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Rotationally symmetric deformations of Cosserat surfaces of revolution(*)

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WITHIN the scope of a linear isothermal theory for an elastic Cosserat surface, the constitutive equations for rotationally symmetric deformations of an elastic Cosserat surface of revolution are written out assuming dependence of all variables on the meridional coordinate only. The differential equations of equilibrium separate into two mutually independent systems describing the torsion and bending problems, respectively. The results are specialized for a cylindrical Cosserat surface and general solutions to the homogeneous differential equations are obtained for the two systems.

W warunkach izotermicznej liniowej teorii sprężystej powierzchni Cosseratów wypisano równania konstytutywne dla obrotowo symetrycznych deformacji obrotowej sprężystej powierzchni Cosseratów przy założeniu, że wszystkie zmienne zależą tylko od współrzędnej południkowej. Równania różniczkowe równowagi rozdzielono na dwa wzajemnie niezależne układy opisujące odpowiednio zagadnienia skręcania i zginania. Rezultaty wyspecyfikowano dla cylindrycznej powierzchni Cosseratów i ogólne rozwiązanie zostało otrzymane dla jednorodnych równań różniczkowych tych dwóch układów.

Выведены определяющие уравнения осесимметрических деформаций поверхностей вращения из упругого материала типа Коссера. Предполагается линейная изотермическая теория, а также зависимость всех переменных лишь от меридиональной координаты. Дифференциальные уравнения равновесия распадаются на две взаимно независимые системы, описывающие соответственно кручение и изгиб. Для цилиндрической поверхности приведены подробности решения. Выведены общие интегралы однородных дифференциальных уравнений указанных систем.

Notations

a radius of cylinder,

$\bar{a}_1, \bar{a}_2, \bar{a}_3$ base vectors,

$a_{\alpha\beta}$ components of first fundamental tensor,

$b_{\alpha\beta}$ components of second fundamental tensor,

k^2 constant (4.13),

m^α $M^{\alpha\beta} - b^\alpha_\lambda M^{\lambda 3}$,

r distance along the radial line,

z distance along the z-axis,

u^θ, u^ξ, u^n components of the displacement vector,

F^θ, F^ξ, F^n body force components,

L^θ, L^ξ, L^n director body force components,

$M^{\alpha i}$ director stress resultants,

$N^{\alpha i}$ stress resultants,

R $\dfrac{r'z'' - z'r''}{\alpha^3}$,

α $\sqrt{r'^2 + z'^2}$,

$\varepsilon_{\alpha\beta}$ components of strain tensor,

$\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_9$ constitutive constants,

$\varkappa_{\alpha\beta}$ $(\delta_\alpha|_\beta - b_{\alpha\beta}\delta_3) - b_{\beta\tau}(u^\tau|_\alpha - b^\tau_\alpha u_3)$,

$\Gamma^\lambda_{\alpha\beta}$ Christoffel symbols,

λ^2 constant (4.9).

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1. Introduction

COSSERAT surface theory treats a deformable surface as a two-dimensional generalized continuum embedded in a Euclidean 3-space taking into account kinematic ingredients other than the ordinary displacement vector. The work on oriented media was orginated by DUHEM [3] and was systematically developed by E. and F. COSSERAT [1] in two and three dimensions. Cosserat's work was motivated by the theory of rods and shells.

Cosserat surface theory, as an exact theory of deformable surfaces and with applications to shells, was given in [7]. In the light of generalized continua, a fairly detailed development of linearized theory of an elastic Cosserat surface, the material of which is homogeneous and possesses a centre of symmetry, is given in [6]. Other recent contributions include [2, 4, 5].

This paper is concerned with rotationally symmetric deformations of Cosserat surfaces of revolution. It treats mainly two aspects: (i) the specialization of the pertinent equations of equilibrium and expressions for kinematic variables, assuming dependence of all quantities on the meridional coordinate only, (ii) particularization of these results for a cylindrical Cosserat surface.

The general solutions of the homogeneous differential equations, which separate into two mutually independent systems of equations for the torsion and bending problems of such cylindrical Cosserat surface, are obtained.

Convected coordinates are used throughout and the partial and covariant derivatives with respect to the metric of the undeformed surface are denoted by a comma and a vertical stroke preceding an index, respectively. Parentheses and square brackets around indices. indicate unique symmetric and skew symmetric components of a tensor, respectively Vectors are denoted by a bar above the symbol and derivatives with respect to the meridional coordinate by a prime above the variable. Symbols are defined where they first occur in the text and are listed, for convenience, under Notations.

2. Notation and basic equations

For the linear, isothermal theory of an elastic Cosserat surface, assume the initial directors $\bar{d}$ to coincide with the unit normals $\bar{n}$ to the surface. Using the notation of [7],

$$\bar{d} = \bar{a}_3 = \bar{n}, \quad d_\alpha = 0, \quad d_3 = 1. \tag{2.1}$$

The kinematic variables are given by

$$\begin{gathered} \varepsilon_{\alpha\gamma} = \frac{1}{2}(u_{\alpha|\gamma} + u_{\gamma|\alpha}) - b_{\alpha\gamma} u_3, \qquad \varkappa_{\gamma\alpha} = (\delta_{\gamma|\alpha} - b_{\alpha\gamma}\delta_3) - b_{\alpha\nu}(u^{\nu}_{|\gamma} - b^{\nu}_{\gamma} u_3), \\ \varkappa_{3\alpha} = (\delta_{3,\alpha} + b^{\tau}_{\alpha}\delta_\tau) - b_{\tau\alpha}\varphi^{\tau}, \end{gathered} \tag{2.2}$$

where $b_{\alpha\lambda}$ are the components of the second fundamental tensor and φ_α denote the surface components of the rotation given by

$$\varphi_\alpha = -(u_{3,\alpha} + b^{\tau}_{\alpha} u_\tau). \tag{2.3}$$

The kinematic variables defined by $(2.2)_{2,3}$ may be written as

$$\varkappa_{\alpha\gamma} = \varrho_{\alpha\gamma} - b_{\alpha\gamma}\hat{\delta}_3, \qquad \varkappa_{3\alpha} = \varrho_{3\alpha} + b^{\tau}_{\alpha}\hat{\delta}_\tau, \tag{2.4}$$

where

$$\varrho_{3\alpha} = \hat{\delta}_{3,\alpha}, \qquad \varrho_{\alpha\tau} = \delta_{\alpha|\tau} - [u_{3|\alpha\tau} + b^{\beta}_{\alpha|\tau} u_\beta + b^{\beta}_{\tau} u_{\beta|\alpha} + b^{\beta}_{\alpha} u_{\beta|\tau} - b_{\alpha\nu} b^{\nu}_{\tau} u_3] \tag{2.5}$$

and

$$\delta_\alpha = d_\alpha = \delta_\alpha - \varphi_\alpha, \quad \hat{\delta}_3 = \delta_3 = d_3 - 1. \tag{2.6}$$

The equations of equilibrium are given in the form

$$\begin{gathered} N^{\alpha\beta}{}_{|\alpha} - b^\beta_\alpha N^{\alpha 3} + \varrho_0 F^\beta = 0, \quad N^{\alpha 3}{}_{|\alpha} + b_{\alpha\beta} N^{\alpha\beta} + \varrho_0 F^3 = 0, \\ M^{\alpha\beta}{}_{|\alpha} - b^\beta_\alpha M^{\alpha 3} - m^\beta = -\varrho_0 L^\beta, \quad M^{\alpha 3}{}_{|\alpha} + b_{\alpha\beta} M^{\alpha\beta} - m^3 = -\varrho_0 L^3, \end{gathered} \tag{2.7}$$

where $N^{\alpha i}$, $M^{\alpha i}$ and m^α associated with the force vector and director force vector are restricted by

$$\epsilon_{\beta\alpha}[N^{\alpha\beta} - M^{\tau\beta} b^\alpha_\tau] = 0, \quad N^{\alpha 3} - M^{\tau 3} b^\alpha_\tau = m^\alpha. \tag{2.8}$$

The constitutive equations for an isothermal isotropic Cosserat surface are given by

$$\begin{gathered} N'^{\alpha\beta} = (\alpha_1 a^{\alpha\beta} a^{\tau\delta} + 2\alpha_2 a^{\alpha\tau} a_{\beta\delta}) \varepsilon_{\tau\delta} + \alpha_9 a^{\alpha\beta} \hat{\delta}_3, \\ M^{(\alpha\beta)} = (\alpha_5 a^{\alpha\beta} a^{\tau\delta} + (\alpha_6 + \alpha_7) a^{\alpha\tau} a^{\beta\delta}) \varrho_{(\tau\delta)}, \\ M^{[\alpha\beta]} = (\alpha_6 - \alpha_7) a^{\alpha\delta} a^{\beta\tau} \varrho_{[\tau\delta]}, \quad M^{\alpha 3} = \alpha_8 a^{\alpha\tau} \varrho_{3\tau}, \quad N^{\alpha 3} = V^\alpha = \alpha_3 a^{\alpha\tau} \hat{\delta}_\tau, \\ V^3 = m^3 - b_{\alpha\tau} M^{\alpha\tau}, \end{gathered} \tag{2.9}$$

where

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} + M^{\tau\alpha} b^\beta_\tau. \tag{2.10}$$

3. Cosserat surfaces of revolution

The position vector of a point P on the meridian curve C of a surface of revolution is

$$\bar{r} = r(\xi)\bar{e}_r + z(\xi)\bar{k}, \tag{3.1}$$

where

$$\bar{e}_r = \bar{i}\cos\theta + \bar{j}\sin\theta \tag{3.2}$$

and where θ is the angle which the plane of the meridian curve C makes with the x-axis, ξ is the distance measured along the curve C, and $\bar{i}$, $\bar{j}$ and $\bar{k}$ are unit base vectors in the Cartesian coordinate system x .

From (3.1), we have

$$\begin{gathered} \bar{a}_1 = \frac{\partial \bar{r}}{\partial \theta} = r(-\bar{i}\sin\theta + \bar{j}\cos\theta), \quad \bar{a}_2 = \frac{\partial \bar{r}}{\partial \xi} = r'\bar{e}_r + z'\bar{k}, \\ \bar{a}_3 = \frac{\bar{a}_1 \times \bar{a}_2}{|\bar{a}_1 \times \bar{a}_2|} = -\bar{k}\frac{r'}{\alpha} + \bar{e}_r \frac{z'}{\alpha}, \end{gathered} \tag{3.3}$$

where a prime above a symbol denotes differentiation with respect to ξ and $\alpha = \sqrt{r'^2 + z'^2}$.

The components of the fundamental forms of the surface are given by

$$a_{11} = r^2, \quad a_{12} = 0, \quad a_{22} = \alpha^2, \tag{3.4}$$

$$b_{11} = -\frac{rz'}{\alpha}, \quad b_{12} = 0, \quad b_{22} = -\alpha^2 R, \tag{3.5}$$

with

$$R = \frac{r'z'' - z'r''}{\alpha^3}. \tag{3.6}$$

Also, we have

$$\Gamma^1_{12} = \frac{r'}{r}, \quad \Gamma^2_{11} = -\frac{rr'}{\alpha^2}, \quad \Gamma^2_{22} = \frac{\alpha'}{\alpha}, \tag{3.7}$$

with all other Christoffel symbols vanishing. It will be assumed that the surface of revolution is under axisymmetric loading and thus all quantities are functions of ξ only, with partial derivatives with respect to θ vanishing.

The equations of equilibrium, Eqs. (2.8), reduce to

$$\begin{gathered}
\frac{d}{d\xi}N^{\xi\theta} + \left(\frac{2r'}{r} + \frac{\alpha'}{\alpha}\right)N^{\xi\theta} + \frac{r'}{r}N^{\theta\xi} + \frac{z'}{\alpha r}N^{\theta n} + \varrho_0 F^\theta = 0,\\
\frac{d}{d\xi}N^{\xi\xi} + \left(\frac{r'}{r} + \frac{2\alpha'}{\alpha}\right)N^{\xi\xi} - \frac{rr'}{\alpha^2}N^{\theta\theta} + RN^{\xi n} + \varrho_0 F^\xi = 0,\\
\frac{d}{d\xi}N^{\xi n} + \left(\frac{r'}{r} + \frac{\alpha'}{\alpha}\right)N^{\xi n} - \frac{rz'}{\alpha}N^{\theta\theta} - \alpha^2 RN^{\xi\xi} + \varrho_0 F^n = 0,\\
\frac{d}{d\xi}M^{\xi\theta} + \left(\frac{2r'}{r} + \frac{\alpha'}{\alpha}\right)M^{\xi\theta} + \frac{r'}{r}M^{\theta\xi} + \frac{z'}{\alpha r}M^{\theta n} - m^\theta = -\varrho L^\theta,\\
\frac{d}{d\xi}M^{\xi\xi} + \left(\frac{r'}{r} + \frac{2\alpha'}{\alpha^2}\right)M^{\xi\xi} - \frac{rr'}{\alpha^2}M^{\theta\theta} + RM^{\xi n} - m^\xi = -\varrho L^\xi,\\
\frac{d}{d\xi}M^{\xi n} + \left(\frac{r'}{r} + \frac{\alpha'}{\alpha}\right)M^{\xi n} - \frac{rz'}{\alpha}M^{\theta\theta} - \alpha^2 RM^{\xi\xi} - m^n = -\varrho L_n.
\end{gathered} \tag{3.8}$$

From Eqs. (2.2)–(2.5) and using Eqs. (3.5), the kinematic variables $\varepsilon_{\alpha\tau}$ and $\varrho_{i\alpha}$ are given by

$$\begin{gathered}
\varepsilon_{\theta\theta} = rr'u^\xi + \frac{rz'}{\alpha}u_n, \quad \varepsilon_{\xi\theta} = \varepsilon_{\theta\xi} = \frac{r^2}{2}\frac{du^\theta}{d\xi}, \quad \varepsilon_{\xi\xi} = \alpha^2\frac{du^\xi}{d\xi} + \alpha\alpha' u^\xi + \frac{r'z'' - z'r''}{\alpha}u_3,\\
\varrho_{\theta\theta} = rr'\delta^\xi + \frac{r'z'}{\alpha}u^\xi + \left(\frac{z'}{\alpha}\right)^2 u_3, \quad \varrho_{\theta\xi} = r^2\frac{d\delta^\theta}{d\xi} + rr'\delta^\theta - rr'Ru^\theta,\\
\varrho_{\xi\theta} = -rr'\delta^\theta + \frac{rz'}{\alpha}\frac{du^\theta}{d\xi} + \frac{r'z'}{\alpha}u^\theta,\\
\varrho_{\xi\xi} = \alpha^2\frac{d\delta^\xi}{d\xi} + \alpha\alpha'\delta^\xi + \alpha^2 R\left(\frac{du^\xi}{d\xi} + \frac{\alpha'}{\alpha}u^\xi + Ru_3\right), \quad \varrho_{n\theta} = 0, \quad \varrho_{n\xi} = \frac{d\delta_3}{d\xi}.
\end{gathered} \tag{3.9}$$

Using Eqs. (3.4)–(3.6) and Eq. $(2.9)_1$ in Eqs. (2.10), one obtains the expressions for the stresses as

$$\begin{aligned}
N^{\theta\theta} = \Bigg[&\frac{\alpha_1\alpha r + \alpha_5 Rz'}{\alpha r^3}\frac{d}{d\xi}\\
&+ \frac{(\alpha_1 + 2\alpha_2)\alpha^2 r^2 r' + \alpha_1\alpha\alpha' r^3 + (\alpha_5 + \alpha_6 + \alpha_7)r'(z')^2 + \alpha_5\alpha' z' r^2 R}{\alpha^2 r^5}\Bigg]u^\xi\\
&+ \frac{z'}{\alpha r}\left[\frac{\alpha_5}{r^2}\frac{d}{d\xi} + \frac{(\alpha_5 + \alpha_6 + \alpha_7)\alpha r' + \alpha_5\alpha' r}{\alpha r^3}\right]\delta^\xi + \frac{\alpha_9}{r^2}\delta_n,
\end{aligned}$$

$$+\frac{(\alpha_1+2\alpha_2)\alpha^2r^2z'+\alpha_1\alpha^3r^3R+(\alpha_5+\alpha_6+\alpha_7)(z')^3+\alpha_5\alpha^2r^2z'}{\alpha^3r^5}u_n,$$

$$N^{\xi\theta}=\frac{z'}{\alpha^3r^3}\left[\alpha_7r^2\frac{d}{d\xi}-(\alpha_6-\alpha_7)rr'\right]\delta^\theta+\frac{1}{\alpha^4r^3}\left[\left(\alpha_6r(z')^2+\alpha_2\alpha^2r^3\right)\frac{d}{d\xi}\right.$$

$$\left.+\alpha_6r'(z')^2-\alpha_7rRr'z'\right]u^\theta,$$

$$N^{\theta\xi}=\left[\frac{\alpha_2\alpha r+\alpha_7Rz'}{\alpha^3r}\frac{d}{d\xi}+R\frac{\alpha_7r'z'-\alpha_6\alpha rr'R}{\alpha^3r^2}\right]u^\theta$$

$$+\left[\frac{\alpha_6R}{\alpha^2}\frac{d}{d\xi}+\frac{(\alpha_6-\alpha_7)r'R}{\alpha^2r}\right]\delta^\theta,$$

$$N^{\xi\xi}=\left[\frac{\alpha_1+2\alpha_2+(\alpha_5+\alpha_6+\alpha_7)R^2}{\alpha^2}\frac{d}{d\xi}\right.$$

$$\left.+\frac{\alpha_1\alpha rr'+(\alpha_1+2\alpha_2)\alpha'r^2+\alpha_5r'z'R+(\alpha_5+\alpha_6+\alpha_7)\alpha'r^2R^2}{\alpha^3r^2}\right]u^\xi$$

$$+R\left[\frac{\alpha_5+\alpha_6+\alpha_7}{\alpha^2}\frac{d}{d\xi}+\frac{\alpha_5\alpha r'+(\alpha_5+\alpha_6+\alpha_7)\alpha'r}{\alpha^3r}\right]\delta^\xi$$

$$+\frac{\alpha_1rz'+(\alpha_1+2\alpha_2)\alpha r^2R+\alpha_5\alpha(z')^2R+(\alpha_5+\alpha_6+\alpha_7)\alpha^3r^2R^3}{\alpha^3r^2}u_n+\frac{\alpha_9}{\alpha^2}\delta_n,$$

$$N^{\theta n}=\alpha_3\left[\delta^\theta-\frac{z'}{\alpha r}u^\theta\right],\qquad N^{\xi n}=\alpha_3\left[\delta^\xi-Ru^\xi+\frac{1}{\alpha^2}\frac{du_n}{d\xi}\right],\tag{3.10}$$

$$V^3=\alpha_4\delta_3+\alpha_9\left[\frac{du^\xi}{d\xi}+\left(\frac{\alpha'}{\alpha}+\frac{r'}{r}\right)u^\xi+\left(\frac{z'}{\alpha r}+R\right)u_n\right],$$

$$M^{\theta\theta}=\left[\frac{\alpha_5}{r^2}\frac{d}{d\xi}+\frac{(\alpha_5+\alpha_6+\alpha_7)\alpha r'+\alpha_5\alpha'r}{\alpha r^3}\right]\delta^\xi+\left[\frac{\alpha_5R}{r^2}\frac{d}{d\xi}\right.$$

$$\left.+\frac{(\alpha_5+\alpha_6+\alpha_7)r'z'+\alpha_5\alpha'r^2R}{\alpha r^4}\right]u^\xi+\frac{(\alpha_5+\alpha_6+\alpha_7)(z')^2+\alpha_5\alpha^2r^2R^2}{\alpha^2r^4}u_n,$$

$$M^{\theta\xi}-M^{\xi\theta}=\frac{\alpha_6-\alpha_7}{\alpha^2r^2}\left[\left(\frac{rz'}{\alpha}\frac{d}{d\xi}+\frac{r'z'+\alpha rr'R}{\alpha}\right)u^\theta-2rr'\delta^\theta-r^2\frac{d\delta^\theta}{d\xi}\right],$$

$$M^{\theta\xi}+M^{\xi\theta}=\frac{\alpha_6+\alpha_7}{\alpha^2r^2}\left[\left(\frac{rz'}{\alpha}\frac{d}{d\xi}+\frac{r'z'-\alpha rr'R}{\alpha}\right)u^\theta+r^2\frac{d\delta^\theta}{d\xi}\right],$$

$$M^{\xi\xi}=\left[\frac{\alpha_5+\alpha_6+\alpha_7}{\alpha^2}R\frac{d}{d\xi}+\frac{\alpha_5r'z'+(\alpha_5+\alpha_6+\alpha_7)\alpha'r^2R}{\alpha^3r^2}\right]u^\xi$$

$$+\left[\frac{\alpha_5(z')^2+(\alpha_5+\alpha_6+\alpha_7)\alpha^2r^2R^2}{\alpha^2r^2}\right]u_n+\left[\frac{\alpha_5+\alpha_6+\alpha_7}{\alpha^2}\frac{d}{d\xi}+\frac{\alpha_5\alpha r'+(\alpha_5+\alpha_6+\alpha_7)\alpha'r}{\alpha^3r}\right]\delta^\xi,$$

$$m^\theta=\alpha_3\left[\delta^\theta-\frac{z'}{\alpha r}u^\theta\right],\qquad m^\xi=\alpha_3\left[\delta^\xi-Ru^\xi+\frac{1}{\alpha^2}\frac{du_\xi}{d\xi}\right]+\frac{\alpha_8R}{\alpha^2}\frac{d\delta_n}{d\xi},$$

$$m^3=V^3-\frac{rz'}{\alpha}M^{\theta\theta}-R\alpha^2M^{\xi\xi}.$$

From Eqs. (3.8) and (3.10), it is observed that the equations of equilibrium separate into two mutually independent systems of differential equations for the displacements and director displacements. Equations $(3.8)_{1,4}$ involve only u^θ and δ and are applicable to torsion problems. Equations $(3.8)_{2,3,5,6}$ involve u^ξ, δ^ξ, u_n, δ_n and are thus associated with the bending problem of surfaces of revolution.

4. A Cosserat cylindrical surface

The meridians are a system of lines parallel to the z-axis. The position vector of a point on the surface is given as

$$\bar{r} = (a\cos\theta, a\sin\theta, z), \tag{4.1}$$

from which

$$\bar{a}_1 = \frac{\partial\bar{r}}{\partial\theta} = (-a\sin\theta, a\cos\theta, 0), \qquad \bar{a}_2 = \frac{\partial\bar{r}}{\partial z} = (0, 0, 1), \tag{4.2}$$

$$\begin{aligned} a_{11} = a^2, \quad a_{12} = 0 = a_{21}, \quad a_{22} = 1, \quad \alpha = 1,\\ b_{11} = -a, \quad b_{12} = 0 = b_{21}, \quad b_{22} = 0. \end{aligned} \tag{4.3}$$

The equations for the kinematic variables, Eqs. (3.9), reduce to

$$\begin{gathered} \varepsilon_{\theta\theta} = au_n, \quad \varepsilon_{\theta z} = \varepsilon_{z\theta} = \frac{a^2}{2}\frac{du^\theta}{dz}, \quad \varepsilon_{zz} = \frac{du^z}{dz},\\ \varrho_{\theta\theta} = u_n, \quad \varrho_{zz} = \frac{d\delta^z}{dz}, \quad \varrho_{\theta z} = a^2\frac{d\delta^\theta}{dz}, \quad \varrho_{z\theta} = a\frac{du^\theta}{dz}, \quad \varrho_{n\theta} = 0, \quad \varrho_{nz} = \frac{d\delta_n}{dz}. \end{gathered} \tag{4.4}$$

The expressions for the stress and director stress resultants, Eqs. (3.10), reduce to

$$N^{\theta\theta} = \frac{\alpha_1}{a^2}\frac{du^z}{dz} + \frac{\alpha_5}{a^3}\frac{d\delta^z}{dz} + \left(\frac{\alpha_1+2\alpha_2}{a^3} + \frac{\alpha_5+\alpha_6+\alpha_7}{a^5}\right)u_n + \frac{\alpha_9}{a^2}\delta_n,$$

$$N^{z\theta} = \left(\alpha_2 + \frac{\alpha_6}{a^2}\right)\frac{du^\theta}{dz} + \frac{\alpha_7}{a}\frac{d\delta^\theta}{dz}, \qquad N^{\theta z} = \alpha_2\frac{du^\theta}{dz},$$

$$N^{zz} = (\alpha_1+2\alpha_2)\frac{du^z}{dz} + \frac{\alpha_1}{a}u_n + \alpha_9\delta_n, \qquad N^{\theta n} = \alpha_3\left(-\frac{u^\theta}{a} + \delta^\theta\right), \qquad N^{zn} = \alpha_3\left(\frac{du_n}{dz} + \delta^z\right),$$

$$V^3 = \alpha_9\left(\frac{du^z}{dz} + \frac{u_n}{a}\right) + \alpha_4\delta_n, \qquad M^{\theta\theta} = \frac{\alpha_5+\alpha_6+\alpha_7}{a^4}u_n + \frac{\alpha_5}{a^2}\frac{d}{dz}\delta^z, \tag{4.5}$$

$$M^{\theta z} - M^{z\theta} = \frac{\alpha_6-\alpha_7}{a}\frac{d}{dz}(u^\theta - a\delta^\theta), \qquad M^{\theta z} + M^{z\theta} = \frac{\alpha_6+\alpha_7}{a}\frac{d}{dz}(u^\theta - a\delta^\theta),$$

$$M^{zz} = \frac{\alpha_5}{a^2}u_n + (\alpha_5+\alpha_6+\alpha_7)\frac{d}{dz}\delta^z, \qquad m^\theta = \alpha_3\left(-\frac{u^\theta}{a} + \delta^\theta\right), \qquad m^z = \alpha_3\left(\frac{du_n}{dz} + \delta^z\right),$$

$$m^n = \alpha_9\frac{du^z}{dz} + \left(\frac{\alpha_9}{a} - \frac{\alpha_5+\alpha_6+\alpha_7}{a^3}\right)u_n - \frac{\alpha_5}{a}\frac{d\delta^z}{dz} + \alpha_4\delta_n,$$

whereas the equations of equilibrium, Eqs. (3.8), simplify to

$$\frac{d}{dz}N^{z\theta}+\frac{N^{\theta n}}{a}+\varrho F^{\theta}=0,\quad \frac{d}{dz}N^{zz}+\varrho F^{z}=0,\quad \frac{d}{dz}N^{zn}-aN^{\theta\theta}+\varrho F^{n}=0, \tag{4.6}$$

$$\frac{d}{dz}M^{z\theta}+\frac{M^{\theta n}}{a}-m^{\theta}=-\varrho L^{\theta},\quad \frac{d}{dz}M^{zz}-m^{z}=-\varrho L^{z},\quad \frac{d}{dz}M^{zn}-aM^{\theta\theta}-m^{n}=-\varrho L^{n}.$$

Substituting Eqs. (4.5) into (4.6), one obtains two mutually independent systems of differential equations with constant coefficients. The two systems of differential equations are solved in the following manner:

1. The system describing the torsion problem, Eqs. $(4.6)_{1,4}$, reduce to

$$\begin{gathered}\left[\left(a^2\alpha_2+\alpha_6\right)\frac{d^2}{dz^2}-\alpha_3\right]\left(\frac{u^{\theta}}{a}\right)+\left(\alpha_7\frac{d^2}{dz^2}+\alpha_3\right)\delta^{\theta}+\varrho F^{\theta}a=0,\\ \left(\alpha_7\frac{d^2}{dz^2}+\alpha_3\right)\left(\frac{u^{\theta}}{a}\right)-\left(\alpha_3-\alpha_6\frac{d^2}{dz^2}\right)\delta^{\theta}+\varrho L^{\theta}=0.\end{gathered} \tag{4.7}$$

Setting F^{θ} and L^{θ} equal to zero and eliminating one of the two variables, results in

$$\frac{d^2}{dz^2}\left(\frac{d^2}{dz^2}-\lambda^2\right)\left(\frac{u^{\theta}}{a},\ \delta^{\theta}\right)=0, \tag{4.8}$$

where

$$\lambda^2=\frac{\alpha_3(2\alpha_7+2\alpha_6+a^2\alpha_2)}{a^2\alpha_2\alpha_6+\alpha_6^2-\alpha_7^2}. \tag{4.9}$$

Thus the general solutions of the homogeneous system, corresponding to Eqs. (4.7), are given by

$$\left(\frac{u^{\theta}}{a},\ \delta^{\theta}\right)=Az+B+Ce^{\lambda z}+De^{-\lambda z}, \tag{4.10}$$

where A, B, C, and D are arbitrary constants to be determined from the boundary conditions.

2. The system describing the bending problem, Eqs. $(4.6)_{2,3,5,6}$, reduce to

$$\begin{gathered}(\alpha_1+2\alpha_2)\frac{d^2}{dz^2}u^{z}+\frac{\alpha_1}{a}\frac{d}{dz}u_n+\alpha_9\frac{d}{dz}\delta_n+\varrho F^{z}=0,\\ \left[-\frac{\alpha_1}{a}\frac{d}{dz}u^{z}+\left(\alpha_3\frac{d^2}{dz^2}-\frac{\alpha_1+2\alpha_2}{a^2}-\frac{\alpha_5+\alpha_6+\alpha_7}{a^4}\right)u_n\right.\\ \left.+\left(\alpha_3-\frac{\alpha_5}{a^2}\right)\frac{d\delta^{z}}{dz}-\frac{\alpha_9}{a}\delta_n+\varrho F_n=0,\right.\\ \left(-\alpha_3+\frac{\alpha_5}{a^2}\right)\frac{du_n}{dz}+\left[(\alpha_5+\alpha_6+\alpha_7)\frac{d^2}{dz^2}-\alpha_3\right]\delta^{z}+\varrho L^{z}=0,\\ -\alpha_9\frac{d}{dz}u^{z}-\frac{\alpha_9}{a}u_n+\left(\alpha_8\frac{d^2}{dz^2}-\alpha_4\right)\delta_n+\varrho L^{n}=0.\end{gathered} \tag{4.11}$$

The displacement vector $\bar{u}=(0,u^{z},u_n)$ and the director displacement vector $\bar{\delta}=(0,\delta^{z},\delta^{n})$ of the homogeneous system corresponding to Eqs. (4.11), satisfy

$$\begin{aligned}\frac{d^2}{dz^2}\Bigg\{\left[(\alpha_5+\alpha_6+\alpha_7)\frac{d^2}{dz^2}-\alpha_3\right]&\left[\alpha_3\alpha_4(\alpha_1+2\alpha_2)\frac{d^2}{dz^2}+k^2\right]\\ &-\alpha_4(\alpha_1+2\alpha_2)\left(\alpha_3-\frac{\alpha_5}{a^2}\right)^2\frac{d^2}{dz^2}\Bigg\}(u^{z},u_n,\delta^{z},\delta_n)=0,\end{aligned} \tag{4.12}$$

where

$$k^2 = \frac{2\alpha_2\alpha_9^2}{a^2} - \frac{4\alpha_2\alpha_4(\alpha_1+\alpha_2)}{a^2} - \frac{\alpha_4(\alpha_1+2\alpha_2)(\alpha_5+\alpha_6+\alpha_7)}{a^4}. \tag{4.13}$$

The general solution of Eq. (4.12) is given by

$$(u^z, u_n, \delta^z, \delta_n) = Az+B+Ce^{\lambda_1 z}+De^{-\lambda_1 z}+Ee^{\lambda_2 z}+Fe^{-\lambda_2 z}, \tag{4.14}$$

where A, B, C, D, E and F are arbitrary constants and λ_1^2, λ_2^2 are roots of

$$\alpha_3\alpha_4(\alpha_1+2\alpha_2)(\alpha_5+\alpha_6+\alpha_7)\lambda^2+[k^2(\alpha_5+\alpha_6+\alpha_7)-\alpha_3^2\alpha_4(\alpha_1+2\alpha_2)]\lambda-\alpha_3 k^2 = 0. \tag{4.15}$$

The general solutions are given by (4.10) and (4.14). Specific boundary value problems can be solved for the case, where the force $\overline{F}$ and director force $\overline{L}$ are specified. The constants can, then, be determined from the boundary conditions. It is also necessary to know the value of the constants specified by Eq. (4.13).

5. Conclusions

Assuming all variables to depend on the meridional coordinate only, the constitutive equations, kinematic relations and equations of equilibrium for a linear isothermal theory of a Cosserat surface of revolution are presented. These equations are specialized for a cylindrical Cosserat surface; the equilibrium equations, which separate into mutually independent systems of differential equations for the torsion and bending problems, respectively, are treated and general solutions to the homogeneous parts of these differential equations are obtained in terms of elementary functions.

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