 1953.
5. C. TRUESDELL, *General and exact theory of waves in finite elastic strain*, Arch. Rat. Mech. Anal., **8**, 263, 1961.
6. A. C. PIPKIN, R. S. RIVLIN, *Small deformations superposed on large deformations in materials with fading memory*, Arch. Rat. Mech. Anal., **8**, 297, 1961.
7. A. C. PIPKIN, *Small finite deformations of viscoelastic solids*, Rev. Mod. Phys., **36**, 1034, 1964.
8. A. C. PIPKIN, D. R. OWEN, *Nearly viscometric flows*, Phys. Fluids, **10**, 836, 1967.
9. B. D. COLEMAN, M. E. GURTIN, *Stability against shear waves of steady flows of non-linear viscoelastic fluids*, J. Fluid Mech., **33**, 165, 1968.
10. M. A. HAYES, R. S. RIVLIN, *The propagation of sinusoidal small-amplitude waves in a deformed viscoelastic solid*, I, J. Acoust. Soc. Amer., **46**, 610, 1969.
11. M. HAYES, R. S. RIVLIN, *Propagation of a plane wave in an isotropic elastic material subjected to pure homogeneous deformation*, Arch. Rat. Mech. Anal., **8**, 15, 1961.
12. C. A. TRUESDELL, R. A. TOUPIN, *The classical field theories*, Handbuch der Physik, vol. III/1, Ed. S. FLÜGGE, Springer Verlag, New York 1960.
13. B. D. COLEMAN, M. E. GURTIN, *Waves in materials with memory*, II. *On the growth and decay of one-dimensional acceleration waves*, Arch. Rat. Mech. Anal., **19**, 239, 1965.
14. B. D. COLEMAN, M. E. GURTIN, I. HERRERA, R., *Waves in materials with memory*, I. *The velocity of one-dimensional shock and acceleration waves*, Arch. Rat. Mech. Anal., **19**, 1, 1965.
15. M. H. SADD, *Small deformations on large deformations in viscoelastic liquids*, Ph.D. Thesis, Ill. Inst. Tech., 1971.
16. B. BERNSTEIN, E. A. KEARSLEY, L. J. ZAPAS, *A study of stress relaxation with finite strain*, Trans. Soc. Rheol., **7**, 391, 1963.
17. B. D. COLEMAN, W. NOLL, *Steady extension if incompressible simple fluids*, Phys. Fluids, **5**, 840, 1962.

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Received April 25, 1972