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Discrete elastic Cosserat media

CZ. WOŹNIAK (WARSZAWA)

IN THE LITERATURE, the theory of elasticity problems of Cosserat media has been investigated in numerous papers. All those problems have been considered on the basis of the theory of continuous media. In the present text, the basic concepts are given of the linear theory of discrete Cosserat media. Such media are defined as countable sets of rigid bodies connected by a specific system of interactions. The theory of the discrete elastic Cosserat media is a part of a discrete elasticity [12] — and is based on the theory of discretized bodies [13].

Literatura z zakresu teorii sprężystości zawiera wiele prac dotyczących ośrodków Cosseratów, rozpatrywanych wyłącznie jako ośrodki ciągłe. Niniejsza praca zawiera podstawowe pojęcia liniowej teorii dyskretnych ośrodków Cosseratów, definiowanych jako przeliczalne zbiory ciał sztywnych, powiązanych pewnymi szczególnymi schematami wzajemnych oddziaływań. Teoria dyskretnych ośrodków Cosseratów jest częścią dyskretnej teorii sprężystości [12] i jest oparta na teorii ciał dyskretyzowanych [13].

В литературе по теории упругости имеется много работ, в которых среды типа Коссера рассматриваются исключительно как сплошные. В данной работе содержатся основные понятия линейной теории дискретных сред типа Коссера, определяемых как счётные множества жестких тел, связанных некоторыми специальными схемами взаимодействий. Теория дискретных сред типа Коссера является частью дискретной теории упругости [12], и опирается на теорию дискретизированных сред [13].

Notations

The indices $\Lambda, \Phi, \ldots$ run over the sequence I, II, ..., m, indices $\lambda, \mu, \nu, \ldots$ take the values 1, 2, 3 and indices $K, L, M, \ldots$ run over the sequence $1, 2, \ldots, N$. We also use the indices k, l, m taking either the values 1, 2 (in Sec. 4) or the values 1, 2, 3. The summation convention holds for all kinds of indices. Partial derivatives with respect to variables x^K are denoted by a comma and the dot denotes differentiation with respect to time coordinate. The functions encountered in the text are assumed to be continuous together with their derivatives of the first and the second order.

1. Introduction

THIS PAPER is based on the concept of discretized bodies, introduced in [13]. The discretized body is a pair $(D, \mathscr{E})$, where D is a set of holonomic dynamical systems $d \in D$, which can interact only in subsets $E \in \mathscr{E}$, where $\mathscr{E}$ is a covering of D. An arbitrary discretized body is an approximate model of a continuous body, which has only a finite or countable number of degrees of freedom [13]. In the present paper, we assume that D is a set of free rigid bodies. Let us denote by $q^a(d, \tau)$, $a = 1, 2, \ldots, 6$, the six independent generalized coordinates of an arbitrary rigid body $d \in D$, where τ is the time coordinate. For each $d \in D$

and each τ, the numbers $q^a(d, \tau)$ are coordinates of a vector in a six-dimensional vector space [13].

The discretized body $(D, \mathscr{E})$, where each $d \in D$ is a free rigid body, is said to be the discrete elastic Cosserat medium, if for each $E \in \mathscr{E}$ there exists the elastic potential ε. Introducing in each E, $\overline{\overline{E}} = n+1$, the coordinate system $f: E \to (d, f_1 d, \ldots, f_m d)$ (cf. [13], Appendix), we can write:

$$\varepsilon = \varepsilon\big(d, q^a(d, \tau), \Delta_A q^a(d, \tau)\big), \quad a = 1,2, \ldots, 6. \tag{1.1}$$

It was also shown in [13] that the Lagrange equations of the second kind of an arbitrary dynamical system $d \in D$(1) have the form:

$$\overline{\Delta}_A T_a{}^A(d, \tau) + t_a(d, \tau) + f_a(d, \tau) = r_a(d, \ldots), \quad d \in D, \tag{1.2}$$

where

$$r_a \stackrel{\text{df}}{=} \frac{d}{d\tau} \frac{\partial T(d, \ldots)}{\partial \dot{q}^a_\cdot(d, \tau)} - \frac{\partial T(d, \ldots)}{\partial q^a(d, \tau)}, \quad d \in D,$$

and $T(d, \ldots) = T(d, q^a(d, \tau), \dot{q}^a(d, \tau))$ is the kinetic energy of the holonomic dynamical system $d \in D$. The symbol $fa(d, \tau)$ denotes the generalized external force acting at d, and $T_a{}^A(d, \tau)$, $t_a(d, \tau)$ are generalized internal forces in the subset $E_d \in \mathscr{E}$, $d \in D_*$ (cf. [13]). The generalized internal forces in the elastic discretized bodies are given by the formulas [13]

$$T_a{}^A(d, \tau) = \frac{\partial \varepsilon(d, \ldots)}{\partial_A q^a(d, \tau)}, \quad t_a(d, \tau) = -\frac{\partial \varepsilon(d, \ldots)}{\partial q^a(d, \tau)}, \quad d \in D. \tag{1.3}$$

If the particles $d \in D$ of the discretized elastic bodies are rigid, we have to deal with the elastic Cosserat discrete media; the index $x'a'$ in (1.1)–(1.3) take the values $1, 2, \ldots, 6$. In what follows, only linear theory of such media will be discussed.

2. Equations of motion and constitutive equations

Let us assume that at the time instant $\tau = \tau_0$ and for each $d \in D$, there is $\varepsilon(d, \ldots) = 0$ and $q^a(d, \tau) = 0$, $a = 1, 2, \ldots, 6$. Let throughout all the motion both the generalized coordinates $q\ (d, \tau)$ and the generalized velocities $\dot{q}^a(d, \tau)$ be sufficiently small in absolute value, so that the problem may be linearized. We can put(2)

$$q^a = \delta^a_k u^k + \delta^{a-3}_k v^k, \quad a = 1, 2, \ldots, 6, \tag{2.1}$$

where $u^k = u^k(d, \tau)$ are components of a displacement vector of a centre of mass of the rigid body $d \in D$, and $v^k = v^k(d, \tau)$ are components of an infinitesimal rotation vector of this body; all those components are related to the inertial Cartesian coordinate system z_k

(1) The subsets $E \in \mathscr{E}$ are said to be discrete elements and holonomic dynamical systems, $d \in D$ are called particles of the discretized body. If the discretized body is the discrete Cosserat medium, particles $d \in D$ can be interpreted as rigid bodies.

(2) In the sequel, the arguments d, τ of the functions considered will be omitted. The difference structure on $(D, \mathscr{E})$ is given [13].

in the physical space. The elastic potential and the kinetic energy will be written now in the form:

$$\varepsilon(d, \ldots) = \frac{1}{2} A_{kl}^{\Lambda\Phi} \gamma_\Lambda^k \gamma_\Phi^l + B_{kl}^{\Lambda\Phi} \gamma_\Lambda^k \varkappa_\Phi^l + \frac{1}{2} F_{kl}^{\Lambda\Phi} \varkappa_\Lambda^k \varkappa_\Phi^l,$$

$$T(d, \ldots) = \frac{1}{2} m\delta_{kl} \dot{u}^k \dot{u}^l + \frac{1}{2} i_{kl} \dot{v}^k \dot{v}^l, \quad d \in D,$$

where

$$\begin{aligned} \gamma_\Lambda^k &= \Delta_\Lambda u^k + \varepsilon^k_{.pr} v^r \Delta_\Lambda \psi^p, \\ \varkappa_\Lambda^k &= \Delta_\Lambda v^k, \quad d \in D_\Lambda, \end{aligned} \tag{2.2}$$

and $z^k = \psi^k(d)$ are coordinates of the place in the physical space occupied at the time instant $\tau = \tau_0$ by the centre of mass of the body $d \in D$. The quantities $A_{kl}^{\Lambda\Phi} = A_{kl}^{\Lambda\Phi}(d)$, $B_{kl}^{\Lambda\Phi} = B_{kl}^{\Lambda\Phi}(d)$, $F_{kl}^{\Lambda\Phi} = F_{kl}^{\Lambda\Phi}(d)$ represent the elastic properties and the quantities $m = m(d)$, $i_{kl} = i_{kl}(d)$ represent the inertial properties of the discrete elastic Cosserat system. It is convenient to assume that $\gamma_\Lambda^k(d, \tau) = \varkappa_\Lambda^k(d, \tau) = 0$ for $d \notin D_\Lambda$, and $A_{kl}^{\Lambda\Phi}(d) = B_{kl}^{\Lambda\Phi}(d) = F_{kl}^{\Lambda\Phi}(d) = 0$ for $d \notin D_\Lambda$ or $d \notin D_\Phi$. The form of elastic potential introduced above is invariant under an arbitrary rigid displacement and infinitesimal rotation of a whole discrete Cosserat system. Denoting $T_a{}^\Lambda = \delta_a{}^k T_k{}^\Lambda + \delta_{a-3}^k M_k{}^\Lambda$, we can express now the constitutive equations $(1.3)_1$ by means of the formulas

$$\begin{aligned} T_k{}^\Lambda &= A_{kl}^{\Lambda\Phi} \gamma_\Phi^l + B_{kl}^{\Lambda\Phi} \varkappa_\Phi^l, \\ M_k{}^\Lambda &= F_{kl}^{\Lambda\Phi} \varkappa_\Phi^l + B_{lk}^{\Phi\Lambda} \gamma_\Phi^l, \quad d \in D. \end{aligned} \tag{2.3}$$

The functions $T_k{}^\Lambda = T_k{}^\Lambda(d, \tau)$, $M_k{}^\Lambda = M_k{}^\Lambda(d, \tau)$ will be called the components of stress, and the functions $\gamma_\Lambda^k = \gamma_\Lambda^k(d, \tau)$, $\varkappa_\Lambda^k = \varkappa_\Lambda^k(d, \tau)$ are said to be the components of strain in the linear discrete Cosserat media. The constitutive equations $(1.3)_2$ in the linear theory reduce to the equalities

$$t_a = \delta_{a-3}^k \varepsilon_{kp}{}^{r}_{.} T_r{}^\Lambda \Delta_\Lambda \psi^p.$$

Using the above equalities and after denotations

$$f_a = \delta_a^k f_k + \delta_{a-3}^k n_k,$$

we obtain from (1.2) the following equations of motion:

$$\begin{aligned} \bar{\Delta}_\Lambda T_k{}^\Lambda + f_k &= m\ddot{u}_k, \\ \bar{\Delta}_\Lambda M_k{}^\Lambda + \varepsilon_{kp}{}^{r}_{.} T_r{}^\Lambda \Delta_\Lambda \psi^p + n_k &= i_{kl} \ddot{v}^l, \quad d \in D. \end{aligned} \tag{2.4}$$

The geometrical equations (2.2), the constitutive equations (2.3) and the equations of motion (2.4) form the basic set of equations of discrete elastic Cosserat media. After simple substitutions, we obtain from (2.2), (2.3), (2.4) the following system of ordinary differential equations for the six unknown functions $u^k(d, \tau)$, $v^k(d, \tau)$, $d \in D$:

$$\begin{aligned} &\bar{\Delta}_\Lambda [A_{kl}^{\Lambda\Phi} (\Delta_\Phi u^l + \varepsilon^l_{.pr} v^r \Delta_\Phi \psi^p) + B_{kl}^{\Lambda\Phi} \Delta_\Phi v^l] + f_k = m\ddot{u}_k, \\ &\bar{\Delta}_\Lambda [F_{kl}^{\Lambda\Phi} \Delta_\Phi v^l + B_{lk}^{\Phi\Lambda} (\Delta_\Phi u^l + \varepsilon^l_{.pr} v^r \Delta_\Phi \psi^p)] + \varepsilon_{kp}{}^{r}_{.} \Delta_\Lambda \psi^p [A_{rm}^{\Lambda\Phi} (\Delta_\Phi u^m + \varepsilon^m_{.st} v^t \Delta_\Phi \psi^s) \\ &\qquad + B_{rm}^{\Lambda\Phi} \Delta_\Phi v^m] + n_k = i_{kl} \ddot{v}^l, \quad d \in D. \end{aligned} \tag{2.5}$$

Denoting by $a^k(d, \tau)$, $b^k(d, \tau)$, $d \in \partial D$, the set of known functions, we put

$$u^k = a^k, \quad v^k = b^k, \quad d \in \partial D \subset D. \tag{2.6}$$

The Eqs. (2.6) are boundary conditions of the discrete Cosserat system. The basic equations (2.2)–(2.4) (or the Eqs. (2.5)) with the boundary conditions (2.6) and with the initial conditions given for each $d \in D$, make it possible to obtain the basic unknown functions $u^k = u^k(d, \tau)$, $v^k = v^k(d, \tau)$, $d \in D$. The uniqueness of the solution results from the fact that both quadratic forms $\varepsilon(d, \ldots)$, $T(d, \ldots)$ are positive definite.

If the difference structure on D is regular — i.e., the relations $f_\Lambda f_\Phi d = f_\Phi f_\Lambda d$ hold for each $d \in D_{\Lambda,\Phi} \cap D_{\Phi,\Lambda}$ [12], it will be possible to represent the basic equations in the form of finite difference equations (cf. Sec. 10). If the conditions

$$A_{kl}^{\Lambda\Phi} = A^{\Lambda\Phi}(d)\,\delta_{kl}, \quad B_{kl}^{\Lambda\Phi} = B^{\Lambda\Phi}(d)\,\delta_{kl}, \quad F_{kl}^{\Lambda\Phi} = F^{\Lambda\Phi}(d)\,\delta_{kl}, \quad d \in D, \tag{2.7}$$

hold, the discrete Cosserat system will be called isotropic.

3. Conditions of compatibility. Static-geometric analogy. Stress functions

Let us confine our considerations to the case $m > 1$. This is the case in which the stress components γ_Λ^k, $\varkappa_\Lambda^k$ are not independent. Let us also confine ourselves to the regular difference structures on D. The conditions $f_\Lambda f_\Phi d = f_\Phi f_\Lambda$, $d \in D_{\Lambda,\Phi} \cap D_{\Phi,\Lambda}$, yield [12]

$$\Delta_{[\Lambda}\Delta_{\Phi]}\varphi = 0, \tag{3.1}$$

for each pair Λ, Φ and for an arbitrary function $\varphi: D \to R$. By virtue of (3.1), from the geometric equations (2.2), we obtain

$$\begin{gathered} \Delta_{[\Phi}\gamma_{\Lambda]}^k + \varepsilon_{\cdot pr}^k \varkappa_{[\Lambda}^r \Delta_{\Phi]}\psi^p = \varepsilon_{\cdot pr}^k \varkappa_{[\Phi}^r \Delta_{[\Phi}\Delta_{\Lambda]}\psi^p, \\ \Delta_{[\Phi}\varkappa_{\Lambda]}^{k]} = 0, \quad d \in D_{\Lambda,\Phi} \cap D_{\Phi,\Lambda}. \end{gathered} \tag{3.2}$$

The Eqs. (3.2) are called the conditions of compatibility of the linear elastic Cosserat media. It may be observed that the additional relations among strain components will hold if the difference operators $\Delta_{\mathrm{I}}, \Delta_{\mathrm{II}}, \ldots, \Delta_m$ are not independent. This problem will not be considered here.

Let us introduce the symbols

$$\in^{\Lambda\Phi} = \begin{cases} 1 \text{ when } \Lambda - \Phi = -1 \text{ or } \Lambda - \Phi = 1 - m, \\ -1 \text{ when } \Lambda - \Phi = 1 \text{ or } \Lambda - \Phi = 1 - m, \\ 0 \text{ in other cases,} \end{cases} \quad l_\Lambda^k = \Delta_\Lambda \psi^k,$$

and let us assume $\Delta_\Phi^{\cdot} l_\Lambda^k \ll l_\Lambda^k$ for each Λ, Φ and each $d \in D_{\Phi,\Lambda}$.

Equations (3.2) yield

$$\in^{\Lambda\Phi}\Delta_\Lambda\gamma_\Phi^k + \varepsilon_{\cdot pr}^k \in^{\Lambda\Phi}\varkappa_\Lambda^r l_\Phi^p = 0, \quad \in^{\Lambda\Phi}\Delta_\Lambda\varkappa^k = 0, \tag{3.3}$$

or

$$\begin{gathered} \Delta_\Lambda\gamma_k{}^\Lambda + \varepsilon_{kp\cdot}^{\ \ r} l_\Lambda^p \varkappa_r{}^\Lambda = 0, \\ \Delta_\Lambda\varkappa_k{}^\Lambda = 0, \quad d \in D', \end{gathered} \tag{3.4}$$

where we have introduced the following notation:

$$\gamma_k{}^\Lambda = \delta_{kl} \in^{\Lambda\Phi}\gamma_\Phi^l, \quad \varkappa_k{}^\Lambda = \delta_{kl} \in^{\Lambda\Phi}\varkappa_\Phi^l.$$

Now, let us recall the form of the equations of motion (2.4) in the quasi-static case and without external forces:

$$
\begin{aligned}
&\bar{\Delta}_A T_k^{\ A} = 0,\\
&\bar{\Delta}_A M_k^{\ A} + \varepsilon_{kp\cdot}^{\ \ r} l_A^p T_r^{\ A} = 0, \quad d \in D.
\end{aligned} \tag{3.5}
$$

The analogy between the form of the Eqs. (3.4) and (3.5) can easily be observed; we can change the Eqs. (3.4) into (3.5) and vice-versa, using the formal scheme

$$
\Delta_A \leftrightarrow \bar{\Delta}_A, \quad \gamma_k^{\ A} \leftrightarrow M_k^{\ A}, \quad \varkappa_k^{\ A} \leftrightarrow T_k^{\ A}. \tag{3.6}
$$

Let us put the geometric equations (2.2) in the form:

$$
\begin{aligned}
\gamma_k^{\ A} &= \delta_{kl} \in^{A\Phi} (\Delta_\Phi u^l + \varepsilon_{\cdot pr}^{l} l_\Phi^p v^r),\\
\varkappa_k^{\ A} &= \delta_{kl} \in^{A\Phi} \Delta_\Phi v^l.
\end{aligned} \tag{3.7}
$$

According to the scheme (3.6), we can write the formulas

$$
\begin{aligned}
M_k^{\ A} &= \delta_{kl} \in^{A\Phi} (\bar{\Delta}_\Phi \varphi^l + \varepsilon_{\cdot pr}^{l} l_\Phi^p \chi^r),\\
T_k^{\ A} &= \delta_{kl} \in^{A\Phi} \bar{\Delta}_\Phi \chi^l,
\end{aligned} \tag{3.8}
$$

where functions φ^k, χ^k are said to be the stress functions in the linear theory of discrete Cosserat media. Substituting the right-hand sides of (3.8) into (3.5), we obtain identities. Introducing the notation

$$
M_\Phi^l = \bar{\Delta}_\Phi \varphi^l + \varepsilon_{\cdot pr}^{l} l_\Phi^p \chi^r, \quad T_\Phi^l = \bar{\Delta}_\Phi \chi^l,
$$

we are able to extend the scheme (3.6)

$$
\varphi^k \leftrightarrow u^k, \quad \chi^k \leftrightarrow v^k, \quad T_\Phi^{\ k} \leftrightarrow \varkappa_\Phi^k, \quad M_\Phi^k \leftrightarrow \gamma_\Phi^k. \tag{3.9}
$$

Denoting by $[\alpha_{kl}^{A\Phi}]$, $[\beta_{A\Phi}^{lkl}]$ the $3m \times 3m$ matrices, and introducing the notations

$$
\begin{aligned}
&\begin{bmatrix} [a_{A\Phi}^{kl}], & [b_{A\Phi}^{kl}] \\ [b_{A\Phi}^{kl}]^T, & [f_{A\Phi}^{kl}] \end{bmatrix} = \begin{bmatrix} [A_{kl}^{A\Phi}], & [B_{kl}^{A\Phi}] \\ [B_{kl}^{A\Phi}]^T, & [F_{kl}^{A\Phi}] \end{bmatrix}^{-1},\\
&a_{kl}^{A\Phi} = \delta_{km}\delta_{lp} \in^{A\Psi} \in^{\Phi\Xi} a_{\Psi\Xi}^{mp},\\
&b_{kl}^{A\Phi} = \delta_{km}\delta_{lp} \in^{A\Psi} \in^{\Phi\Xi} b_{\Psi\Xi}^{mp},\\
&f_{kl}^{A\Phi} = \delta_{km}\delta_{lp} \in^{A\Psi} \in^{\Phi\Xi} f_{\Psi\Xi}^{mp},
\end{aligned} \tag{3.10}
$$

we put the constitutive equations (2.3) into the inverse form

$$
\begin{aligned}
\gamma_k^{\ A} &= a_{kl}^{A\Phi} T_\Phi^l + b_{kl}^{A\Phi} M_\Phi^l,\\
\varkappa_k^{\ A} &= f_{kl}^{A\Phi} M_\Phi^l + b_{lk}^{\Phi A} T_\Phi^l.
\end{aligned} \tag{3.11}
$$

In deriving the equations for the stress functions φ^k, χ^k, we proceed from the Eqs. (3.4), (3.8) and (3.11). After some substitutions, we arrive at the equations

$$
\begin{aligned}
&\Delta_A [a_{kl}^{A\Phi} \bar{\Delta}_\Phi \chi^l + b_{kl}^{A\Phi} (\bar{\Delta}_\Phi \varphi^l + \varepsilon_{\cdot pr}^{l} \chi^r l_\Phi^p)] + \varepsilon_{kp\cdot}^{\ \ r} l_A^p [f_{rl}^{A\Phi} (\bar{\Delta}_\Phi \varphi^l + \varepsilon_{\cdot st}^{l} \chi^t l_\Phi^s)\\
&\qquad\qquad + b_{lr}^{\Phi A} \bar{\Delta}_\Phi \chi^l] = 0,\\
&\Delta_A [f_{kl}^{A\Phi} (\bar{\Delta}_\Phi \varphi^l + \varepsilon_{\cdot pr}^{l} \chi^r l_\Phi^p + b_{lk}^{\Phi A} \bar{\Delta}_\Phi \chi^l] = 0, \quad d \in D.
\end{aligned} \tag{3.12}
$$

Equations (3.12) hold in the quasi-static problems in which external forces are equal to zero. The analogy between the Eqs. (2.5) and the Eqs. (3.12) will be established if we complete the schemes (3.6), (3.9) by the scheme:

$$F_{kl}^{\Lambda\Phi} \leftrightarrow a_{kl}^{\Lambda\Phi}, \quad B_{lk}^{\Phi\Lambda} \leftrightarrow b_{kl}^{\Lambda\Phi}, \quad A_{kl}^{\Lambda\Phi} \leftrightarrow f_{kl}^{\Lambda\Phi}. \tag{3.13}$$

The formulas (3.6), (3.9) and (3.13) express the static-geometric analogy in the linear theory of the discrete elastic Cosserat media.

4. Plane and plate problems([3])

Let us consider the discrete Cosserat system in which centres of mass of all rigid bodies belonging to the set D are placed, at the time instant $\tau = \tau_0$, in the plane $z^3 = 0$. The constitutive equations (2.3) can be rewritten in the form:

$$\begin{aligned} T_k^{\ \Lambda} &= A_{k\ l}^{\Lambda\Phi}\gamma_\Phi^l + A_{k\cdot}^{\Lambda\Phi}\gamma_\Phi + B_{k\ l}^{\Lambda\Phi}\varkappa_\Phi^l + B_{k\cdot}^{\Lambda\Phi}\varkappa_\Phi, \\ T_{\cdot}^{\Lambda} &= A_{\cdot\ l}^{\Lambda\Phi}\gamma_\Phi^l + A_{\cdot\cdot}^{\Lambda\Phi}\gamma_\Phi + B_{\cdot\ l}^{\Lambda\Phi}\varkappa_\Phi^l + B_{\cdot\cdot}^{\Lambda\Phi}\varkappa_\Phi, \\ M_k^{\ \Lambda} &= F_{k\ l}^{\Lambda\Phi}\varkappa_\Phi^l + F_{k\cdot}^{\Lambda\Phi}\varkappa_\Phi + B_{l\ k}^{\Phi\Lambda}\varkappa_\Phi^l + B_{\cdot\ k}^{\Phi\Lambda}\varkappa_\Phi, \\ M^{\Lambda} &= F_{\cdot\ l}^{\Lambda\Phi}\varkappa_\Phi^l + F^{\Lambda\Phi}\varkappa_\Phi + B_{l\cdot}^{\Phi\Lambda}\varkappa_\Phi^l + B_{\cdot\cdot}^{\Phi\Lambda}\varkappa_\Phi, \end{aligned} \tag{4.1}$$

where the index 3 has been omitted. Let us put

$$A_{k\cdot}^{\Lambda\Phi} = F_{k\ l}^{\Lambda\Phi} = 0, \quad B_{k\ l}^{\Lambda\Phi} = 0, \quad B_{\cdot\cdot}^{\Lambda\Phi} = 0, \quad i_{k3} = 0, \tag{4.2}$$

for each $d \in D$ and each Λ, Φ. It follows that the basic equations of the discrete Cosserat media considered can be separated into two independent systems of equations. The first of these has the form:

$$\begin{aligned} \gamma_\Phi^l &= \Delta_\Phi u^l + \varepsilon_{\cdot p}^{l} v \Delta_\Phi \psi^p, \quad \varkappa_\Phi = \Delta_\Phi v, \\ \bar{\Delta}_\Lambda T_k^{\ \Lambda} + f_k &= m\ddot{u}_k, \quad \bar{\Delta}_\Lambda M^\Lambda + \varepsilon_{p\cdot}^{\ r} T_r^{\ \Lambda}\Delta_\Lambda \psi^p + n = i\ddot{v}, \\ T_k^{\ \Lambda} &= A_{k\ l}^{\Lambda\Phi}\gamma_\Phi^l + B_{k\cdot}^{\Lambda\Phi}\varkappa_\Phi, \quad M^\Lambda = F^{\Lambda\Phi}\varkappa_\Phi + B_{l\cdot}^{\Phi\Lambda}\gamma_\Phi^l. \end{aligned} \tag{4.3}$$

As the basic unknowns in (4.3), we can take the three functions u^k, v. The second system of equations can be written as follows:

$$\begin{aligned} \gamma_\Phi &= \Delta_\Phi u + \varepsilon_{pr} v^r \Delta_\Phi \psi^p, \quad \varkappa_\Phi^l = \Delta_\Phi v^l, \\ \bar{\Delta}_\Lambda T^\Lambda + f &= m\ddot{u}, \quad \bar{\Delta}_\Lambda M_k^{\ \Lambda} + \varepsilon_{kp} T^\Lambda \Delta_\Lambda \psi^p + n_k = i_{kl}\ddot{v}^l, \\ T^\Lambda &= A^{\Lambda\Phi}\gamma_\Phi + B_{\cdot\ l}^{\Lambda\Phi}\varkappa_\Phi^l, \quad M_k^{\ \Lambda} = F_{k\ l}^{\Lambda\Phi}\varkappa_\Phi^l + B_{\cdot\ k}^{\Phi\Lambda}\gamma_\Phi. \end{aligned} \tag{4.4}$$

The three functions u, v^k are the basic unknowns in (4.4). The problem of the theory of discrete Cosserat media described by the Eqs. (4.3) will be called the plane problem, and that described by the Eqs. (4.4) is said to be the plate problem.

5. Principle of virtual work. Laws of conservation

Let us consider the set of m functions $\varphi^\Lambda : D \to R$ and put $\bar{\varphi}^\Lambda(d) = \varphi^\Lambda(f_{-\Lambda}d)$ for each $d \in D_{-\Lambda}$ (the summation convention does not hold). We have

$$\Delta_\Lambda(\bar{\varphi}^\Lambda \zeta) = \varphi^\Lambda \Delta_\Lambda \zeta + \zeta \Delta_\Lambda \bar{\varphi}^\Lambda, \quad \Delta_\Lambda \bar{\varphi}^\Lambda = \bar{\Delta}_\Lambda \varphi^\Lambda, \quad d \in D', \tag{5.1}$$

([3]) In the formulas of this Section the indices $k, l, \ldots$ take the values 1, 2.

where $\zeta: D \to R$ is an arbitrary function; the formulas (5.1) will also be valid if the summation convention in (5.1) does not hold.

Let $\delta_0 u^k$, $\delta_0 v^k$ be the variations of the functions u^k, v^k, respectively, due to a change in the functional form of these functions. To obtain the principle of virtual work, we shall proceed from the expression

$$\begin{aligned} T_k{}^A\delta_0\gamma^k_A + M_k{}^A\delta_0\varkappa^k_A &= T_k{}^A\Delta_A\delta_0 u_k + T_k{}^A\varepsilon^k_{\cdot pr}\delta_0 v^r\Delta_A\psi^p + M_k{}^A\Delta_A\delta_0 v^k \\ &= \Delta_A(\bar{T}_k{}^A\delta_0 u^k + \bar{M}_k{}^A\delta_0 v^k) - \Delta_A\bar{T}_k{}^A\delta_0 u^k - \Delta_A\bar{M}_k{}^A\delta_0 v^k - \varepsilon^r_{\cdot kp}T_r{}^A\delta_0 v^k\Delta_A\psi^p. \end{aligned}$$

By virtue of the equations of motion (2.4) and the identity $(5.1)_2$, we arrive at:

$$T_k{}^A\delta_0\gamma^k_A + M_k{}^A\delta_0\varkappa^k_A = \Delta_A(\bar{T}_k{}^A\delta_0 u^k + \bar{M}_k{}^A\delta_0 v^k) + (f_k - m\ddot{u}_k)\delta_0 u^k + (n_k - i_{kl}\ddot{v}^l)\delta_0 v^k. \tag{5.2}$$

Let $D \overset{\mathrm{df}}{=} \cap (D_A \cap D_{-A}) \neq \phi$ [12]. For an arbitrary subset $K \subseteq D'$, we define now the set ΔK, called the Δ-boundary of K, assuming $d \in \Delta K$ if, and only if $\left[(d \in K) \wedge \left(\bigvee_A f_{-A} d \notin K\right)\right] \vee \left[(d \notin K) \wedge \left(\bigvee_A f_{-A} d \in K\right)\right]$. It can easily be verified that the following identity holds:

$$\sum_k \Delta_A^{\cdot}\varphi^A = \sum_{\Delta k} \varphi^A N_A, \tag{5.3}$$

where

$$N_A = N_A(d) = \begin{cases} 1 \text{ when } (d \notin K) \wedge (f_{-A} d \in K), \\ -1 \text{ when } (d \in K) \wedge (f_{-A} d \notin K), \\ 0 \text{ in other cases.} \end{cases} \tag{5.4}$$

By virtue of (5.3), we derive from (5.2) the equality:

$$\begin{aligned} \sum_k (T_k{}^A\delta_0\gamma^k_A + M_k{}^A\delta_0\varkappa^k_A) = \sum_{\Delta k}(T^{(\mathrm{N})}_k\delta_0 u^k + M^{(\mathrm{N})}_k\delta_0 v^k) + \sum_k [(f_k - m\ddot{u}_k)\delta_0 u^k \\ + (n_k - i_{kl}\ddot{v}^l)\delta_0 v^k], \end{aligned} \tag{5.5}$$

where

$$T^{(\mathrm{N})}_k = \bar{T}_k{}^A N_A, \quad M^{(\mathrm{N})}_k = \bar{M}_k{}^A N_A. \tag{5.6}$$

Let us denote $E(K) = \sum_k \varepsilon$. Since we have $\delta_0\varepsilon = T_k{}^A\delta_0\gamma^k_A + M_k{}^A\delta_0\varkappa^k_A$, it follows that

$$\delta_0 E(K) = \sum_{\Delta k}(T^{(\mathrm{N})}_k\delta_0 u^k + M^{(\mathrm{N})}_k\delta_0 v^k) + \sum_k [(f_k - m\ddot{u}_k)\delta_0 u^k + (n_k - i_{kl}\ddot{v}^l)\delta_0 v^k]. \tag{5.7}$$

The formula (5.7) represents the principle of virtual work in the theory of discrete elastic Cosserat media.

The conservation laws of momentum and the moment of momentum we can derive directly from the equations of motion (2.4), using the formulas (5.3) and (5.1). Introducing (5.6), we find that

$$\begin{aligned} \frac{d}{d\tau}\sum_k m\dot{u}_k &= \sum_{\Delta k} \boldsymbol{T}^{(\mathrm{N})}_k + \sum_k f_k, \\ \frac{d}{d\tau}\sum_k (i_{kl}\dot{v}^l + \varepsilon_{kp\cdot}{}^r\psi^p m\dot{u}_r) &= \sum_{\Delta k}(M^{(\mathrm{N})}_k + \varepsilon_{kp\cdot}{}^r\psi^p T^{(\mathrm{N})}_r) + \sum_k (n_k + \varepsilon_{kp\cdot}{}^r\psi^p f_r). \end{aligned} \tag{5.8}$$

The quantities on the left-hand sides of (5.8) are the time derivatives of the momentum and the moment of momentum, respectively. The law of conservation of energy we derive from (5.7), by replacing the functions $\delta_0 u^k$, $\delta_0 v^k$, $\delta_0 \gamma^k_\Lambda$, $\delta_0 \varkappa^k_\Lambda$ by the functions $\dot{u}^k$, $\dot{v}^k$, $\dot{\gamma}^k_\Lambda$, $\dot{\varkappa}^k_\Lambda$, respectively. The left-hand side of (5.7) will then be the time derivative

$$\dot{E}(K) = \sum_{\Delta k} (T^{(\mathrm{N})}_k \dot{u}^k + M^{(\mathrm{N})}_k \dot{v}^k) + \sum_k [(f_k - m\ddot{u}_k)\dot{u}^k + (n_k - i_{kl}\ddot{v}^l)\dot{v}^k].$$

In view of

$$m\ddot{u}_k\dot{u}^k + i_{kl}\ddot{v}^l\dot{v}^k = \frac{d}{d\tau}\left(\frac{1}{2}m\delta_{kl}\dot{u}^k\dot{u}^l + \frac{1}{2}i_{kl}\dot{v}^k\dot{v}^l\right),$$

we obtain finally

$$(5.9)\qquad \frac{d}{d\tau}\sum_k\left(\frac{1}{2}(m\delta_{kl}\dot{u}^k\dot{u}^l + \frac{1}{2}i_{kl}\dot{v}^k\dot{v}^l + \varepsilon\right) = \sum_{\Delta k}(T^{(\mathrm{N})}_k \dot{u}^k + M^{(\mathrm{N})}_k \dot{v}^k) + \sum_k (f_k\dot{u}^k + n_k\dot{v}^k).$$

The quantity on the left-hand side of (5.9) is the time derivative of the internal energy $E(K) = \Sigma\varepsilon$ and the kinetic energy of the set $K \subset D'$.

The laws of conservation can be obtained also from the variational approach (cf. Sec. 7).

6. Principle of Betti. Somigliana formulas

Let us consider now the quasi-static case, in which on one discrete elastic Cosserat system there act independently two groups of external forces. The first group of forces will be denoted by f_k, n_k, and the second by $\overset{*}{f}_k$, $\overset{*}{n}_k$. We denote displacements, rotations and stress and strain components induced by these two groups of forces by u_k, v_k, γ^k_Λ, $\varkappa^k_\Lambda$, $T_k{}^\Lambda$, $M_k{}^\Lambda$ and $\overset{*}{u}{}^k$, $\overset{*}{v}{}^k$, $\overset{*}{\gamma}{}^k_\Lambda$, $\overset{*}{\varkappa}{}^k_\Lambda$, $\overset{*}{T}_k{}^\Lambda$, $\overset{*}{M}_k{}^\Lambda$, respectively. By virtue of (2.3), we obtain the identity

$$T_k{}^\Lambda\overset{*}{\gamma}{}^k_\Lambda + M_k{}^\Lambda\overset{*}{\varkappa}{}^k_\Lambda = \overset{*}{T}_k{}^\Lambda\gamma^k_\Lambda + \overset{*}{M}_k{}^\Lambda\varkappa^k_\Lambda.$$

Substituting into the above identity the right-hand sides of the geometrical equations (2.2), and using the formulas (5.1), we find that

$$\Delta_\Lambda(\overline{T}_k{}^\Lambda\overset{*}{u}{}^k + \overline{M}_k{}^\Lambda\overset{*}{v}{}^k) - \overline{\Delta}_\Lambda T_k{}^\Lambda\overset{*}{u}{}^k - \overline{\Delta}_\Lambda M_k{}^\Lambda\overset{*}{v}{}^k + \varepsilon^r_{\cdot pk}T_r{}^\Lambda\overset{*}{v}{}^k\Delta_\Lambda\psi^p$$
$$= \Delta_\Lambda(\overset{*}{\overline{T}}_k{}^\Lambda u_k + \overset{*}{\overline{M}}_k{}^\Lambda v^k) - \overline{\Delta}_\Lambda\overset{*}{T}_k{}^\Lambda u^k - \overline{\Delta}_\Lambda\overset{*}{M}_k{}^\Lambda v^k + \varepsilon^r_{\cdot pk}\overset{*}{T}_r{}^\Lambda v^k\Delta_\Lambda\psi^{\cdot p}.$$

But then, according to the equations of motion in the quasi-static case

$$\Delta_\Lambda(\overline{T}_k{}^\Lambda\overset{*}{u}{}^k + \overline{M}_k{}^\Lambda\overset{*}{v}{}^k) + f_k\overset{*}{u}{}^k + n_k\overset{*}{v}{}^k = \Delta_\Lambda(\overset{*}{\overline{T}}_k{}^\Lambda u^k + \overset{*}{\overline{M}}_k{}^\Lambda v^k) + \overset{*}{f}_k u^k + \overset{*}{n}_k v^k,$$

and using (5.3), (5.6), we obtain finally the Betti principle:

$$(6.1)\qquad \sum_{\Delta k}(T^{(\mathrm{N})}_k\overset{*}{u}{}^k + M^{(\mathrm{N})}_k\overset{*}{v}{}^k) + \sum_k(f_k\overset{*}{u}_k + n_k\overset{*}{v}{}^k)$$
$$= \sum_{\Delta k}(\overset{*}{T}{}^{(\mathrm{N})}_k u^k + \overset{*}{M}{}^{(\mathrm{N})}_k v^k) + \sum_k(\overset{*}{f}_k u^k + \overset{*}{n}_k v^k), \qquad K \subseteq D'.$$

It can be observed that all global theorems proved above — i.e., the principle of virtual work, the laws of conservation and the Betti principle — have a form similar to the known integral theorems of the theory of continuous Cosserat media [11].

Let us assume now that at the discrete Cosserat media acts only one external force $\overset{*}{f}_k(d_0) = \delta_{kl}$, where d_0 and l are given. This force induces displacements $\overset{*}{u}_k = U^k_{(l)}(d_0, d)$ and rotations $\overset{*}{v}^k = V^k_{(l)}(d_0, d)$ of an arbitrary body $d \in D$. From the Betti principle we derive the expression

$$(6.2) \qquad u^k(d_0) = \sum_k (f_k U^k_{(l)} + n_k V^k_{(l)}) + \sum_{\Delta k} (T^{(N)}_k U^k_{(l)} + M^{(N)}_k V^k_{(l)}) - \sum_{\Delta k} (\overset{*}{T}{}^{(N)}_k u^k + \overset{*}{M}{}^{(N)}_k v^k),$$

where $\overset{*}{T}{}^{(N)}_k(d_0, d)$, $\overset{*}{M}{}^{(N)}_k(d_0, d)$ are caused by the force $\overset{*}{f}_k(d_0) = \delta_{kl}$. In the same way we obtain:

$$(6.3) \qquad v^k(d_0) = \sum_k (f_k{}'U^k_{(l)} + n_k{}'V^k_{(l)}) + \sum_{\Delta k} (T^{(N)}_k{}'U^k_{(l)} + M^{(N)}_k{}'V^k_{(l)}) - \sum_{\Delta k} ('\overset{*}{T}{}^{(N)}_k u^k + '\overset{*}{M}{}^{(N)}_k v_k),$$

where $'U^k_{(l)}(d_0, d)$, $'V^k_{(l)}(d_0, d)$, $'\overset{*}{T}{}^{(N)}_k(d_0, d)$, $'\overset{*}{M}{}^{(N)}_k(d_0, d)$ are caused by the couple $\overset{*}{n}_k(d_0) = \delta_{kl}$ (d_0 and l are given). The Eqs. (6.2) and (6.3) are said to be Somigliana formulas in the theory of discrete Cosserat media. If we have $u^k = v^k = 0$ on ΔK, and the functions $U^k_{(l)}(d_0, d)$, $V^k_{(l)}(d_0, d)$, $'U^k_{(l)}(d_0, d)$, $'V^k_{(l)}(d_0, d)$ have been determined, then the Eqs. (6.2), (6.3) will represent the solution of the quasi-static problem of discrete Cosserat media.

7. Variational formulation

The equations of motion and the laws of conservation can also be derived using the variational approach. The action functional in the theory of linear elastic discrete Cosserat media can be assumed in the form:

$$(7.1) \qquad \mathscr{W}(K) = \sum_K \int_{\tau_0}^{\tau_1} \left(\frac{1}{2} m\delta_{kl}\dot{u}^k\dot{u}^l + \frac{1}{2} i_{kl}\dot{v}^k\dot{v}^l - \varepsilon \right) d\tau.$$

Let $\delta_0\mathscr{W}$ be the variation of the action functional due to a change in the functional form of the functions $u^k(d, \tau)$, $v^k(d, \tau)$. After performing the operation δ_0 on (7.1), we arrive at:

$$(7.2) \qquad \delta_0\mathscr{W} = \sum_k \int_{\tau_0}^{\tau_1} (m\delta_{kl}\dot{u}^k\delta_0 u^l + i_{kl}\dot{v}^k\delta_0\dot{v}^l - \delta_0\varepsilon) d\tau$$

$$= -\sum_k \int_{\tau_0}^{\tau_1} (m\delta_{kl}\ddot{u}^k\delta_0 u^l + i_{kl}\ddot{v}^k\delta_0 v^l) d\tau + \sum_k [\![m\delta_{kl}\dot{u}^k\delta_0 u^l + i_{kl}\dot{v}^k\delta_0 v^l]\!]_{\tau_0}^{\tau_1} - \sum_k \int_{\tau_0}^{\tau_1} \delta_0\varepsilon d\tau.$$

Calculating $\delta_0 \varepsilon$, we find:

$$(7.3)\qquad \sum_K \delta_0 \varepsilon = \sum_K (T_k^{\ \Lambda} \delta_0 \gamma_\Lambda^k + M_k^{\ \Lambda} \delta_0 \varkappa_\Lambda^k) = \sum_{\Delta k} (T_k^{(\mathrm{N})} \delta_0 u^k + M_k^{(\mathrm{N})} \delta_0 v^k)$$
$$- \sum_k \Delta_\Lambda \overline{T}_k^{\ \Lambda} \delta_0 u^k - \sum_k (\Delta_\Lambda \overline{M}_k^{\ \Lambda} - \varepsilon^r_{\cdot kp} T_r^{\ \Lambda} \Delta_\Lambda \psi^p) \delta_0 v^k.$$

Substituting the right-hand side of (7.3) into (7.2), we obtain:

$$(7.4)\qquad \delta_0 \mathscr{W} = \sum_K \int_{\tau_0}^{\tau_1} \{(\Delta_\Lambda \overline{T}_k^{\ \Lambda} - m\delta_{kl} \ddot{u}^l) \delta_0 u^k + (\Delta_\Lambda \overline{M}_k^{\ \Lambda} + \varepsilon^r_{\cdot kp} T_r^{\ \Lambda} \Delta_\Lambda \psi^p$$
$$- i_{kl} \ddot{v}^l) \delta_0 v^k\} d\tau + \sum_K [\![m\delta_{kl} \dot{u}^k \delta_0 u^l + i_{kl} \dot{v}^k \delta_0 v^l]\!]_{\tau_0}^{\tau_1} - \sum_{\Delta K} \int_{\tau_0}^{\tau_1} (T_k^{(\mathrm{N})} \delta_0 u^k + M_k^{(\mathrm{N})} \delta_0 v^k) d\tau.$$

Denoting by δ_τ the symbol of variation resulting from the variation $\delta\tau$ of the time coordinate: $\delta_\tau u^k = \dot{u}^k \delta\tau$, $\delta_\tau \dot{v}^k = \dot{v}^k \delta\tau$, we derive the following expression for $\delta_\tau \mathscr{W}$

$$(7.5)\qquad \delta_\tau \mathscr{W} = \sum_K \left[\!\!\left[\left(\frac{1}{2} m\delta_{kl} \dot{u}^k \dot{u}^l + \frac{1}{2} i_{kl} \dot{v}^k \dot{v}^l - \varepsilon \right) \delta\tau \right]\!\!\right]_{\tau_0}^{\tau_1}.$$

The total variation $\delta\mathscr{W}$ is the sum $\delta_0 \mathscr{W} + \delta_\tau \mathscr{W}$. By virtue of $(5.1)_2$, we obtain:

$$(7.6)\qquad \delta\mathscr{W} = \sum_K \int_{\tau_0}^{\tau_1} \{(\overline{\Delta}_\Lambda T_k^{\ \Lambda} - m\delta_{kl} \ddot{u}^l) \delta_0 u^k + (\overline{\Delta}_\Lambda M_k^{\ \Lambda} + \varepsilon_{kp}^{\ \ r} T_r^{\ \Lambda} \Delta_\Lambda \psi^p - i_{kl} \ddot{v}^l) \delta_0 v^k\} d\tau$$
$$+ \sum_K \left[\!\!\left[m\delta_{kl} \dot{u}^k \delta_0 u^l + i_{kl} \dot{v}^k \delta_0 v^l + \left(\frac{1}{2} m\delta_{kl} \dot{u}^k \dot{u}^l + \frac{1}{2} i_{kl} \dot{v}^k \dot{v}^l - \varepsilon \right) \delta\tau \right]\!\!\right]_{\tau_0}^{\tau_1}$$
$$- \sum_{\Delta K} \int_{\tau_0}^{\tau_1} (T_k^{(\mathrm{N})} \delta_0 u^k + M_k^{(\mathrm{N})} \delta_0 v^k) d\tau.$$

According to the principle of stationary action in the form given in [15] — i.e., after introducing the external forces f_k, n_k — we derive the equations of motion (2.4). If these equations are satisfied, the total variation of the action functional will be equal to

$$(7.7)\qquad \delta\mathscr{W} = - \sum_K \int_{\tau_0}^{\tau_1} (f_k \delta_0 u^k + n_k \delta_0 v^k) d\tau + \sum_K \left[\!\!\left[m\delta_{kl} \dot{u}^k \delta_0 u^l + i_{kl} \dot{v}^k \delta_0 v^l \right.\right.$$
$$\left.\left. + \left(\frac{1}{2} m\delta_{kl} \dot{u}^k \dot{u}^l + \frac{1}{2} i_{kl} \dot{v}^k \dot{v}^l - \varepsilon \right) \delta\tau \right]\!\!\right]_{\tau_0}^{\tau_1} - \sum_{\Delta K} \int_{\tau_0}^{\tau_1} (T_k^{(\mathrm{N})} \delta_0 u^k + M_k^{(\mathrm{N})} \delta_0 v^k) d\tau.$$

From the invariance properties of the action functional it follows that $\delta\mathscr{W} = 0$ for $\delta\tau = \in$, $\delta u^k = \delta_0 u^k + \dot{u}^k \in = \in^k + \in^{kl} \psi_l$, $\delta v^k = \delta_0 v^k + \dot{v}^k \in = \frac{1}{2} \varepsilon^k_{\cdot lm} \in^{lm}$, where $\in$, $\in^k$, $\in^{kl} = - \in^{lk}$ are arbitrary infinitesimal constants. Making use of (7.7), we derive the weak conservation laws [15] in the form (5.8), (5.9). On the other hand, using (7.7), we derive the strong conservation laws [15], representing the conditions of existence of the weak conservation laws. These conditions are satisfied, because the Lagrange function in (7.1) is invariant with respect to an arbitrary infinitesimal translation and rotation of the physical space and with respect to an arbitrary translation of time [15].

In general considerations, the action functional of the discrete Cosserat media can be assumed in the form:

$$\mathscr{W}(K) = \sum_{K} \int_{\tau_0}^{\tau_1} L(d, \tau, u^k, v^k, \Delta_A u^k, \Delta_A v^k, \dot{u}^k \dot{v}^k) d\tau. \tag{7.8}$$

Denoting

$$T_k{}^A = -\frac{\partial L}{\partial \Delta_A u^k}, \quad M_k{}^A = -\frac{\partial L}{\partial \Delta_A v^k}, \tag{7.9}$$

we obtain:

$$\begin{aligned}\delta\mathscr{W} = \sum_{K} \int_{\tau_0}^{\tau_1} \Big\{ \Big[\frac{\partial L}{\partial u^k} - \Big(\frac{\partial L}{\partial \dot{u}^k}\Big)^{\cdot} + \bar{\Delta}_A T_k{}^A \Big] \delta_0 u^k + \Big[\frac{\partial L}{\partial v^k} - \Big(\frac{\partial L}{\partial \dot{v}^k}\Big)^{\cdot} \\ + \bar{\Delta}_A M_k{}^A \Big] \delta_0 v^k \Big\} d\tau - \sum_{\Delta K} \int_{\tau_0}^{\tau_1} (T_k^{(N)} \delta_0 u^k + M_k^{(N)} \delta_0 v^k) d\tau + \sum_{K} \Big[\!\Big[\frac{\partial L}{\partial \dot{u}^k} \delta_0 u^k \\ + \frac{\partial L}{\partial \dot{v}^k} \delta_0 v^k + L d\tau \Big]\!\Big]_{\tau_0}^{\tau_1}.\end{aligned} \tag{7.10}$$

From the principle of stationary action it follows that

$$\bar{\Delta}_A T_k{}^A + \frac{\partial L}{\partial u^k} + f_k = \Big(\frac{\partial L}{\partial \dot{u}^k}\Big)^{\cdot}, \quad \bar{\Delta}_A M_k{}^A + \frac{\partial L}{\partial v^k} + n_k = \Big(\frac{\partial L}{\partial \dot{v}^k}\Big)^{\cdot}. \tag{7.11}$$

Substituting the external forces f_k, n_k into (7.10), we obtain:

$$\begin{aligned}\delta\mathscr{W} = -\sum_{K} \int_{\tau_0}^{\tau_1} (f_k \delta_0 u^k + n_k \delta_0 v^k) d\tau - \sum_{\Delta K} \int_{\tau_0}^{\tau_1} (T_k^{(N)} \delta_0 u^k + M_k^{(N)} \delta_0 v^k) d\tau \\ + \sum_{K} \Big[\!\Big[\frac{\partial L}{\partial \dot{u}^k} \delta_0 u^k + \frac{\partial L}{\partial \dot{v}^k} \delta_0 v^k + L\delta\tau \Big]\!\Big]_{\tau_0}^{\tau_1}.\end{aligned} \tag{7.12}$$

If we put $\delta\tau = \epsilon$, $\delta_0 u^k = \epsilon^k + \epsilon^{kl}\psi_l - \dot{u}^k \epsilon$, $\delta_0 v^k \frac{1}{2} \varepsilon^k_{\cdot lm} \epsilon^{ml} - \dot{v}^k \epsilon$ into (7.10) and (7.12), we shall obtain $\delta\mathscr{W} = 0$ for the arbitrary parameters $\epsilon^k, \epsilon^{kl} = -\epsilon^{lk}, \epsilon$. From (7.12) follow the weak conservation laws:

$$\Big[\!\Big[\sum_{K} \frac{\partial L}{\partial \dot{u}^k} \Big]\!\Big]_{\tau_0}^{\tau_1} = \int_{\tau_0}^{\tau_1} \Big(\sum_{\Delta K} T_k^{(N)} + \sum_{K} f_k \Big) d\tau, \tag{7.13}$$

(7.13) [cont.]

$$\left[\!\!\left[\sum_{K}\left(\frac{\partial L}{\partial \dot{v}^k}+\varepsilon_{kp.}^{\;\;\;r}\psi^p\frac{\partial L}{\partial \dot{u}^r}\right)\right]\!\!\right]_{\tau_0}^{\tau_1}=\int_{\tau_0}^{\tau_1}\Big\{\sum_{\Delta K}(M_k^{(N)}+\varepsilon_{kp.}^{\;\;\;r}\psi^p T_r^{(N)})+\sum_{K}(f_k+\varepsilon_{kp.}^{\;\;\;r}\psi^p n_r)\Big\}\,d\tau,$$

$$\left[\!\!\left[\sum_{K}\left(\frac{\partial L}{\partial \dot{u}^k}\dot{u}^k+\frac{\partial L}{\partial v^k}\dot{v}^k-L\right)\right]\!\!\right]_{\tau_0}^{\tau_1}=\int_{\tau_0}^{\tau_1}\Big\{\sum_{\Delta K}(T_k^{(N)}\dot{u}^k+M_k^{(N)}\dot{v}^k)+\sum_{K}(f_k\dot{u}^k+n_k)\dot{v}^k\Big\}\,d\tau,$$

and from (7.10) the strong conservation laws [15]:

$$\int_{\tau_0}^{\tau_1}\sum_{K}\frac{\partial L}{\partial u^k}\,d\tau=0,\qquad \int_{\tau_0}^{\tau_1}\sum_{K}\left(\frac{\partial L}{\partial v^k}+\varepsilon_{k}^{\cdot lm}\frac{\partial L}{\partial \Delta_A u^l}\Delta_A\psi_m\right)d\tau=0,$$
$$\tag{7.14}\int_{\tau_0}^{\tau_1}\sum_{K}\frac{\partial L}{\partial \tau}\,d\tau=0.$$

By virtue of $(7.14)_{1,2}$, the equations of motion (7.11) may also be written as follows:

$$\tag{7.15}\overline{\Delta}_A T_k^{\;A}+f_k=\left(\frac{\partial L}{\partial \dot{u}^k}\right)^{\cdot},\qquad \overline{\Delta}_A M_k^{\;A}+\varepsilon_{kp.}^{\;\;\;r}T_r^{\;A}\Delta_A\psi^p+n_k=\left(\frac{\partial L}{\partial \dot{v}^k}\right)^{\cdot},$$

and from $(7.14)_{1,3}$ we find that the Lagrange function $L(d, \ldots)$ does not depend on τ and u^k. The conservation laws (7.13) and (7.14) have to be satisfied for an arbitrary subset $K \subset D'$ and for arbitrary time instants τ_0, τ_1.

8. Equations in general coordinates

Equations of discrete Cosserat media can be transformed to a more general form after introducing, for each $d \in D$, the separate Cartesian coordinate system in the physical space [12]. Let the Cartesian coordinate system assigned to the body $d \in D$ be obtained from the Cartesian coordinate system z^k by the 3×3 non-singular matrix $[A_k^\lambda(d)]$. Ue obtain the equations in such bundles of coordinate systems by replacing the indices $k, l, \ldots$ by $\lambda, \mu, \ldots$, and replacing the differences Δ_γ, $\overline{\Delta}_A$ by the absolute differences δ_A, $\overline{\delta}_A$ [12, 13], where

$$\tag{8.1}\begin{aligned}
\delta_A u^\lambda(d) &= \Delta_A u^\lambda(d)+G_{A\mu}^{\;\;\lambda}(d)u^\mu(f_A d), && d\in D_A,\\
\overline{\delta}_A u^\lambda(d) &= \overline{\Delta}_A u^\lambda(d)+G_{\mu A}^{\lambda}(d)u^\mu(f_{-A} d), && d\in D_{-A},\\
\delta_A v_\lambda(d) &= \Delta_A v_\lambda(d)+\overset{*}{G}{}_{A\lambda}^{\;\;\mu}(d)v_\mu(f_A d), && d\in D_A,\\
\overline{\delta}_A v_\lambda(d) &= \overline{\Delta}_A v_\lambda(d)+\overset{*}{G}{}_{\lambda A}^{\mu}(d)v_\mu(f_{-A} d), && d\in D_{-A},
\end{aligned}$$

and where $G_{\Lambda\mu}^{\lambda}$, $G_{\mu\Lambda}^{\lambda}$, $\overset{\bullet}{G}{}_{\Lambda\mu}^{\lambda}$, $\overset{\bullet}{G}{}_{\mu\Lambda}^{\lambda}$ are called connexion objects. The connexion objects can be calculated using the formulas:

$$
\begin{aligned}
& G_{\Lambda\lambda}^{\mu}(d) = A_k^{\mu}(d)\Delta_\Lambda A_\lambda^k(d), \quad \overset{\bullet}{G}{}_{\Lambda\lambda}^{\mu}(d) = A_\lambda^k(d)\Delta_\Lambda A_k^{\mu}(d), \quad d \in D_\Lambda, \\
(8.2)\qquad & [\delta_\nu^\mu + G_{\Lambda\nu}^{\mu}(d)]G_{\lambda\Lambda}^{\nu}(d) = G_{\Lambda\lambda}^{\mu}(f_{-\Lambda}d), \quad d \in D_\Lambda \cap D_{-\Lambda}, \\
& [\delta_\nu^\mu + \overset{\bullet}{G}{}_{\Lambda\nu}^{\mu}(d)]\overset{\bullet}{G}{}_{\lambda\Lambda}^{\nu}(d) = \overset{\bullet}{G}{}_{\Lambda\lambda}^{\mu}(f_{-\Lambda}d), \quad d \in D_\Lambda \cap D_{-\Lambda}.
\end{aligned}
$$

The concept of absolute differences was first introduced in [2] and then generalized in [13].

The equations of motion (2.4) and the geometric equations (2.2) transformed to the general coordinates have the form:

$$
(8.3)\qquad \overline{\delta}_\Lambda T_\lambda^{\ \Lambda} + f_\lambda = m\ddot{u}_\lambda, \quad \overline{\delta}_\Lambda M_\lambda^{\ \Lambda} + \varepsilon_{\lambda\pi\cdot}^{\ \ \ \rho} T_\rho^{\ \Lambda}\delta_\Lambda\psi^\pi + n_\lambda = i_{\lambda\mu}\ddot{v}^\mu, \quad d \in D',
$$

and

$$
(8.4)\qquad \gamma_\Lambda^\lambda = \delta_\Lambda u^\lambda + \varepsilon_{\cdot\pi\rho}^{\lambda} v^\rho \delta_\Lambda \psi^\pi, \quad \varkappa_\Lambda^\lambda = \delta_\Lambda v^\lambda, \quad d \in D_\Lambda.
$$

The constitutive equations (2.3) can be rewritten in general coordinates by means of known transformation formulas

$$
\begin{aligned}
& T_\lambda^{\ \Lambda} = A_{\lambda\mu}^{\Lambda\Phi}\gamma_\Phi^\mu + B_{\lambda\mu}^{\Lambda\Phi}\varkappa_\Phi^\mu, \\
(8.5)\qquad & M_\lambda^{\ \Lambda} = F_{\lambda\mu}^{\Lambda\Phi}\varkappa_\Phi^\mu + B_{\mu\lambda}^{\Phi\Lambda}\gamma_\Phi^\mu, \quad d \in D.
\end{aligned}
$$

Other equations considered in preceding Sections can be transformed in a similar way. The static-geometric analogy holds also in general coordinates because $\delta_{[\Lambda}\delta_{\Phi]}u^\nu = \delta_{[\Lambda}\delta_{\Phi]}v^\nu = 0$ [13]. It is convenient to apply general coordinates solving some special problems of discrete elasticity. In considerations concerning the fundamentals of the theory, it is not necessary to introduce general coordinates.

9. Alternative form of basic equations

The equations of motion will be transformed into the symmetric form, if we introduce the following strain components:

$$
\begin{aligned}
& \eta_\Lambda^k(d,\tau) = \Delta_\Lambda u^k(d,\tau) + \frac{1}{2}\varepsilon_{\cdot pr}^{k}[v^r(d) + v^r(f_\Lambda d)]\Delta_\Lambda\psi^p(d), \\
(9.1)\qquad & \varkappa_\Lambda^k(d,\tau) = \Delta_\Lambda v^k(d,\tau), \quad d \in D_\Lambda.
\end{aligned}
$$

By virtue of $\eta_\Lambda^k = \gamma_\Lambda^k + \frac{1}{2}\varepsilon_{\cdot pr}^{k}\varkappa_\Lambda^r\Delta_\Lambda\psi^p$, the strain energy function (the elastic potential) is now represented by the expressions:

$$
\begin{aligned}
& \pi = \frac{1}{2}A_{kl}^{\Lambda\Phi}\eta_\Lambda^k\eta_\Phi^l + H_{kl}^{\Lambda\Phi}\eta_\Lambda^k\varkappa_\Phi^k + \frac{1}{2}C_{kl}^{\Lambda\Phi}\varkappa_\Lambda^k\varkappa_\Phi^l, \\
(9.2)\qquad & H_{kl}^{\Lambda\Phi} = B_{kl}^{\Lambda\Phi} - \frac{1}{4}A_{kr}^{\Lambda\Phi}\varepsilon_{\cdot pl}^{r}l^{\Phi p} - \frac{1}{4}A_{kr}^{\Lambda\Phi}\varepsilon_{\cdot pl}^{r}l^{\Lambda p}, \quad l^{\Lambda k} \equiv \Delta_\Lambda\psi^k, \\
& C_{kl}^{\Lambda\Phi} = F_{kl}^{\Lambda\Phi} + \frac{1}{4}A_{st}^{\Lambda\Phi}\varepsilon_{\cdot pk}^{t}\varepsilon_{\cdot rl}^{s}l^{\Phi p}l^{\Lambda r}.
\end{aligned}
$$

Let us introduce the following stress components:

$$T_k^{\Lambda} = \frac{\partial \pi}{\partial \eta_\Lambda^k} = \frac{\partial \varepsilon}{\partial \gamma_\Lambda^k}, \qquad G_k^{\Lambda} = \frac{\partial \pi}{\partial \varkappa_\Lambda^k} = \frac{\partial \varepsilon}{\partial \varkappa_\Lambda^k} - \frac{1}{2}\frac{\partial \varepsilon}{\partial \gamma_\Lambda^k}\varepsilon^{l}_{.pk}\Delta_\Lambda \psi^p;$$

the summation convention with respect to the index Λ does not hold. The constitutive equations will be given by

$$(9.3)\qquad \begin{aligned} T_k^{\Lambda} &= A_{kl}^{\Lambda\Phi}\eta_\Phi^{l} + H_{kl}^{\Lambda\Phi}\varkappa_\Phi^l,\\ G_k^{\Lambda} &= C_{kl}^{\Lambda\Phi}\varkappa_\Phi^l + H_{lk}^{\Phi\Lambda}\gamma_\Phi^l. \end{aligned}$$

By virtue of

$$(9.4)\qquad G_k^{\Lambda} = M_k^{\Lambda} - \frac{1}{2}T_l^{\Lambda}\varepsilon^{l}_{.pk}l_\Lambda^p, \qquad \bar{l}_\Lambda^p(d) = l_\Lambda^p(f_{-\Lambda}d) = \bar{\Delta}_\Lambda\psi^p(d)$$

(the summation convention with respect to Λ does not hold), we can transform the equations of motion (2.4) to the symmetric form:

$$(9.5)\qquad \begin{aligned} &\frac{1}{2}(\bar{\Delta}_\Lambda T_k^{\Lambda} + \Delta_\Lambda \bar{T}_k^{\Lambda}) + f_k = m\ddot{u}_k,\\ &\frac{1}{2}(\bar{\Delta}_\Lambda M_k^{\Lambda} + \Delta_\Lambda \bar{M}_k^{\Lambda}) + \frac{1}{2}\varepsilon_{kl.}^{\;\;\,m}(l_\Lambda^l T_m^{\Lambda} + \bar{l}_\Lambda^l \bar{T}_m^{\Lambda}) + n_k = \dot{I}_{kl}\ddot{v}^l. \end{aligned}$$

The equations (9.1), (9.3) and (9.5) are the alternative form of the basic equations of the discrete elastic Cosserat media. In some special cases, we obtain $H_{kl}^{\Lambda\Phi} = 0$ [9, 10].

10. Classes of discrete Cosserat media

Let E^N, $N \leqslant m$, be the N-dimensional space of points $\mathbf{x}$, with the vector basis $\mathbf{t}_1, \ldots, \mathbf{t}_N$. Denote by $\mathbf{t}_\Lambda = \alpha_\Lambda^K \mathbf{t}_K$ the set of m different vectors, where $\alpha_\Lambda^K = \delta_\Lambda^K$ for $\Lambda \leqslant N$ and α_Λ^K for $\Lambda > N$ are integers.

Let $K \subset D$, where $K_{,\Lambda} \overset{\mathrm{df}}{=} \cap (K_{,\Lambda} \cap K_{,-\Lambda}) \neq \Phi$ (cf. [12, 13]).

Let us assume next that there exists the mapping ξ^{-1}: $K \to E^N$ having an inverse ξ, $\xi_0\xi^{-1} = id$, and satisfying the conditions $(\xi^{-1}(d) = x) \Rightarrow (\xi^{-1}(f_\Lambda d) = \mathbf{x} + \mathbf{t}_\Lambda)$ for each $d \in K_\Lambda$ and each Λ. The mapping ξ is said to be a parametrization of the subset $K \subset D$ with respect to a difference structure given on $(D, \mathscr{E})$. If for each $d \in D$ there exists the subset $K \subset D$ satisfying the conditions given above and $d \in K$, the basic equations of discrete elasticity will be represented in the form of finite difference equations. The argument d in the Eqs. (2.2) (2.3), (2.4) can then be replaced by the argument $\mathbf{x} = \xi^{-1}(d)$ (where $d \in K_{,\Lambda}$ in the Eqs. (2.2), (2.3) and $d \in K'$ in the Eqs. (2.4) (and $\Delta_\Lambda\varphi(\mathbf{x}) = \varphi(\mathbf{x}+t_\Lambda) - \varphi(\mathbf{x})$, $\mathbf{x} \in \xi^{-1}(K_\Lambda)$; $\bar{\Delta}_\Lambda\varphi(\mathbf{x}) = \varphi(\mathbf{x}) - \varphi(\mathbf{x}-t_\Lambda)$, $\mathbf{x} \in \xi^{-1}(K_{-\Lambda})$.

Let us consider now the whole class of discrete elastic Cosserat media described by the following finite difference equations:

$$(10.1)\qquad \begin{aligned} \gamma_\Lambda^k(\mathbf{x}, \tau) &= \Delta_\Lambda u^k(\mathbf{x}, \tau) + \varepsilon^k_{.pr}v^r(\mathbf{x}, \tau)\Delta_\Lambda\psi^p(\mathbf{x}),\\ \varkappa_\Lambda^k(\mathbf{x}, \tau) &= \Delta_\Lambda v^k(\mathbf{x}, \tau), \qquad \mathbf{x} \in \Omega_\Lambda, \end{aligned}$$

$$(10.2)\qquad \begin{aligned} T_k{}^{\Lambda}(\mathbf{x}, \tau) &= A_{kl}^{\Lambda\Phi}(\mathbf{x})\gamma_{\Phi}^{l}(\mathbf{x}, \tau)+B_{kl}^{\Lambda\Phi}(\mathbf{x})\,\varkappa_{\Phi}^{l}(\mathbf{x}, \tau),\\ M_k{}^{\Lambda}(\mathbf{x}, \tau) &= F_{kl}^{\Lambda\Phi}(\mathbf{x})\varkappa_{\Phi}^{l}(\mathbf{x}, \tau)+B_{lk}^{\Phi\Lambda}(\mathbf{x})\gamma_{\Phi}^{l}(\mathbf{x}, \tau), \quad \mathbf{x}\in\Omega, \end{aligned}$$

$$(10.3)\qquad \begin{aligned} &\overline{\Delta}_{\Lambda} T_k{}^{\Lambda}(\mathbf{x}, \tau)+f_k(\mathbf{x}, \tau) = m(\mathbf{x})\ddot{u}_k(\mathbf{x}, \tau),\\ &\overline{\Delta}_{\Lambda} M_k{}^{\Lambda}(\mathbf{x}, \tau)+\varepsilon_{kp}{}^{r}_{\cdot} T_{r\blacktriangle}{}^{\Lambda}(\mathbf{x}, \tau)\Delta_{\Lambda}\psi^{p}_{\cdot}(\mathbf{x})+n_k(\mathbf{x}, \tau) = i_{kl}(\mathbf{x})\ddot{v}^{l}(\mathbf{x}, \tau), \; \mathbf{x}\in\Omega', \end{aligned}$$

where $\Omega_{\Lambda}, \Omega = U(\Omega_{\Lambda}\cup\Omega_{-\Lambda}), \Omega' = \cap(\Omega_{\Lambda}\cap\Omega_{-\Lambda})$ are regions in E^N, and $(\mathbf{x}\in\Omega_{-\Lambda}) \leftrightarrow \Leftrightarrow (\mathbf{x}-\mathbf{t}_{\Lambda}\in\Omega_{\Lambda})$ for each Λ. Moreover, let us assume that all functions in (10.1), (10.2), (10.3) are differentiable functions of the point x and satisfy the conditions:

$$(10.4)\qquad \Delta_{\Lambda}\varphi(\mathbf{x}) \approx \varphi_{,K}(\mathbf{x})t_{\Lambda}^{K}, \quad \varphi(\mathbf{x}\pm t_{\Lambda}) \approx \varphi(\Lambda\mathbf{x}), \quad \mathbf{x}\in\Omega'.$$

The functional finite difference equations (10.1)–(10.3), defining the class of discrete elastic Cosserat media considered, can be written now in the form of partial differential equations [12]. By virtue of (10.4), we obtain from (10.1)–(10.3) the following set of equations:

$$(10.5)\qquad \begin{aligned} \gamma_L^l &= u^l{}_{,L}+\varepsilon^{l}_{\cdot pr}v^{r}\psi^{p}_{,L},\\ \varkappa_L^l &= v^l{}_{,L}; \end{aligned}$$

$$(10.6)\qquad \begin{aligned} T_k{}^{K} &= A_{kl}^{KL}\gamma_L^l+B_{kl}^{KL}\varkappa_L^l,\\ M_k{}^{K} &= F_{kl}^{KL}\varkappa_L^l+B_{lk}^{LK}\gamma_L^l; \end{aligned}$$

$$(10.7)\qquad \begin{aligned} T_k{}^{K}{}_{,K}+F_k &= M\ddot{u}_k,\\ M_k{}^{K}{}_{,K}+\varepsilon_{kp}{}^{r}_{\cdot}T_r{}^{K}\psi^{p}{}_{,K}+N_k &= I_{kl}\ddot{v}^l, \end{aligned}$$

where we have denoted

$$(10.8)\qquad \begin{gathered} A_{kl}^{KL} = \frac{1}{V}t_{\Lambda}^{K}t_{\Phi}^{L}A_{kl}^{\Lambda\Phi}, \qquad B_{kl}^{KL} = \frac{1}{V}t_{\Lambda}^{K}t_{\Phi}^{L}B_{kl}^{\Lambda\Phi}, \qquad F_{kl}^{KL} = \frac{1}{V}t_{\Lambda}^{K}t_{\Phi}^{L}F_{kl}^{\Lambda\Phi},\\ F_k = \frac{1}{V}f_k, \quad N_k = \frac{1}{V}n_k, \quad M = \frac{m}{V}, \quad I_{kl} = \frac{1}{V}i_{kl} \end{gathered}$$

and V is the volume of the parallelopiped in E^N given by the vectors $\mathbf{t}_I, \dots, \mathbf{t}_N$. The partial differential equations (10.6)–(10.8) describe the whole class of discrete elastic Cosserat media and have to be satisfied in the region $\Omega\subset E^N$. On the boundary of the region Ω, values of the functions $u^k(\mathbf{x}, \tau)$ $v^k(\mathbf{x}, \tau)$ can be given, cf. [12]. By virtue of the conditions (10.4), we obtain $\eta_{\Lambda}^{k}\approx\gamma_{\Lambda}^{k}$; it follows that $\varepsilon\approx\pi$, $G_k{}^{\Lambda}\approx M_k{}^{\Lambda}$, and the difference between the equations given in Sec. 2 and Sec. 9 can be disregarded.

Let us consider now the special case $N = 3$. We can now take as independent variables in the Eqs. (10.5)–(10.7) the Cartesian coordinates z^k, using the differentiable mapping $z^k = \psi^k(\mathbf{x})$, $\mathbf{x} = \xi^{-1}(d)\in\Omega$. Denoting $a_{KL} = \psi^{k}_{,K}\psi^{l}_{,L}\delta_{kl}$, $a = \det a_{KL}$, $K, L, \ldots = 1, 2, 3$, we transform the Eqs. (10.5)–(10.7) to the form:

$$(10.9)\qquad \begin{aligned} \gamma^l{}_p &= u^l_{,p}+\varepsilon^{l}_{\cdot pm}v^m,\\ \varkappa^l{}_p &= v^l_{,p}, \end{aligned}$$

$$(10.10)\qquad \begin{aligned} T_k{}^{l} &= A_{k\cdot p}^{\cdot l\cdot m}\gamma^{p}m+B_{k\cdot p}^{\cdot l\cdot m}\varkappa^{p}m,\\ M_k{}^{l} &= F_{k\cdot p}^{\cdot l\cdot m}\varkappa^{p}{}_{m}+B_{p\cdot l}^{\cdot m\cdot k}\gamma^{p}{}_{m}, \end{aligned}$$

$$(10.11)\qquad \begin{aligned} T_k{}^{l}{}_{,p}+b_k &= \mu\ddot{u}_k,\\ M_k{}^{l}{}_{,l}+\varepsilon_{kp}{}^{r}_{\cdot}T_r{}^{p}+h_k &= \varrho_{kl}\ddot{v}^l, \quad z\in\psi(\Omega), \end{aligned}$$

where

$$
(10.12)\qquad
\begin{aligned}
&A^{\cdot l\ \cdot m}_{k\cdot p} = \frac{1}{\sqrt{a}} A^{KL}_{kp}\psi^l{}_{,K}\psi^m{}_{,L}, \quad B^{\cdot l\ \cdot m}_{k\cdot p} = \frac{1}{\sqrt{a}} B^{KL}_{kp}\psi^l{}_{,K}\psi^m{}_{,L},\\
&F^{\cdot l\ \cdot m}_{k\cdot p} = \frac{1}{\sqrt{a}} F^{KL}_{kp}\psi^l{}_{,K}\psi^m{}_{,L}, \quad b_k = \frac{F_k}{\sqrt{a}}, \quad h_k = \frac{N_k}{\sqrt{a}}, \quad \mu = \frac{M}{\sqrt{a}}, \quad \varrho_{kl} = \frac{T_{kl}}{\sqrt{a}}.
\end{aligned}
$$

The Eqs. (10.9)–(10.11) have to be satisfied in the region $\psi(\Omega)$ of the physical space, and independent variables in these equations are Cartesian coordinates in the physical space. It can easily be observed that the case $N = 3$ leads to the equations of the Cosserat linear elastic continuous media [11].

Let us now consider the second special case, in which $N = 2$ and Ω is a region in E^2. The functions $z^k = \psi^k(\mathbf{x})$, $\mathbf{x} = \xi^{-1}(d) \in \Omega$, determine now the surface in the physical space. Let us denote by $a_{KL} = \psi^k_{,K}\psi^l_{,L}\delta_{kl}$, $K, L, \ldots = 1, 2$, the components of the first fundamental tensor of the surface $\psi(\Omega)$, and let b_{KL} be the components of the second fundamental tensor of this surface. After some calculations given in [14] (p. 47–50), we transform the Eqs. (10.5)–(10.7) to the form:

$$
(10.13)\qquad
\begin{aligned}
&\gamma_K{}^L = u^L|_K - b^L_K u + \epsilon^{\cdot L}_{K} v, \quad \gamma_K = u|_K + b^L_K u_L + \epsilon_{KL} v^L,\\
&\varkappa_K{}^L = v^L|_K - b^L_K v, \quad \varkappa_K = v|_K + b^L_K v_L;
\end{aligned}
$$

$$
(10.14)\qquad
\begin{aligned}
p^K{}_L &= A^{K\cdot M}_{\cdot L\cdot N}\gamma_M{}^N + A^{K\cdot M}_{\cdot L}\gamma_M + B^{K\cdot M}_{\cdot L\cdot N}\varkappa_M{}^N + B^{K\cdot M}_{\cdot L}\varkappa_M,\\
p^K &= A^{KM}_{\cdot\cdot N}\gamma_M{}^N + A^{KM}\gamma_M + B^{KM}_{\cdot\cdot}\varkappa_M{}^N + B^{KM}\varkappa_M,\\
m^K{}_L &= F^{K\cdot M}_{\cdot L\cdot N}\varkappa_M{}^N + F^{K\cdot M}_{\cdot L}\varkappa_M + B^{M\cdot K}_{\cdot N\cdot L}\gamma_M{}^N + B^{M\cdot K}_{\cdot L}\gamma_M,\\
m^K &= F^{KM}_{\cdot\cdot N}\varkappa_M{}^N + F^{KM}\varkappa_M + B^{M\cdot K}_{\cdot N}\varkappa_M{}^N + B^{MK}\varkappa_M;
\end{aligned}
$$

$$
(10.15)\qquad
\begin{aligned}
&p^K{}_L|_K - b_{LK}p^K + b_L = \mu\ddot{u}_L,\\
&p^K|_K + b^{\cdot L}_{K}{}_p{}^K{}_L + b = \mu\ddot{u},\\
&m^K{}_L|_K - b_{LK}m^K + \epsilon_{LK}p^K + h_L = \varrho_{LK}\ddot{v}^K + \varrho_L\ddot{v},\\
&m^K_{\,|}|_K + \epsilon^{\cdot L}_{K}p^K{}_L + b^{\cdot L}_{K}m^K{}_L + h = \varrho\ddot{v} + \varrho_K\ddot{v}^K, \quad K, L = 1, 2,\ x \in \Omega,
\end{aligned}
$$

where the vertical lines denote the covariant derivative in the metric a_{KL}, ϵ_{KL} are components of the Ricci bivector, and

$$
(10.16)\qquad
\begin{aligned}
&A^{K\cdot M}_{\cdot L\cdot N} = \frac{1}{\sqrt{a}}A^{KM}_{kl}\psi^k{}_{,L}\psi^l{}_{,N}, \quad A^{K\cdot M}_{\cdot L} = \frac{1}{\sqrt{a}}A^{KM}_{kl}\psi^k{}_{,L}\gamma^l, \quad A^{KM} = \frac{1}{\sqrt{a}}A^{KM}_{kl}\nu^k\nu^l, \ldots,\\
&a = \det a_{KL}, \quad \nu^k = \frac{1}{2}\epsilon^{LM}\varepsilon^k_{\cdot lm}\psi^l{}_{,L}\psi^m{}_{,M},\\
&b_L = \frac{1}{\sqrt{a}}F_k\psi^k{}_{,L}, \quad b = \frac{1}{\sqrt{a}}F_k\nu^k, \quad h_L = \frac{1}{\sqrt{a}}N_k\psi^k{}_{,L}, \quad h = \frac{1}{\sqrt{a}}N_k\nu^k,\\
&\mu = \frac{1}{\sqrt{a}}M, \quad \varrho_{KL} = \frac{1}{\sqrt{a}}I_{kl}\psi^k{}_{,K}\psi^l{}_{,L}, \quad \varrho_K = \frac{1}{\sqrt{a}}I_{kl}\psi^k{}_{,K}\nu^l, \quad \varrho = \frac{1}{\sqrt{a}}I_{kl}\nu^k\nu^l,\\
&u_K = \mu\psi^k{}_{,K}, \quad u = u_k\nu^k, \quad v_k = v_k\psi^k{}_{,K}, \quad v = v_k\nu^k.
\end{aligned}
$$

The Eqs. (10.13)–(10.15) represent the two-dimensional Cosserat continuous media, immersed in the physical space. The continuous Cosserat media considered describe certain classes of discrete elastic Cosserat systems.

11. Applications

The theory of discrete elastic Cosserat media can be applied, among other problems, to that of lattice-type structures; [1, 3–10]. The set D of rigid bodies is then a set of the rigid nodes of the lattice constructed of thin linear elastic rods. The difference structure on the set D will be determined if we assume $d' = f_A d$ when the nodes d, d' are connected by a single rod. In what follows, we shall consider a special case assuming that we can disregard the dimensions of nodes, all rods are prismatic, external loads act on the nodes only, and the mass of the whole structure can be approximately replaced by the masses concentrated at the nodes only. For this case, the elastic potential has been calculated in [9, 10]. Let us denote by $t_A^k(d)$ the components of the unit vector normal to the cross-section of the rod connecting the nodes $d, f_A d$, and let $'t_A^k(d)$, $''t_A^k(d)$ be the components of the unit vectors directed along the principal axes of this cross-section. Let us denote by $A_A(d)$ the area of the cross-section and by $C_A(d), I'_A(d), I''_A(d)$ the torsional rigidity, and the moments of inertia with respect to axes given by the vectors $'t_A(d)$, $''t_A(d)$, respectively. Using the known approximated formulas of the theory of structure, we obtain:

$$
\begin{aligned}
A_{kl}^{A\Phi} &= \delta^{A\Phi}\left(\frac{E_A A_A}{l_A} t_k^A t_l^A + \frac{12E_A I'_A}{l_A^3}\, ''t_k^A\, ''t_l^A + \frac{12E_A I''_A}{l_A^3}\, 't_k^A\, 't_l^A\right), \qquad l_A = \sqrt{l_A^k l_A^l \delta_{kl}},\\
(11.1)\qquad B_{kl}^{A\Phi} &= \delta^{A\Phi}\left(\frac{6E_A I'_A}{l_A^2}\, ''t_k^A\, 't_l^A - \frac{6E_A I''_A}{l_A^2}\, 't_k^A\, ''t_l^A\right)\\
F_{kl}^{A\Phi} &= \delta^{A\Phi}\left(\frac{C_A}{l_A} t_k^A t_l^A + \frac{4E_A I'_A}{l_A}\, 't_k^A\, 't_l^A + \frac{4E_A I''_A}{l_A}\, ''t_k^A\, ''t_l^A\right).
\end{aligned}
$$

For the equations considered in Sec. 9, we have [10]

$$
\begin{aligned}
A_{kl}^{A\Phi} &= \delta^{A\Phi}\left(\frac{E_A A_A}{l_A} t_k^A t_l^A + \frac{12E_A I'_A}{l_A^3}\, ''t_k^A\, ''t_l^A + \frac{12E_A I''_A}{l_A^3}\, 't_k^A\, 't_l^A\right),\\
(11.2)\qquad H_{kl}^{A\Phi} &= 0,\\
C_{kl}^{A\Phi} &= \delta^{A\Phi}\left(\frac{C_A}{l_A} t_k^A t_l^A + \frac{E_A I'_A}{l_A}\, 't_k^A\, 't_l^A + \frac{E_A I''_A}{l_A}\, ''t_k^A\, ''t_l^A\right).
\end{aligned}
$$

In the special case in which equations of lattice type structures are finite difference equations, the problem considered has been analysed in, among other works [1, 3]. The Eqs. (11.1) and (11.2) are valid only if the material of each rod is isotropic and homogeneous, $E_A(d)$ being the Young's modulus of the rod connecting the nodes $d, f_A d$. A more general form of (11.1) and (11.2) has been given in [9].

The form of constitutive equations for classes of lattice type structures can be obtained from (11.2) or (11.1) by virtue of (10.8), (10.12) and (10.16) [4, 5, 10]. Classes of lattice-type structures described by partial differential equations (cf. Sec. 10) have been analysed in several papers [4–10, 14], in the case $N = 2$, which corresponds to what are called lattice-type plates and shells. The full list of references can be found in [14].

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INSTITUTE OF MECHANICS
UNIVERSITY OF WARSAW

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