CHAPTER XXXIV.

CALCULUS OF VARIATIONS. (SECTION I.)

1482. To ascertain the greatest or least values of which a given function is susceptible under specific conditions, it has been found necessary in the Differential Calculus to *allow it to grow,* and then to find the magnitude attained when the rate of growth stops. And methods have been formulated by which this rate of variation can be ascertained and tests constructed for the discrimination of maxima values from minima values and from other stationary values which the method may discover.

The functions considered in the *Differential Calculus* have all been expressed directly or indirectly in terms of a set of one or more independent variables not usually involving signs of integration, and if any dependent variables have occurred in the functions under discussion their connection with the independent ones has always been specified and known.

We now have a problem of different nature. We are to consider the maximum or minimum value of a function usually expressed by an integration, in which the integrand contains not only an independent variable or set of independent variables, but also one or more dependent variables and their differential coefficients, *for which the relationship between the dependent ones with the independent ones is not specified, but remains to be discovered,* in order that a stationary value of the integral may result under any conditions with regard to the limits of the integration which may be imposed.

1483. **Preliminary Ideas as to the Mode of Procedure.**

As before, it will be necessary to allow the function to grow and to ascertain the rate of its growth under the imposed conditions when the variables it contains are made to vary in an arbitrary and independent manner consistent with the retention of the continuity of the function and consistent with the imposed conditions.

We shall first take the case of one independent variable only, viz. x, and we shall suppose that the form of the relationship between x and the dependent variable y is required which shall be such that the integral with respect to x of a *given* function V of $x, y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \ldots$, viz. $\int V\,dx$, acquires a stationary value. For amongst the stationary values the maxima and minima values lie. To fix the ideas we may regard x and y as the Cartesian coordinates of a point. And here it will be observed that y is to be regarded as a function of x, but that the form of this functional connecting relation is unknown and is to be the subject of investigation.

The form of V is supposed known. The limits of the integration may be regarded as being from a point P, (x_0, y_0), to a point $P_1, (x_1, y_1)$, which will be referred to as the terminal points or terminals, and which may be specified either as *fixed* points, or as *points which lie on specific loci*.

It is then our object to discover the relationship between x and y which will compass the object of making $\int V\,dx$ assume a stationary value with such terminal conditions.

1484. For instance, if we require to find the shortest path in the plane x-y from the given line $x+y=2a$ to the circle $x^2+y^2=a^2$, we have to make $\int ds$, or what is the same thing $\int \sqrt{1+y'^2}\,dx$, assume a minimum value, where the things at our choice are (i) the positions of the terminal points on their respective loci, (ii) the nature of the path from one terminal to the other. And the solution we should expect will be that there is a linear relation $y=mx+n$ between x and y, and that the values of m and n will be such that the line cuts both the terminal loci at right angles; which we shall presently find to be the case.

1485. **The Symbol δ of Arbitrary Variation.**

When a known and definite relation exists between x and y, say $y=f(x)$, and when we pass from a definite point P_1, (x, y), on the graph to an adjacent point P_2, $(x+dx, y+dy)$, travelling along the curve, *there is a relation between the differentials dx, dy*, viz. $dy=f'(x)\,dx$, to the first order of infinitesimals, where $f'(x)$ represents the differential coefficient of $f(x)$ with regard to x.

We may, however, assign quite arbitrary independent infinitesimal variations to x and y, and thus pass from the point P_1 to a point Q_1, *not necessarily upon the curve* $y=f(x)$, but indefinitely close to P_1, and we shall denote such independent and unconnected arbitrary variations by δx and δy. Thus, in Fig. 431, P_1P_2P being the graph of $y=f(x)$ and P_1N_1, P_2N_2, Q_1M_1 perpendiculars upon the axis and P_1SR a parallel to the x-axis cutting Q_1M_1 and P_2N_2 at S and R respectively, we have $dx=N_1N_2$, $dy=RP_2$, $\delta x=N_1M_1$, $\delta y=SQ_1$.

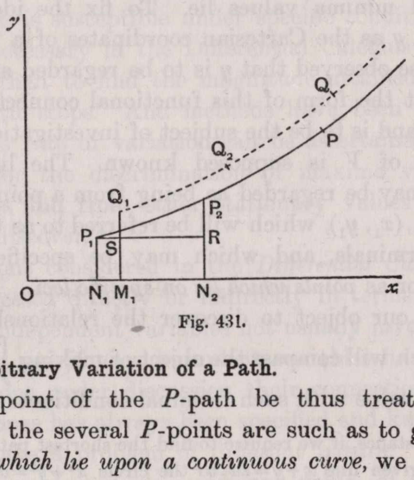

Fig. 431.

1486. Arbitrary Variation of a Path.

If every point of the P-path be thus treated and the variations of the several P-points are such as to give a series of Q-points *which lie upon a continuous curve*, we may regard the P-path as being deformed in an arbitrary manner from point to point into an indefinitely close Q-path, and the arbitrariness in the deformation is such that the deformation at P_1 from P_1 to Q_1 does not in any way fix the law by which the position of P_2 is deformed into the position Q_2, the only restriction upon the removals of the various points $P_1, P_2, \dots P$ upon the P-path to the corresponding points $Q_1, Q_2, \dots Q$ upon the Q-path being that each such removal shall be through an infinitesimal distance, and that the aggregate of the Q-points shall form a continuous curve. This deformation of the P-path, whatever that path may be, whether $f(x)$ be a function of known form or not, is therefore entirely, point by point, at our choice along the whole path of P, with the exception of the terminals, which in any particular case may have definite loci assigned to them, where there will be definite relations between the terminal values of δx and δy at each end, but the variations at one terminal being quite independent of those at the other.

The processes of the Calculus of Variations are essentially conducted by means of the consideration of such arbitrary differential variations as the δx, δy here defined.

1487. Results of the Differential Calculus which do not involve the nature of the connection between the variables occurring remain the same with the one set of variations dx, $dy, \ldots$ as with the other $\delta x, \delta y, \ldots$. Thus, if V be a function of any set of variables $x_1, x_2\ x_3, \ldots$, say, $V = \phi(x_1, x_2, x_3, \ldots)$, and if these variables receive two sets of variations,

$$(dx_1, dx_2, dx_3, \ldots) \quad \text{and} \quad (\delta x_1, \delta x_2, \delta x_3, \ldots),$$

then, if dV and δV be to the first order the corresponding changes in V, we have, whether the variables be connected in any way or not,

$$dV = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{dx_2} dx_2 + \ldots \quad \text{and} \quad \delta V = \frac{\partial \phi}{\partial x_1} \delta x_1 + \frac{\partial \phi}{\partial x_2} \delta x_2 + \ldots.$$

1488. **δ and d Commutative.**

We shall now prove that $d(\delta x) = \delta(dx)$.

Let AA_1 be any curve $y = \phi(x)$, and let P, P_1 be contiguous

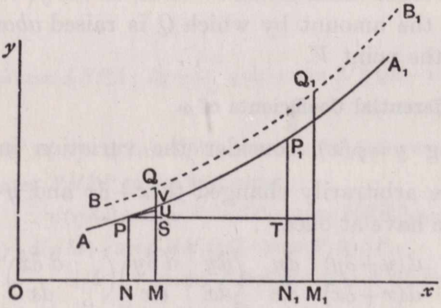

Fig. 432.

points upon it, viz. (x, y) and $(x + dx, y + dy)$ respectively. Let the curve AA_1 be deformed to a contiguous curve BB_1

so that the arbitrary point to point deformation displaces P to Q, P_1 to Q_1, etc. Let the ordinates NP, N_1P_1, MQ, M_1Q_1 be drawn, and PST parallel to the x-axis cutting the ordinates of Q and P_1 at S and T, and let PU, the tangent at P, cut the ordinate of Q at U, and let V be the point in which the ordinate of Q cuts the curve AA_1. Then $NN_1 = dx$, $NM = \delta x$. The change in NM due to a change from x to $x+dx$ is $d(NM)$, *i.e.* $d(\delta x)$. But $d(NM) = N_1M_1 - NM = MM_1 - NN_1$, which is the arbitrary change in NN_1 due to the deformation of the curve, and is therefore $\delta(dx)$. Hence $d(\delta x) = \delta(dx)$.

1489. It follows that $\delta d(dx) = d\delta(dx) = dd(\delta x)$, etc., and generally $\delta d^n V = d^m \delta d^{n-m} V = d^n \delta V$; and so on. (See Lacroix, *Calc. Diff.*, ii., p. 658.)

1490. **δ Commutative with regard to the Sign of Integration.**

Let $z = \int V\,dx$. Then $dz = V\,dx$, and $d\delta z = \delta dz = \delta(V\,dx)$.

Therefore integrating $\delta z = \int \delta(V\,dx)$.

That is $$\delta \int V\,dx = \int \delta(V\,dx).$$

1491. **The Quantity ω.**

Again, $UQ = SQ - SU = \delta y - y'\delta x$, where y' stands for $\frac{dy}{dx}$, or the tangent of the slope of the curve at P. We shall call this quantity ω. It is the amount by which Q is raised by the variation δy above the tangent line at P, and the distance UV is a second-order infinitesimal. Thus, *to the first order*, ω or $\delta y - y'\delta x$ is the amount by which Q is raised *above the curve* $y = \phi(x)$ at the point V.

1492. **Differential Coefficients of ω.**

Supposing $y = \phi(x)$, consider the variation in $\frac{dy}{dx}$, where x and y are arbitrarily changed to $x + \delta x$ and $y + \delta y$ respectively. We have at once

$$\delta \frac{dy}{dx} = \frac{d(y+\delta y)}{d(x+\delta x)} - \frac{dy}{dx} = \left(\frac{dy}{dx} + \frac{d\,\delta y}{dx}\right)\left(1 + \frac{d\,\delta x}{dx}\right)^{-1} - \frac{dy}{dx}$$

$$= \frac{d}{dx}\delta y - y'\frac{d}{dx}\delta x,$$

to the first order of infinitesimals.

Hence

$$\delta y' - y''\,\delta x = \frac{d}{dx}\delta y - y'\frac{d}{dx}\delta x - y''\delta x = \frac{d}{dx}(\delta y - y'\,\delta x) = \frac{d\omega}{dx} = \omega', \text{ say.}$$

Similarly, $\delta y'' - y'''\delta x = \omega''$, $\delta y''' - y^{\text{iv}}\,\delta x = \omega'''$; and so on.

1493. **Geometrical Proof.**

Let $\eta = f(x)$ be a curve such that $\int_a^x \eta\,dx = y$, *i.e.* y represents the area bounded by the curve AP (Fig. 433), the ordinates AL, PN, viz. $X=a$ and $X=x$, and the x-axis.

Let the curve APP_1 be displaced by an arbitrary infinitesimal point to point deformation to the curve BQQ_1, A going to B, P to Q, P_1 to Q_1, etc.

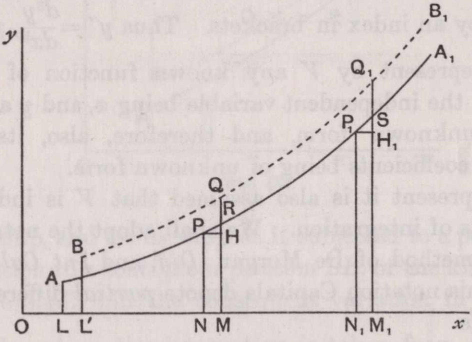

Fig. 433.

Let (x, η), $(x+\delta x, \eta+\delta\eta)$, $(x+dx, \eta+d\eta)$ be the coordinates of P, Q, P_1 respectively, and draw the ordinates AL, BL', etc., and PH, P_1H_1 parallel to the x-axis.

Then

$$y = \int_a^x \eta\,dx = \text{area } LNPA\ ;\ \delta y = \delta\int_a^x \eta\,dx = \text{area } L'MQB - \text{area } LNPA,$$

and

$$d(\delta y) = d(\text{area } L'MQB) - d(\text{area } LNPA) = \text{area } MM_1Q_1Q - \text{area } NN_1P_1P. \quad (1)$$

Also $\eta\,\delta x = $ area $NMRP$ to the first order ;

$$\therefore\ d(\eta\,\delta x) = \text{area } N_1M_1SP_1 - \text{area } NMRP. \quad \text{..................}(2)$$

Hence $d(\delta y) - d(\eta\,\delta x) = $ area $MM_1Q_1Q - $ area $N_1M_1SP_1$

$$- \text{area } NN_1P_1P + \text{area } NMRP = \text{area } RSQ_1Q,$$

i.e.

$$d\left[\delta\int_a^x \eta\,dx - \eta\,\delta x\right] = \text{area } RSQ_1Q,$$

and to the first order $RQ = \delta\eta - \eta'\,\delta x$; and

$$MM_1 = NN_1 + N_1M_1 - NM = dx + \delta(x+dx) - \delta x = dx + \delta\,dx.$$

So that to the second order, area $RSQ_1Q=(\delta\eta-\eta'\,\delta x)\,dx$;

$$\therefore\ \frac{d}{dx}\left[\delta\int_a^x \eta\,dx-\eta\,\delta x\right]=\delta\eta-\eta'\,\delta x,\quad\text{and}\quad \eta=y',\ \eta'=y'';$$

$$\therefore\ \frac{d}{dx}[\delta y-y'\,\delta x]=\delta y'-y''\,\delta x,\quad\text{and}\quad \delta y'-y''\,\delta x=\omega'.$$

This geometrical proof appears to be due to the late Dr. E. J. Routh.

1494. **Notation.**

We shall use accents to denote differentiations with regard to the independent variable x, and when accents become inconvenient by their number, we shall replace them as elsewhere by an index in brackets. Thus $y'''=\dfrac{d^3y}{dx^3}$, $y^{(n)}=\dfrac{d^ny}{dx^n}$. We shall represent by V any known function of x, y, y', y'', ..., $y^{(n)}$; the independent variable being x, and y a function of x of unknown form, and therefore, also, its several differential coefficients being of unknown form.

For the present it is also assumed that V is independent of the limits of integration. We shall adopt the notation and follow the method of De Morgan (*Diff.* and *Int. Calc.*, p. 449, etc.). In this notation Capitals denote *partial* differentiations of V. Thus

$$X\equiv\frac{\partial V}{\partial x},\quad Y\equiv\frac{\partial V}{\partial y},\quad Y_{,}\equiv\frac{\partial V}{\partial y'},\quad Y_{,,}\equiv\frac{\partial V}{\partial y''},\ \text{etc.},$$

the suffixes indicating the particular differential coefficient of y with regard to which the partial differentiation of V is effected. Also accents will be used in these cases also to denote *total* differentiations with regard to x. Thus

$$Y_{,,}'''\equiv\frac{d^3}{dx^3}\left(\frac{\partial V}{\partial y''}\right),\ \text{etc.}$$

Lagrange, to whom this Calculus is in the first place due, uses a different notation, convenient when no differential coefficients of y beyond the second order occur, but not so convenient otherwise. In Lagrange's notation p stands for y', q for y'', etc., and

$$N\equiv\frac{\partial V}{\partial y}\equiv Y,\quad P\equiv\frac{\partial V}{\partial p}\equiv Y_{,},\quad Q\equiv\frac{\partial V}{\partial q}\equiv Y_{,,},\ \text{etc.}$$

1495. Variation of $\int V\,dx$.

Supposing $V \equiv \phi\{x, y, y', y'', \ldots, y^{(n)}\}$, where the relationship of y and x is unassigned and held in abeyance, remaining to be chosen to suit circumstances which may arise, let us take AA_1 (Fig. 434) as the graph of a supposititious case of such

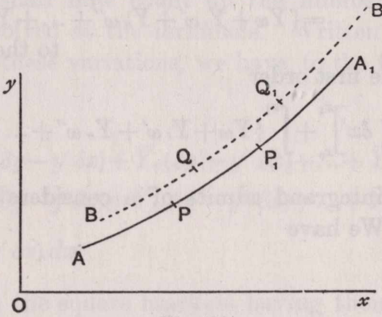

Fig. 434.

relationship, and let us suppose it subjected to a point to point deformation to a contiguous position BB_1 of the kind described. Then we shall find the consequent variation in the integral $u \equiv \int V\,dx$, where the integration is taken from one terminal point A to another terminal point A_1, which, like other points on the curve, may be subject to small variations of position, which may, however, in these terminal cases be partially prescribed by the terminal circumstances, A going to B, P to Q, P_1 to Q_1, etc. Then, since δ is commutative with regard to an integral sign,

$$\delta u = \delta\int V\,dx = \int \delta(V\,dx) = \int(\delta V\,dx + V\delta\,dx) = \int(\delta V\,dx + V d\,\delta x)$$

$$= \int \delta V\,dx + [V\,\delta x] - \int \delta x\,dV = \Big[V\,\delta x\Big]_0^1 + \int(\delta V\,dx - dV\,\delta x),$$

the integral being taken throughout the whole length of the curve from A to A_1, and the square brackets $\Big[\ \Big]_0^1$ or $\Big[\ \Big]_{x_0}^{x_1}$ round the integrated portion indicating that the included portion is to be taken between the same limits, viz. (x_0, y_0) the coordinates of A to (x_1, y_1) the coordinates of A_1. Now to the first order,

$$\delta V = X\,\delta x + Y\,\delta y + Y_{,}\,\delta y' + Y_{,,}\,\delta y'' + \ldots + Y_{(n)}\,\delta y^{(n)},$$

and $\quad dV = X\,dx + Y\,dy + Y_{,}\,dy' + Y_{,,}\,dy'' + \ldots + Y_{(n)}\,dy^{(n)};$

$$\begin{aligned}\therefore\ \delta V\,dx - dV\,\delta x &= Y(\delta y - y'\,\delta x)\,dx + Y_{,}(\delta y' - y''\,\delta x)\,dx \\ &\qquad + Y_{,,}(\delta y'' - y'''\,\delta x)\,dx + \ldots \\ &= \{Y\omega + Y_{,}\,\omega' + Y_{,,}\,\omega'' + \ldots + Y_{(n)}\omega^{(n)}\}\,dx\end{aligned}$$

to the second order.

Hence to the first order

$$\delta \int V\,dx = \Big[V\,\delta x\Big]_{x_0}^{x_1} + \int_{x_0}^{x_1} \{Y\omega + Y_{,}\,\omega' + Y_{,,}\,\omega'' + \ldots + Y_{(n)}\omega^{(n)}\}\,dx.$$

1496. The integrand admits of a considerable amount of integration. We have

$$\begin{aligned}\int Y\omega\,dx &= \int Y\omega\,dx, \\ \int Y_{,}\,\omega'\,dx &= Y_{,}\,\omega - \int Y_{,}'\,\omega\,dx, \\ \int Y_{,,}\,\omega''\,dx &= Y_{,,}\,\omega' - Y_{,,}'\,\omega + \int Y_{,,}''\,\omega\,dx, \\ \int Y_{,,,}\,\omega'''\,dx &= Y_{,,,}\,\omega'' - Y_{,,,}'\,\omega' + Y_{,,,}''\,\omega - \int Y_{,,,}'''\,\omega\,dx,\end{aligned}$$

....

$$\int Y_{(n)}\omega^{(n)}\,dx = Y_{(n)}\omega^{(n-1)} - Y_{(n)}'\omega^{(n-2)} + \ldots + (-1)^{n-1}Y_{(n)}^{(n-1)}\omega + (-1)^n \int Y_{(n)}^{(n)}\omega\,dx.$$

Now make a further abbreviation, and write

$$\begin{aligned}K \equiv \overline{Y} &\equiv Y - Y_{,}' + Y_{,,}'' - Y_{,,,}''' + \ldots + (-1)^n Y_{(n)}^{(n)}, \\ \overline{Y}_{,} &\equiv Y_{,} - Y_{,,}' + Y_{,,,}'' - \ldots + (-1)^{n-1} Y_{(n)}^{(n-1)}, \\ \overline{Y}_{,,} &\equiv Y_{,,} - Y_{,,,}' + \ldots + (-1)^{n-2} Y_{(n)}^{(n-2)},\end{aligned}$$

etc.; we then have

$$\delta \int V\,dx = \Big[V\,\delta x + \overline{Y}_{,}\,\omega + \overline{Y}_{,,}\,\omega' + \overline{Y}_{,,,}\,\omega'' + \ldots + \overline{Y}_{(n)}\,\omega^{(n-1)}\Big]_{x_0}^{x_1} + \int_{x_0}^{x_1} \overline{Y}\omega\,dx,$$

which may be written for short as

$$\delta \int V\,dx = H_1 - H_0 + \int K\omega\,dx \quad \text{or} \quad [H]_0^1 + \int K\omega\,dx,$$

which gives the variation of the integral to the first order.

Terms of the second and higher orders of the variation are not needed for the present. We shall recur to a consideration of such terms later when we come to formulate an analytical test for the discrimination between maxima and minima values. But in a large number of cases the nature of the stationary result found will be obvious from the circumstances of the problem without any formal analytical discriminatory test.

1497. We shall now count up the number of first-order variations involved at the terminals. Written at full length to exhibit all these variations, we have, to the first order,

$$\begin{aligned}\delta\int_{x_0}^{x_1} V\,dx &= [V\,\delta x+\overline{Y}_{,}(\delta y-y'\,\delta x)+\overline{Y}_{,,}(\delta y'-y''\,\delta x)+\ldots+\overline{Y}_{(n)}(\delta y^{(n-1)}-y^{(n)}\,\delta x)]_1\\ &\quad-[V\,\delta x+\overline{Y}_{,}(\delta y-y'\,\delta x)+\overline{Y}_{,,}(\delta y'-y''\,\delta x)+\ldots+\overline{Y}_{(n)}(\delta y^{(n-1)}-y^{(n)}\,\delta x)]_0\\ &\quad+\int_{x_0}^{x_1}\overline{Y}(\delta y-y'\,\delta x)\,dx,\end{aligned}$$

the suffixes to the square brackets having their usual significance. There are in each square bracket $n+1$ variations, viz. $\delta x, \delta y, \delta y', \ldots \delta y^{(n-1)}$; but these are not necessarily all independent.

(i) If the terminals be fixed we have *four* equations of condition, viz. $\delta x=0$ and $\delta y=0$ at each end, and $n-1$ arbitrary variations are left in each bracket, viz. $\delta y', \delta y'', \ldots, \delta y^{(n-1)}$, depending upon the direction of the tangent to the path, the curvature, etc., at each terminal.

(ii) If the terminals be not fixed but constrained to lie upon assigned curves, say $y=\chi_0(x)$, $y=\chi_1(x)$, then $\delta y_0=\chi_0'(x_0)\,\delta x_0$, $\delta y_1=\chi_1'(x_1)\,\delta x_1$; so that two conditions are imposed and *two* variations, viz. δy_0 and δy_1, cease to be arbitrary, which leaves n independent arbitrary terminal variations in each bracket.

(iii) Other terminal stipulations may be made. For instance, if the end x_0, y_0 is to be fixed, and also the direction of departure from that point and the curvature at that point also fixed, this will entail $\delta x_0=0$, $\delta y_0=0$, $\delta y_0'=0$, $\delta y_0''=0$, and the number of arbitrary variations left in that bracket is $n-3$. Similarly, any specific data may be assigned for the other extremity.

Thus, on the whole, there are in the two brackets $2n+2$ terminal variations. Every imposed terminal condition expressible by one equation, such as $x_0=a$, $y_0''=c$, etc., which is to hold at a terminal, reduces the number of independent terminal variations by unity. Hence, if there be p equations of condition, there are $2n+2-p$ independent terminal variations. *E.g.* if the terminal (x_0, y_0) be given, and the abscissa of x_1, and the direction and curvature of the direction of approach to (x_1, y_1) be given, there are 5 equations of condition and $2n-3$ independent terminal variations.

1498. In the remaining part of the total variation, viz.

$$\int K\omega\, dx \quad \text{or} \quad \int \overline{Y}(\delta y - y'\,\delta x)\,dx,$$

there are an infinite number of variations, each pair δx, δy indicating the displacement of a point (x, y) of the curve to be found to a hypothetical adjacent position. The function $\overline{Y}$ or K is a linear function of the total differential coefficients with regard to x of the partial differential coefficients of V, standing for $Y - Y_i' + Y_{ii}'' - \ldots + (-1)^n Y_{(n)}^{(n)}$.

In general $Y_{(n)}$ itself contains $y^{(n)}$, and therefore in general $\overline{Y}$ contains a term $y^{(2n)}$. Hence, if $\overline{Y}$ be equated to zero, as we shall see will be necessary in a search for a stationary value of $\int V\,dx$, $\overline{Y}=0$ is in general a differential equation of order $2n$, *i.e.* of double the order of the highest order differential coefficient occurring in V. The solution of such a differential equation will contain $2n$ arbitrary constants. This is less by 2 than the number of terminal conditions + the number of independent terminal variations, which is $2(n+1)$.

1499. **Conditions for a Stationary Value of** $\int V\,dx$.

The same line of argument as that employed in the *Differential Calculus* (Art. 496), in searching for the maxima and minima values of a function of several variables, will now apply in a search for the stationary values of $\int_{x_0}^{x_1} V\,dx$. It follows that the first order terms of the variation of this integral, viz. $[H]_0^1 + \int_{x_0}^{x_1} \omega K\,dx$, must vanish, and further that the coefficients of the several independent arbitrary variations contained in it must separately vanish.

Now one system of choices of these independent variations will be that in which all variations at each terminal are fixed so that H is made zero at each end. Therefore we must have in all cases $\int_{x_0}^{x} K(\delta y - y'\,\delta x)\,dx = 0$. Moreover, as $\delta y - y'\,\delta x$ is arbitrary at every point of the path, it follows that K must vanish as a primary condition. Hence the aggregate of the terms in $[H]_0^1$ must also vanish in any case. And further, since it has been seen that if the number of prescribed terminal conditions be p, the number of independent terminal variations is $2n+2-p$, there will be $2n+2-p$ relations arising from equating to zero the coefficients of these independent terminal variations.

It has been seen that the solution of the differential equation $K=0$ contains in general $2n$ arbitrary constants (Art. 1498).

It then appears that as the conditions for a stationary value of $\int_{x_0}^{x_1} V\,dx$, we have

(1) $\overline{Y}$ or $K=0$, the solution containing $2n$ arbitrary constants,

(2) $2n+2-p$ independent equations arising from $[H]_0^1=0$,

(3) p terminal equations.

Thus we have $2n+2$ terminal equations in all to find the $2n$ constants, which fix the nature of the path and two other quantities, usually the abscissae of the terminals. The problem is therefore in general completely determinate, as will be seen when we come to discuss examples of the method.

1500. **Cases of Integrability of $K=0$.**

The chief difficulty in this problem lies in the solution of the differential equation $K=0$, and often this cannot be obtained.

(1) There is one case in which at least a first integration can be effected in general terms, viz. when V does not explicitly contain x; *i.e.* $V=\phi(y, y', y'', \dots y^{(n)})$.

For now

$$X=0 \quad \text{and} \quad \frac{dV}{dx} = Yy' + Y_{\prime}y'' + Y_{\prime\prime}y''' + \dots + Y_{(n)}y^{(n+1)}.$$

But

$$\int Yy'\,dx = \int Yy'\,dx,$$

$$\int Y_{\prime}y''\,dx = Y_{\prime}y' - \int Y'_{\prime}y'\,dx,$$

$$\int Y_{\prime\prime}y'''\,dx = Y_{\prime\prime}y'' - Y'_{\prime\prime}y' + \int Y''_{\prime\prime}y'\,dx,$$

....

$$\int Y_{(n)}y^{(n+1)}\,dx = Y_{(n)}y^{(n)} - Y'_{(n)}y^{(n-1)} + \ldots + (-1)^{n-1}Y^{(n-1)}_{(n)}y' + (-1)^n \int Y^{(n)}_{(n)}y'\,dx.$$

Hence $V = \{\overline{Y}_{\prime}y' + \overline{Y}_{\prime\prime}y'' + \overline{Y}_{\prime\prime\prime}y''' + \ldots + \overline{Y}_{(n)}y^{(n)}\} + C$,

for the coefficient of y' in the integrand of the unintegrated part is K, which vanishes.

(2) Another case of integrability (to a first integral) of the equation $K=0$ is obvious, viz. when V does not contain y, so that Y does not appear. For $K=0$ then becomes

$$Y'_{\prime} - Y''_{\prime\prime} + Y'''_{\prime\prime\prime} - \ldots = 0, \text{ of which a first integral is}$$
$$Y_{\prime} - Y'_{\prime\prime} + Y''_{\prime\prime\prime} - \ldots = \text{const., } i.e.\ \overline{Y}_{\prime} = C'.$$

(3) If V contains neither x nor y explicitly, we have also

$$V = C'y' + C + \overline{Y}_{\prime\prime}y'' + \overline{Y}_{\prime\prime\prime}y''' + \ldots + \overline{Y}_{(n)}y^{(n)}.$$

1501. A very Common Case.

If $V=\phi(y, y')$, in which x does not explicitly occur, and no differential coefficients of y beyond the first, we have $V=Y_{\prime}y'+C$, with the condition $V\,\delta x + Y_{\prime}(\delta y - y'\,\delta x)=0$ at each terminal, *i.e.*

$$[C\,\delta x + Y_{\prime}\,\delta y]_0 = 0 \quad\text{and}\quad [C\,\delta x + Y_{\prime}\,\delta y]_1 = 0.$$

(1) If the terminal points be fixed, the terminal conditions are identically satisfied, and the two constants which will be present in the final integration of $V=Y_{\prime}y'+C$ will be determined by making the curve obtained pass through the specified points, whose coordinates are in that case known.

(2) If the terminal points are to lie on specific loci

$$y=\chi_0(x), \quad y=\chi_1(x),$$

we have $\delta y_0 = \chi_0'(x_0)\,\delta x_0, \quad \delta y_1 = \chi_1'(x_1)\,\delta x_1,$

and therefore

$$[C + Y_{\prime}\chi_0'(x_0)]_0 = 0 \quad\text{and}\quad [C + Y_{\prime}\chi_1'(x_1)]_1 = 0.$$

And supposing $y=F(x, C, C')$, the solution of the equation $K=0$, the substitutions of this value of y in the above equations, together with the equations

$$\chi_0(x_0)=F(x_0, C, C'), \quad \chi_1(x_1)=F(x_1, C, C'),$$

suffice to determine the values of the two constants of the differential equation and the abscissae of the terminals of the path. (See Art. 1499.)

1502. Illustrative Examples.

1. *Let us apply the rule to find the nature of the shortest distance between two given points* (x_0, y_0), (x_1, y_1), the result to be expected being of course obvious. (See Art. 1484.)

Here $\int ds \equiv \int \sqrt{1+y'^2}\,dx$ is to be a minimum.

We have

$$V=\sqrt{1+y'^2}, \quad X=0, \quad Y=0, \quad Y_{,}=y'/\sqrt{1+y'^2}, \quad V=Y_{,}y'+C.$$

Thus $\sqrt{1+y'^2}=y'^2/\sqrt{1+y'^2}+C$, *i.e.* $\sqrt{1+y'^2}=1/C$ or $y'=\text{const.}=m$, say.

Then $y=mx+n$, m and n to be determined so that the straight-line path indicated shall pass through the terminals, *i.e.*

$$\begin{vmatrix} x, & y, & 1 \\ x_0, & y_0, & 1 \\ x_1, & y_1, & 1 \end{vmatrix}=0.$$

2. *Suppose we require the shortest distance from the curve* $y=\chi_0(x)$ *to the curve* $y=\chi_1(x)$.

Then, in addition to the above, we have terminal conditions at each end, viz. $V\,\delta x+Y_{,}(\delta y-y'\,\delta x)=0$, *i.e.* $C\,\delta x+y'C\,\delta y=0$ or $1+y'\dfrac{\delta y}{\delta x}=0$ at each end, *i.e.* the straight line is to cut the terminal curves at right angles at each end

Also the equations

$$1+m\chi_0'(x_0)=0, \quad 1+m\chi_1'(x_1)=0, \quad mx_0+n=\chi_0(x_0), \quad mx_1+n=\chi_1(x_1)$$

determine the four quantities m, n, x_0, x_1.

It will be noted that maxima as well as minima distances are included in the solution. The discrimination depends upon the nature of the terminal curves, but in particular cases the nature of the result will usually be obvious without formal test.

3. *Let us enquire next the nature of the curve for which, with specific terminal conditions,* $\int\left(\dfrac{d^2y}{dx^2}\right)^2 dx$ *attains a minimum value.* [Lacroix, *Calc. D.*, p. 704.]

Here $\quad V=y''^2, \quad X=Y=Y_{,}=0, \quad Y_{,,}=2y'', \quad Y_{,,,}=0$, etc.

$K=0$ gives

$$\frac{d^2}{dx^2}(2y'')=0, \text{ i.e. } \frac{d^4y}{dx^4}=0 \text{ or } y=C_0+C_1\frac{x}{1!}+C_2\frac{x^2}{2!}+C_3\frac{x^3}{3!}. \quad \text{......(1)}$$

The terminal variation conditions are for each end

$$V\,\delta x+(Y_{,}-Y'_{,,})(\delta y-y'\,\delta x)+Y_{,,}(\delta y'-y''\,\delta x)=0. \quad \ldots\ldots\ldots\ldots(2)$$

If we impose the condition that the curve is to pass through (0, 0), $(a, 0)$ and its tangent to make with the x-axis angles $\tan^{-1}\alpha$, $\tan^{-1}\alpha'$ at these points, equation (2) is satisfied and

$$0=C_0,\quad 0=C_1\frac{a}{1}+C_2\frac{a^2}{2\,!}+C_3\frac{a^3}{3\,!},\quad \alpha=C_1,\quad \alpha'=C_1+C_2a+C_3\frac{a^2}{2\,!};$$

whence $C_0=0,\quad C_1=\alpha,\quad C_2=-2(2\alpha+\alpha')/a,\quad C_3=6(\alpha+\alpha')/a^2$;

and we have $\qquad y=\alpha x-(2\alpha+\alpha')x^2/a+(\alpha+\alpha')x^3/a^2.$

If $\alpha'=-\alpha$, this becomes the parabola $ay=\alpha x(a-x)$, in which case $y''=-2\alpha/a$, and is constant throughout the curve.

4. *In the case of a bead sliding freely on a smooth wire in a vertical plane under the action of gravity, to find the form of the wire so that the time of descent from one point of the wire to another is the least possible. This curve is called a brachistochrone.*

The energy equation is $v^2=2gy$, where y is the vertical distance of the bead at time t from the horizontal line of zero velocity. This gives

$$t=\frac{1}{\sqrt{2g}}\int\frac{ds}{\sqrt{y}}=\frac{1}{\sqrt{2g}}\int\frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx,$$

which is to be a minimum.

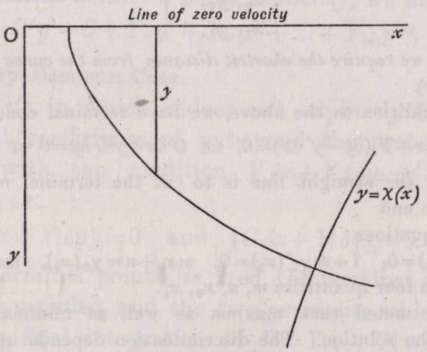

Fig. 435.

Here

$$V=\sqrt{1+y'^2}/\sqrt{y},\quad X=0,\quad Y=-\sqrt{1+y'^2}/2y^{\frac{3}{2}},\quad Y_{,}=y'/\sqrt{y}\sqrt{1+y'^2}.$$

$V=Y_{,}y'+C$ gives $C\sqrt{y}\sqrt{1+y'^2}=1$; or, writing

$$y'=\tan\psi \quad\text{and}\quad C=1/\sqrt{2a},$$

$$y=2a\cos^2\psi \quad\text{or}\quad 2a-y=2a\sin^2\psi, \quad \ldots\ldots\ldots\ldots(1)$$

which indicates an arc of a cycloid with cusps on $y=0$, *i.e.* on the line of zero velocity. (*D.C.*, Art. 395.)

At each terminal $V\,\delta x + Y_{\prime}(\delta y - y'\,\delta x) = 0$, *i.e.*

$$C\,\delta x + Y_{\prime}\,\delta y = 0 \quad \text{or} \quad \delta x + y'\,\delta y = 0. \qquad (2)$$

(i) If the terminal points be fixed, equation (2) is identically satisfied.

Equation (1) is only a first integral, but sufficient to determine the nature of the curve.

To proceed with it, $\dfrac{dy}{dx} = \tan\psi = \sqrt{\dfrac{2a-y}{y}}$,

and putting $y = a(1+\cos\theta)$, we have

$$dx = -a(1+\cos\theta)\,d\theta, \quad i.e. \quad x - C' = -a(\theta + \sin\theta).$$

So the equations of the curve are

$$\left.\begin{aligned} x &= C' - a(\theta+\sin\theta), \\ y &= \phantom{C' -{}} a(1+\cos\theta). \end{aligned}\right\}$$

Moreover, as $y = a(1+\cos\theta)$ and also $= a(1+\cos 2\psi)$, we have $\theta = 2\psi$. If the curve is to pass through (x_0, y_0) and (x_1, y_1), both supposed fixed, we have two equations to determine C' and a, *i.e.* the position of the cusp and the magnitude of the curve.

If the bead is to start *from rest* at (x_0, y_0) this point must lie on the line of zero velocity, *i.e.* $y_0 = 0$, and this point is then a cusp of the cycloid.

But if the end (x_0, y_0) be fixed, and the other end (x_1, y_1) is a point only known to lie on a definite locus $y = \chi(x)$, we have $\delta x_0 = \delta y_0 = 0$, $\delta y_1 = \chi'(x_1)\,\delta x_1$, and the terminal equation at (x_1, y_1) gives $\delta x + y'\,\delta y = 0$ at that point, *i.e.* $y'\,\dfrac{\delta y}{\delta x} = -1$, and the path cuts $y = \chi(x)$ orthogonally, and the same is true if (x_1, y_1) be fixed and (x_0, y_0) lies on a fixed locus $y = \chi(x)$, viz. the path must be such as to cut orthogonally the line from which it starts.

If both ends are to lie on fixed curves, viz. $y = \chi_0(x)$, $y = \chi_1(x)$, we have the conditions $y'\,\dfrac{\delta y}{\delta x} = -1$ at each end, and therefore each terminal curve is to be cut orthogonally.

If, for instance, the terminal curves be (1) the line of zero velocity, (2) a vertical line at a distance b from the starting point, the starting point is the cusp of the cycloid, and the other terminal is the vertex. The value of a is then found from the equation $b = \pi a$, *i.e.* $a = b/\pi$, and the constant C is $\sqrt{\pi/2b}$. It will be noted that the starting velocity from (x_0, y_0) on the first curve must be that due to a fall to that point from the line of zero velocity, *i.e.* $\sqrt{2gy_0}$. Paths starting from any other given horizontal line, and therefore with the *same* velocity, and describing paths in the least time to a given curve cut the curve at right angles, but not the straight line, except in the case when the line is the line of zero velocity itself.

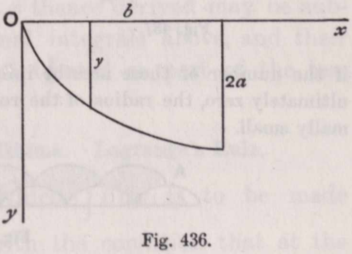

Fig. 436.

The problem just discussed is the celebrated problem of John Bernoulli which gave rise to the Calculus of Variations. It was proposed in the Acta Eruditorum, 1696 (see Cajori, *Hist. of Math.*, p. 234). The general problem of brachistochronism for any conservative system of forces will be considered later (Arts. 1537 to 1544).

5. *Taking two given points A, B as terminals to find a curve connecting them such that the area bounded by the arc AB, the radii of curvature at A and B and the intercepted arc of the evolute is least.* [De Morgan.]

Here $\frac{1}{2}\int \rho\, ds \equiv \frac{1}{2}\int \frac{(1+y'^2)^2}{y''}\,dx$ is to be a minimum.

$$V=(1+y'^2)^2/y'',\quad X=Y=0,\quad Y_{,}=4y'(1+y'^2)/y'',\quad Y_{,,}=-(1+y'^2)^2/y''^2,$$

and $V=2C_1y'+2C_2+Y_{,,}y''$ gives $(1+y'^2)^2/y''=C_1y'+C_2$;

or, putting $y'=\tan\psi$, $\rho=C_1\sin\psi+C_2\cos\psi=A\sin(\psi+B)$, say.

The curve is therefore a cycloid.

The terminal conditions are $V\delta x+\overline{Y}_{,}(\delta y-y'\delta x)+\overline{Y}_{,,}(\delta y'-y''\delta x)=0$ at each end, and since $\delta x=\delta y=0$ at each end, this reduces to $\overline{Y}_{,,}\delta y'=0$ at each end.

Also $\overline{Y}_{,,}=Y_{,,}=-(1+y'^2)^2/y''^2$, and the values of $\delta y'$ at each end are arbitrary. Hence y'' must be ∞ at each end, and the radii of curvature must therefore vanish. The terminals must therefore be cusps of the cycloid.

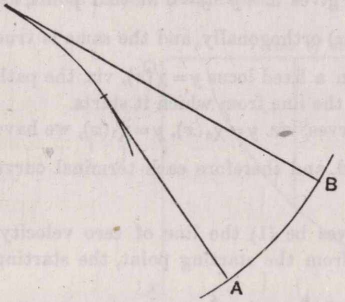

Fig. 437.

If a condition be added that these are *consecutive* cusps the cycloid is then determinate, the length of the chord AB being given, say l, the radius of the rolling circle must be $l/2\pi$. If the cusps be not necessarily consecutive the area might be that contained between a set of such cycloidal arcs as shown in Fig. 438, and their cycloidal evolutes, and it will be obvious that if the number of these arcs be infinite, the area thus bounded becomes ultimately zero, the radius of the rolling circle having become infinitesimally small.

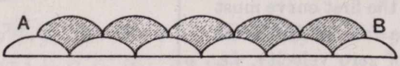

Fig. 438.

If the terminals A, B be not fixed but constrained to move on given curves, there is a relation between δx and δy at each end, but the values of $\delta y'$ are still independent and arbitrary; therefore $Y_{,,}$ still vanishes at each end, which are cusps of the cycloidal path, which may or may not be consecutive; and other relations also arise by equating to zero the coefficients of δx for each end after substitution of the terminal conditions which give δy in terms of δx.

1503. The Case when V depends upon the Terminals.

If V contains the coordinates x_0, y_0 and x_1, y_1 of the terminals and differential coefficients of y_0 and y_1, in addition to x, y, y', etc., *i.e.*

$$V=\phi(x, y, y', y'', \dots, x_0, x_1, y_0, y_1, y_0', y_1', \dots),$$

the variation δV will include terms in addition to those of Art. 1495, and now

$$\delta V = X\delta x + Y\delta y + Y_1\delta y' + \dots$$
$$+\frac{\partial V}{\partial x_0}\delta x_0+\frac{\partial V}{\partial x_1}\delta x_1+\frac{\partial V}{\partial y_0}\delta y_0+\frac{\partial V}{\partial y_1}\delta y_1+\frac{\partial V}{\partial y_0'}\delta y_0'+\dots,$$

and these additional terms in the variation $\delta\int V dx$ give rise to

$$\delta x_0\int\frac{\partial V}{\partial x_0}dx+\delta x_1\int\frac{\partial V}{\partial x_1}dx+\delta y_0\int\frac{\partial V}{\partial y_0}dx+\delta y_1\int\frac{\partial V}{\partial y_1}dx$$
$$+\delta y_0'\int\frac{\partial V}{\partial y_0'}dx+\dots,$$

the variations δx_0, δx_1, δy_0, etc., not being functions of x but only of the limiting values of x, and the integrations being from x_0 to x_1 as before. These extra terms are all to be added to the terminal variation portion of the total variation $\delta\int V dx$. The differential equation will be unaltered, and the general value of y in terms of x thence derived may be substituted in the several additional integrals above, and their values may then be found and treated as part of the terminal variation $[H]$.

1504. Relative Maxima and Minima. Lagrange's Rule.

Many problems occur in which $\int V dx$ is to be made a maximum or a minimum with the condition that at the same time a second integral $\int W dx$ is to acquire a given value a, where W, like V, is also a function of x, y, y', y'', etc. For instance, we might require the curve joining two specified points, such that with the x-axis and the terminal ordinates a maximum area is to be enclosed *whilst the length of the arc between the terminals is given.*

Lagrange solves this relative species of maxima and minima problems by making $\delta\int(V+\lambda W)\,dx=0$ unconditionally, where λ is some constant to be determined.

For clearly this gives $\delta\int V\,dx+\lambda\delta\int W\,dx=0$, *i.e.* $\delta\int V\,dx$ vanishes for all such relations between y and x as make $\int W\,dx$ *any* constant quantity. Now, upon solving this unconditional problem in the way described in the preceding articles, we shall get a relation involving λ as well as the constants of integration, say $y=\phi(\lambda, x, C_1, C_2, C_3, \ldots)$. Then substituting for y in $\int W\,dx$ and integrating, we are to make such a choice of λ as will give the integral $\int W\,dx$ the stipulated value a.

We then have $\delta\int V\,dx+\lambda\,\delta a=0$, *i.e.* $\delta\int V\,dx=0$, and the variation of $\int V\,dx$ is zero, and the integral has a stationary value for such a relation between x and y as gives to $\int W\,dx$ the prescribed constant value a. The constants of integration are to be determined as described before from the terminal conditions.

1505. **Illustrative Examples.**

1. *To two points A, B given in position, whose distance apart is $2c$, an inextensible thread is attached by its ends, whose length is $2ca \operatorname{cosec} \alpha$. To examine in what curve the thread must be arranged so that the area enclosed by the thread and the chord AB shall be as great as possible.*

Taking the mid-point of AB as origin and OA as x-axis, we are to make $\frac{1}{2}\int p\,ds$ a maximum with a condition $\int ds=2ca \operatorname{cosec} \alpha$.

By Lagrange's rule we are to make $u\equiv\int(p+2\lambda)\,ds=$ a maximum, *i.e.* in Cartesians

$$u\equiv\int\left(y-xy'+2\lambda\sqrt{1+y'^2}\right)dx \text{ is to be a maximum.}$$

Here $V=y-xy'+2\lambda\sqrt{1+y'^2}$, $X=-y'$, $Y=1$, $Y_{,}=-x+2\lambda y'/\sqrt{1+y'^2}$, $Y_{,,}=0$, etc. Along the path we are to have

$$\overline{Y}\equiv Y-Y'_{,}=0 \quad\text{or}\quad 1=-1+2\lambda\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}.$$

Hence

$$\frac{y'}{\sqrt{1+y'^2}}=\frac{x-a}{\lambda}\quad\text{and}\quad dy=\frac{(x-a)\,dx}{\sqrt{\lambda^2-(x-a)^2}},\quad \textit{i.e.}\ (x-a)^2+(y-b)^2=\lambda^2.$$

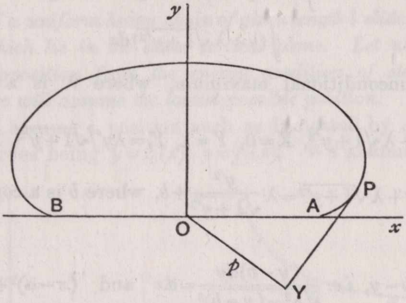

Fig. 439.

Thus the thread must lie on a circular arc of radius $\pm\lambda$ of which AB is a chord. Therefore the centre lies upon the y-axis and $a=0$.

Let D be the centre and $A\hat{D}O=\beta$. Then $\lambda=\pm c\operatorname{cosec}\beta$, and the length of the arc $=2(\pi-\beta)c\operatorname{cosec}\beta$, which is to be $2ca\operatorname{cosec}\alpha$; whence

$$\beta=\pi-\alpha,\quad \lambda=\pm c\operatorname{cosec}\alpha\quad\text{and}\quad b=\pm\lambda\cos\beta=-c\cot\alpha.$$

The equation of the arc is therefore $x^2+(y+c\cot\alpha)^2=c^2\operatorname{cosec}^2\alpha$.

In the limiting case when $c=0$, $\alpha=\pi$, and if r be the radius

$$Lt\, c\cot\alpha=Lt\, r\cos\alpha=-r\quad\text{and}\quad Lt\, c^2(\operatorname{cosec}^2\alpha-\cot^2\alpha)=c^2=0,$$

and the equation becomes $x^2+y^2=2ry$, where $2\pi r=l$, the length of the thread. The thread then forms a complete circle $x^2+y^2=ly/\pi$.

Incidentally this shows that the closed *curve of given perimeter and greatest area* is a circle. The process is the same if we require the *curve of least perimeter with a given area*, which is therefore also a circle.

Note also that if the length of the thread exceeds πc, the curve will cut the ordinates drawn at A and B and lie partly outside them. For this reason we did not express the area as $\int y\,dx$, for in that case the limits $-c$ to $+c$ for x would not contain the whole area bounded, but only so much of it as lies between the ordinates at A and B, and there would be the difficulty of assigning such limits for the integration as would give the whole area.

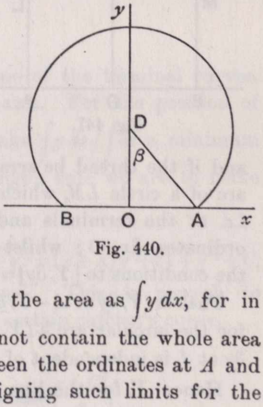

Fig. 440.

A Case of Discontinuity.

If the condition be superimposed that the thread in the above example *is not allowed to extend beyond the ordinates at A and B*, we should prefer to begin by expressing the area as $\int_{-c}^{c} y\,dx$. But when $l > \pi c$ a discontinuity will be introduced by the imposition of the new condition. We still have the condition $\int \sqrt{1+y'^2}\,dx =$ the given length $= l$. Hence

$$\int (y + \lambda\sqrt{1+y'^2})\,dx$$

is to be an unconditional maximum, where λ is a constant to be determined.

Here $Y = y + \lambda\sqrt{1+y'^2}$, $X=0$, $Y=1$, $Y_{,} = \lambda y'/\sqrt{1+y'^2}$, $Y_{,,}=0$, etc.;

$$\therefore\ y + \lambda\sqrt{1+y'^2} = \lambda\frac{y'^2}{\sqrt{1+y'^2}} + b, \text{ where } b \text{ is a constant.} \quad \text{........(1)}$$

Hence

$$\frac{\lambda}{\sqrt{1+y'^2}} = b - y, \text{ i.e. } \frac{(y-b)\,dy}{\sqrt{\lambda^2-(y-b)^2}} = dx \quad \text{and} \quad (x-a)^2 + (y-b)^2 = \lambda^2.$$

So long as $l \ngtr \pi c$ this will lead to the same solution as before. But the arc is now, by the new condition, precluded from lying outside the ordinates at A and B, and therefore, for the case where $\lambda > \pi c$, we must re-examine the problem. Now, it has been assumed in the reduction of equation (1) and in integrating, that y' is finite throughout.

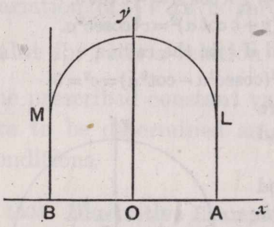

Fig. 441.

But equation (1) can be satisfied by making y' infinite, which indicates that part of the boundary of the area may be a straight line perpendicular to AB. Examine next the limiting conditions along the ordinates AL, BM at the extremities of the chord; δx is to be zero, but δy is arbitrary. Now, for the terms involving the terminal variations

$$[V\delta x + Y_{,}(\delta y - y'\,\delta x)] = 0,$$

and if the thread be arranged as AL and BM, straight portions, with an arc of a circle LM, which satisfies equation (1), we have at A, L, M, B, *i.e.* at the terminals and at the points where the thread leaves the ordinates, $\delta x = 0$; whilst at A and B, δy is also zero. This reduces the conditions to $[Y_{,}\,\delta y] = 0$.

That is ($Y_{,}\,\delta y$ at A $-$ $Y_{,}\,\delta y$ at L) for the line AL + ($Y_{,}\,\delta y$ at L $-$ $Y_{,}\,\delta y$ at M) for the circular arc + ($Y_{,}\,\delta y$ at M $-$ $Y_{,}\,\delta y$ at B) for the line $MB = 0$, and δy at L is independent of δy at M.

Hence $Y_{,}$ for the line AL at $L = Y_{,}$ for the circle at L
and $Y_{,}$ for the line BM at $M = Y_{,}$ for the circle at M.

But in each case $Y_,/\lambda \equiv \dfrac{y'}{\sqrt{1+y'^2}}$ becomes 1 for the lines, y' being infinite. Hence $\dfrac{y'}{\sqrt{1+y'^2}}=1$ for the circle also, both at L and at M. Therefore $y'=\infty$ for the circle at L and M, and the circle *touches* both the ordinates. The area in question is therefore that of a rectangle surmounted by a semicircle, and is such that $l=AL+MB+\frac{1}{2}\pi AB$, which gives the lengths of the straight portions as $\frac{1}{2}(l-\pi c)$, when $l>\pi c$.

2. *The ends of a uniform heavy chain of given length l slide freely upon two smooth curves which lie in the same vertical plane. Let us investigate its form on the supposition from the energy condition of stability that the centroid of the arc will assume the lowest possible position.*

Let the chain assume a position such as indicated by AB in Fig. 442, the terminal curves being $y=f_0(x)$, $y=f_1(x)$. We assume it as obvious

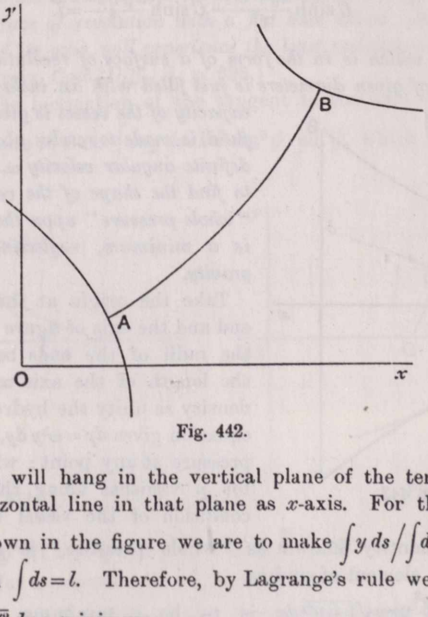

Fig. 442.

that the chain will hang in the vertical plane of the terminal curves. Take any horizontal line in that plane as x-axis. For the position of this x-axis shown in the figure we are to make $\int y\,ds \Big/ \int ds$ a minimum with condition $\int ds=l$. Therefore, by Lagrange's rule we are to make $\int (y+\lambda)\sqrt{1+y'^2}\,dx$ a minimum.

The equation $V=Y_,y'+C$ gives $(y+\lambda)\sqrt{1+y'^2}=(y+\lambda)y'^2/\sqrt{1+y'^2}+C$, *i.e.* $y+\lambda=C\sqrt{1+y'^2}=C\sec\psi$, where $y'=\tan\psi$. This is enough to indicate that the chain is to lie in the arc of a certain catenary curve.

Proceeding further with the integration,

$$\frac{C\,dy}{\sqrt{(y+\lambda)^2-C^2}}=dx, \quad i.e.\ \frac{y+\lambda}{C}=\cosh\frac{x+C'}{C},$$

where C' is a new constant. The catenary is therefore one with its vertex at $(-C', -\lambda+C)$ and with parameter C.

As to the terminals, we are to have $[V\delta x + Y_r(\delta y - y'\delta x)]=0$.

But $\delta y_1 = f_1'(x_1)\,\delta x_1$, $\delta y_0 = f_0'(x_0)\,\delta x_0$, so that only two of the four variations at the terminals are independent, and we have $C\,\delta x + Cy'\,\delta y = 0$ at each end, *i.e.* $1+y'\dfrac{\delta y}{\delta x}=0$ at each end, and therefore each of the terminal curves is cut at right angles by the curve of the chain.

The seven quantities $x_0, y_0, x_1, y_1, C, C'$ and λ are determinable from the seven equations

$$y_0 = f_0(x_0),\quad y_1 = f_1(x_1),\quad \frac{y_0+\lambda}{C}=\cosh\frac{x_0+C'}{C},\quad \frac{y_1+\lambda}{C}=\cosh\frac{x_1+C'}{C},$$

$$f_0'(x)\sinh\frac{x_0+C'}{C}=-1,\quad f_1'(x_1)\sinh\frac{x_1+C'}{C}=-1,$$

$$C\sinh\frac{x_0+C'}{C}\sim C\sinh\frac{x_1+C'}{C}=l.$$

3. *A vessel which is in the form of a surface of revolution with parallel circular ends of given diameters is just filled with an inelastic fluid. The capacity of the vessel is given and the whole fluid is made to revolve about the axis at a definite angular velocity* ω. *It is required to find the shape of the vessel so that the "whole pressure" upon the curved surface is a minimum, neglecting the effect of gravity.*

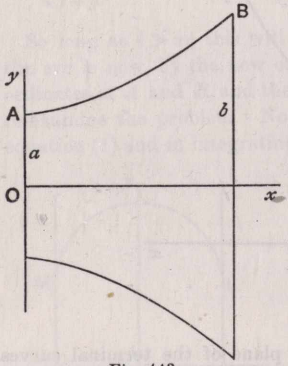

Fig. 443.

Take the origin at the centre of one end and the axis of figure as x-axis. Let the radii of the ends be a and b and the length of the axis x_1. Taking the density as unity the hydrostatic pressure equation gives $dp=\omega^2 y\,dy$, where p is the pressure at any point; whence $p=\frac{1}{2}\omega^2y^2$, for p vanishes along the axis by the condition of the vessel being *just* full. Now, the quantity known as "whole pressure" is given by $\int p\,dS$, where S is an element of surface.

Thus $\displaystyle\int\frac{\omega^2y^2}{2}2\pi y\sqrt{1+y'^2}\,dx$ is to be a minimum with condition $\int \pi y^2dx =$ a given quantity.

Hence $\int (y^3\sqrt{1+y'^2}+\lambda y^2)dx$ is to be an unconditional minimum.

So $y^3\sqrt{1+y'^2}+\lambda y^2 = y^3y'^2/\sqrt{1+y'^2}+C$, *i.e.* $y^3/\sqrt{1+y'^2}+\lambda y^2=C$, and for the terminals $[V\delta x + Y_r(\delta y - y'\delta x)]=0$, and at the end through the origin δx and δy both vanish, whilst at the other end $\delta y=0$, for the radius is fixed, *i.e.* $C\,\delta x=0$, and therefore as δx is not necessarily zero, $C=0$.

Hence $y/\sqrt{1+y'^2}=-\lambda$ or $y\cos\psi=-\lambda$, where $y'=\tan\psi$. This indicates that the arc of the generating curve is a catenary with parameter $-\lambda$, and directrix along the axis of revolution.

The constants of the catenary and the value of λ are determinable from the facts that the curve is to pass through $(0, a)$, (x_1, b), and that the vessel is to have a given capacity U.

If the abscissa of the vertex be ξ, we have for the equation of the curve $\frac{y}{-\lambda}=\cosh\frac{x-\xi}{-\lambda}$.

Hence $\frac{a}{-\lambda}=\cosh\frac{\xi}{\lambda}$, $\frac{b}{-\lambda}=\cosh\frac{x_1-\xi}{-\lambda}$, $\pi\int_0^{x_1}\lambda^2\cosh^2\left(\frac{x-\xi}{-\lambda}\right)dx=U$, three equations to determine ξ, x_1 and λ.

4. *If the assumption be adopted that the pressure upon a small element dS moving with uniform velocity u in a still fluid is normal to dS, and proportional to the square of the normal velocity, it is required to find the form of a surface of revolution with a flat base which, when it moves in the direction of its axis, will experience the least resistance upon its curved surface.* (Lacroix, *Calc. Diff.*, ii., p. 698.)

Let ψ be the inclination of the tangent to the axis of figure. The resolved pressure is then $\int 2\pi y\,ds\,.\,ku^2\sin^2\psi\,.\,\sin\psi$, which $\propto\int\frac{yy'^3\,dx}{1+y'^2}$.

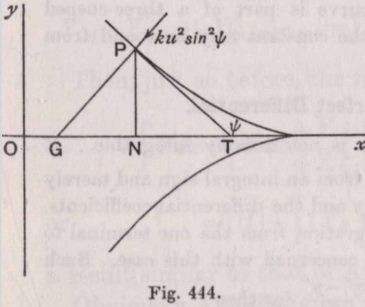

Fig. 444.

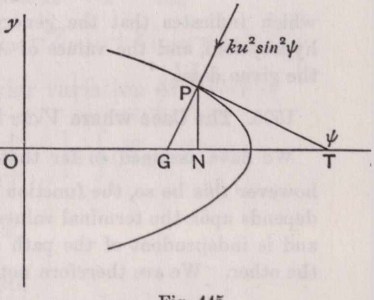

Fig. 445.

Here $V=yy'^3/(1+y'^2)$, $Y_1=y(3y'^2+y'^4)/(1+y'^2)^2$.

Therefore for a minimum $V=Y_1y'+\text{const.}$ yields

$$yy'^3/(1+y'^2)^2=\text{const.}\quad\text{or}\quad y\cos\psi\propto\operatorname{cosec}^3\psi.$$

That is, the generating curve must be such that the projection of the ordinate upon the normal varies as the cube of the secant of the inclination of the normal to the axis

If we add the condition that the flat base is to be of given area, and that the volume of the solid is to be given, we have the conditional equation

$$\pi\int y^2dx=\text{a given constant.}$$

Then $V = yy'^3/(1+y'^2) + \lambda y^2$, $Y_1 = y(3y'^2 + y'^4)/(1+y'^2)^2$; whence

$$\lambda y^2 - \frac{2yy'^3}{(1+y'^2)^2} = C, \quad i.e.\ \lambda y^2 - 2y\sin^3\psi\cos\psi = C. \quad \text{..........(1)}$$

For the terminals $[V\delta x + Y_1(\delta y - y'\delta x)] = 0$, *i.e.* $[C\,\delta x + Y_1\delta y] = 0$.

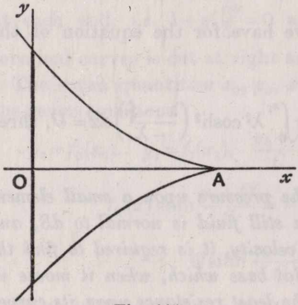
Fig. 446.

The origin being taken at the centre of the flat base (Fig. 446), and the base being given, we have δx and δy both zero for the terminal of the generating curve which lies on the y-axis. Also $C\,\delta x + Y_1\delta y$ must vanish at the other terminal. Rejecting the supposition of a discontinuous flat-nosed surface, this other terminal must be on the x-axis and $\delta y = 0$. But δx is arbitrary. Hence $C = 0$. Rejecting also the solution of an end-on straight line experiencing zero resistance, we have

$$y = \frac{2}{\lambda}\sin^3\psi\cos\psi.$$

It follows that $\dfrac{ds}{d\psi} = \dfrac{ds}{dy}\dfrac{dy}{d\psi} = -\dfrac{1}{\sin\psi}\cdot\dfrac{2}{\lambda}(3\sin^2\psi - 4\sin^4\psi) = -\dfrac{2}{\lambda}\sin 3\psi$

and
$$s = \frac{2}{3\lambda}\cos 3\psi + \text{const.},$$

which indicates that the generating curve is part of a three-cusped hypocycloid, and the values of λ and the constant may be found from the given data.

1506. The Case where $V\,dx$ is a Perfect Differential.

We have assumed so far that $\int V\,dx$ is not directly integrable. If however this be so, the function is free from an integral sign and merely depends upon the terminal values of x, y and the differential coefficients, and is independent of the path of integration from the one terminal to the other. We are therefore not much concerned with this case. Such a case would occur if, for instance, $V = \dfrac{xy'' - y'}{x^2}$, for then

$$\int_{x_0}^{x_1} V\,dx = \int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{y'}{x}\right)dx = \left[\frac{y'}{x}\right]_{x_0}^{x_1}.$$

1507. Tests of Integrability.

Our method of procedure, however, yields a test of integrability. For supposing V to be the differential coefficient of some function of form $F\{x, y, y', \dots y^{(n-1)}\}$,

$$\delta\int_{x_0}^{x_1} V\,dx = \delta\Big[F\{x, y, y', \dots y^{(n-1)}\}\Big]_{x_0}^{x_1},$$

and assuming the variation to be one which does not affect the terminal values of the variables, this vanishes independently of any assigned relation between x and y. That is, the relation $\bar{Y}=0$ is identically satisfied. And the converse is also true, and the condition is sufficient as well as necessary.

For the demonstration of this converse the student may be referred to Todhunter, *Int. Calc.*, p. 365.

1508. **Two or more Dependent Variables.**

Let V be a function of one independent variable x and two or more dependent variables y, z with their differential coefficients with regard to x, and suppose we are to search for the nature of this dependence which will give a stationary value to $\int V\,dx$.

Here $V=F\left(x, \begin{matrix} y, y', y'', \dots \\ z, z', z'', \dots \end{matrix}\right)$. We may proceed to find the first order variation of the integral exactly as before, but it is necessary to extend our notation.

Let $$\frac{\partial V}{\partial x}=X, \quad \frac{\partial V}{\partial y^{(n)}}=Y_{(n)}, \quad \frac{\partial V}{\partial z^{(n)}}=Z_{(n)},$$

$$\eta^{(n)}=\delta y^{(n)}-y^{(n+1)}\,\delta x, \quad \zeta^{(n)}=\delta z^{(n)}-z^{(n+1)}\,\delta x,$$

$$Y_{(n)}-Y'_{(n+1)}+Y''_{(n+2)}-\dots=\bar{Y}_{(n)}, \quad Z_{(n)}-Z'_{(n+1)}+Z''_{(n+2)}-\dots=\bar{Z}_{(n)}.$$

Then, just as before, the first order variation of $\int V\,dx$ is

$$\delta\int V\,dx=\left[V\,\delta x \begin{matrix} +\bar{Y}_{\prime}\eta+\bar{Y}_{\prime\prime}\eta'+\dots \\ +\bar{Z}_{\prime}\zeta+\bar{Z}_{\prime\prime}\zeta'+\dots \end{matrix}\right]+\int(\bar{Y}\eta+\bar{Z}\zeta)\,dx$$

or $$=[H]+\int(\bar{Y}\eta+\bar{Z}\zeta)\,dx,$$

a result similar to that of Art. 1496.

Obviously, a similar form will hold however many dependent variables there may be.

1509. **The Subsequent Procedure.**

As in the case of one dependent variable, in a search for the forms of the functions y and z which will give $\int V\,dx$ a stationary value, we are to put $\delta\int V\,dx=0$, and now two cases arise, viz.

(i) When y and z are independent functional forms;

(ii) when they are connected by an equation $L=0$.

(i) In the first case, $\eta \equiv \delta y - y'\delta x$ and $\zeta \equiv \delta z - z'\delta x$ are independent variations, and we get $\overline{Y}=0$ and $\overline{Z}=0$ separately, which form two differential equations to determine y and z in terms of x.

(ii) In the second case, η and ζ are not independent variations, but we have $\overline{Y}\eta+\overline{Z}\zeta=0$, together with $L=0$.

We shall consider these cases in detail.

1510. **Case I. y and z independent.**

Here

$$\overline{Y} \equiv Y - Y'_{,} + Y''_{,,} - \ldots = 0, \quad \overline{Z} = Z - Z'_{,} + Z''_{,,} - \ldots = 0.$$

Besides these equations, in the event of V not explicitly containing x, we have, as in Art. 1500,

$$V = (\overline{Y}_{,}y' + \overline{Y}_{,,}y'' + \ldots) + (\overline{Z}_{,}z' + \overline{Z}_{,,}z'' + \ldots) + C.$$

And further special cases arise. For instance, if y and z are also absent from V, we have

$$Y'_{,} - Y''_{,,} + \ldots = 0 \quad \text{and} \quad Z'_{,} - Z''_{,,} + \ldots = 0,$$

whence $\overline{Y}_{,} = C_1$ and $\overline{Z}_{,} = C_2$;

$$\therefore \quad V = C_1 y' + C_2 z' + C + \overline{Y}_{,,}y'' + \ldots + \overline{Z}_{,,}z'' + \ldots;$$

and similarly in other cases.

Also, if other dependent variables be present, a corresponding modification of these results will obviously hold.

1511. **Case II. The Case when the Path lies on a Specified Surface.**

Before considering Case II. in detail, viz. y and z independent, we may point out one very useful case which follows immediately from what has been said, viz. the case where the equation $L=0$ is a relation between x, y and z alone. This equation is that of a surface on which the path to be discovered must necessarily lie. And the case is useful for the very large class of problems dealing with maxima or minima conditions for lines drawn upon a given surface.

In addition to $\overline{Y}\eta+\overline{Z}\zeta=0$, we have

$$L_x dx + L_y dy + L_z dz = 0 \quad \text{and} \quad L_x \delta x + L_y \delta y + L_z \delta z = 0.$$

Multiplying the first by $\delta x/dx$ and subtracting, we have $L_y\eta + L_z\zeta = 0$; whence, eliminating η and ζ, $\overline{Y}/L_y = \overline{Z}/L_z$ and $L=0$ for all such cases.

1512. Next suppose the equation of condition to contain x, y, z and differential coefficients of y and z with regard to x, viz.

$$L \equiv f\left(x, \begin{matrix} y, y', y'', \dots \\ z, z', z'', \dots \end{matrix}\right) = 0.$$

Lagrange adopts a method similar to that of Art. 1504, and makes

$$\delta \int (V + \lambda L)\, dx = 0 \text{ without condition, } \dots\dots\dots\dots(1)$$

where he regards λ as a function of x only.

It is clear that this will make $\delta \int V\, dx$ vanish for all such values of the variables as make $L = 0$, which is what we require.

Now

$$\delta \int \lambda L\, dx = \int (L\, dx \,.\, \delta\lambda + \lambda\, dx \,.\, \delta L + \lambda L \,.\, d\,\delta x)$$

$$= [\lambda L\, \delta x] + \int \lambda(\delta L\, dx - dL\, \delta x) + \int L(\delta\lambda\, dx - \delta x\, d\lambda).$$

The first term is a function of the variables and variations at the terminals only, and vanishes with L.

The third term is the only one in which variations of λ appear. And it will be noticed that if λ be regarded as a function of x only, say $\lambda = \chi(x)$, then since $d\lambda = \chi'(x)\, dx$ and $\delta\lambda = \chi'(x)\, \delta x$, we have $\delta\lambda\, dx - \delta x\, d\lambda = 0$, so that the suppositions (i) $L = 0$, (ii) $\lambda = \chi(x)$ produce in that term the same result. Therefore, in finding the variation $\delta \int (V + \lambda L) dx$ without condition, it is unnecessary to consider variations of λ when we consider λ to be a function of x alone. The variation of $\int \lambda L\, dx$ therefore produces in the unintegrated part of $\delta \int (V + \lambda L)\, dx$, the additional term $\int \lambda \left(\delta L - \frac{dL}{dx}\, \delta x\right) dx$.

1513. Regarding λ therefore as a function of x alone, and writing $V + \lambda L$ instead of V, let us put

$$[Y] \equiv \frac{\partial}{\partial y}(V + \lambda L), \quad [\overline{Y}_1] \equiv \frac{\partial}{\partial y'}(V + \lambda L), \text{ etc.,}$$

the square brackets indicating that the substitution of $V+\lambda L$ for V has been made therein. Thus

$$\delta\int(V+\lambda L)\,dx=\left\{[V]\,\delta x\begin{matrix}+[\overline{Y}_{\prime}]\,\eta+[\overline{Y}_{\prime\prime}]\,\eta'+\dots\\+[\overline{Z}_{\prime}]\,\zeta+[\overline{Z}_{\prime\prime}]\,\zeta'+\dots\end{matrix}\right\}$$
$$+\int([\overline{Y}]\,\eta+[\overline{Z}]\,\zeta)\,dx\,;$$

and as the variation is unconditional, we have η and ζ independent and $[\overline{Y}]=0$, $[\overline{Z}]=0$; that is

$$\frac{\partial}{\partial y}(V+\lambda L)-\frac{d}{dx}\frac{\partial}{\partial y'}(V+\lambda L)+\frac{d^2}{dx^2}\frac{\partial}{\partial y''}(V+\lambda L)-\dots=0$$

and $$\frac{\partial}{\partial z}(V+\lambda L)-\frac{d}{dx}\frac{\partial}{\partial z'}(V+\lambda L)+\frac{d^2}{dx^2}\frac{\partial}{\partial z''}(V+\lambda L)-\dots=0,$$

i.e. λ being a function of x alone,

$$\left.\begin{aligned}&\overline{Y}+\lambda\frac{\partial L}{\partial y}-\frac{d}{dx}\left(\lambda\frac{\partial L}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\lambda\frac{\partial L}{\partial y''}\right)-\dots=0\\ \text{and}\quad&\overline{Z}+\lambda\frac{\partial L}{\partial z}-\frac{d}{dx}\left(\lambda\frac{\partial L}{\partial z'}\right)+\frac{d^2}{dx^2}\left(\lambda\frac{\partial L}{\partial z''}\right)-\dots=0,\\ \text{which, with}\quad&\qquad\qquad\qquad\qquad\qquad\qquad L=0,\end{aligned}\right\}$$

give three equations to determine y, z and λ as functions of x.

1514. It will be observed that the terms after the first in the first and second of these equations, are those which accrue from the treatment of the term

$$\int\lambda\left(\delta L-\frac{dL}{dx}\,\delta x\right)dx$$

in the variaton of $\int\lambda L\,dx$, after the manner of Art. 1496.

We may note further that when L does not contain differential coefficients of y or z with respect to x, these equations reduce to

$$\overline{Y}+\lambda L_y=0,\quad \overline{Z}+\lambda L_z=0,\quad L=0,$$

and therefore give again the result of Art. 1511, viz.

$$\overline{Y}/L_y=\overline{Z}/L_z\quad\text{and}\quad L=0.$$

1515. Illustrative Examples.

1. As an example of Case I. of Art. 1509, *let us find the shortest distance from the surface $F(x, y, z)=0$ to the surface $f(x, y, z)=0$ without any further condition as to the path.* This should obviously be a straight line cutting both surfaces perpendicularly.

We are to make $\int ds = \int \sqrt{1+y'^2+z'^2}\,dx$ a minimum, with specific terminal conditions. Here

$$V=\sqrt{1+y'^2+z'^2},\quad X=0,\quad Y=0,\quad Y_{,}=\frac{y'}{\sqrt{1+y'^2+z'^2}},\quad Z=0,$$

$$Z_{,}=\frac{z'}{\sqrt{1+y'^2+z'^2}},\quad \overline{Y}=-\frac{d}{dx}Y_{,},\quad \overline{Y}_{,}=Y_{,},\quad \overline{Z}=-\frac{d}{dx}Z_{,},\quad \overline{Z}_{,}=Z_{,}.$$

The equations $\overline{Y}=0$, $\overline{Z}=0$ give

$$Y_{,}=C_1,\quad Z_{,}=C_2,\quad i.e.\ \frac{dy}{ds}=C_1,\quad \frac{dz}{ds}=C_2,$$

and therefore
$$\frac{dx}{ds}=\sqrt{1-C_1{}^2-C_2{}^2}.$$

That is, the tangent to the path is in a constant direction, and the path itself is a straight line.

At the terminals we have

$$[V\,\delta x+\overline{Y}_{,}(\delta y-y'\,\delta x)+\overline{Z}_{,}(\delta z-z'\,\delta x)]=0,\quad i.e.\ \left[\frac{\delta x+y'\,\delta y+z'\,\delta z}{\sqrt{1+y'^2+z'^2}}\right]=0,$$

and the variations at one end are independent of those at the other, *i.e.* $\delta x+y'\,\delta y+z'\,\delta z$ must be zero at each end, *i.e.*

$$\frac{dx}{ds}\,\delta x+\frac{dy}{ds}\,\delta y+\frac{dz}{ds}\,\delta z=0$$

at each end. But the variations δx, δy, δz must refer to displacements in the tangent planes of the terminal surfaces, for which

$$F_x\,\delta x+F_y\,\delta y+F_z\,\delta z=0\quad\text{and}\quad f_x\,\delta x+f_y\,\delta y+f_z\,\delta z=0.$$

Hence the path sought must cut each surface orthogonally.

2. As an example of Case II. of Art. 1509, *examine by aid of these equations Lagrange's first rule*, Art. 1504, *where we have to find a function y such that $\delta\int V\,dx=0$ under condition $\int W\,dx=$ a constant a.*

Putting $z=\int W\,dx$, we may write this as $L\equiv z'-W=0$.

Then we make $\delta\int\{V+\lambda(z'-W)\}\,dx=0$, λ being a function of x alone.

We have
$$\left.\begin{aligned}&\overline{Y}+\frac{\partial}{\partial y}\lambda(z'-W)-\frac{d}{dx}\frac{\partial}{\partial y'}\lambda(z'-W)+\ldots=0\\ \text{and}\quad&\overline{Z}+\frac{\partial}{\partial z}\lambda(z'-W)-\frac{d}{dx}\frac{\partial}{\partial z'}\lambda(z'-W)+\ldots=0.\end{aligned}\right\}$$

But
$$\frac{\partial}{\partial y}\lambda(z'-W)=-\lambda\frac{\partial W}{\partial y},\quad \frac{\partial}{\partial y'}\lambda(z'-W)=-\lambda\frac{\partial W}{\partial y'},\ \text{etc.}$$

$$\overline{Z}=0,\quad \frac{\partial}{\partial z}\lambda(z'-W)=0,\quad \frac{\partial}{\partial z'}\lambda(z'-W)=\lambda.$$

Hence these equations become

$$Y-Y'_{,}+Y''_{,,}-\ldots-\left\{\lambda\frac{\partial W}{\partial y}-\frac{d}{dx}\left(\lambda\frac{\partial W}{\partial y'}\right)+\ldots\right\}=0\quad\text{and}\quad -\frac{d\lambda}{dx}=0.$$

The second shows that λ does not contain x, and is a constant; and the first may then be written

$$Y - Y'_{,} + Y''_{,,} - \ldots - \lambda\left(\frac{\partial W}{\partial y} - \frac{d}{dx}\frac{\partial W}{\partial y'} + \ldots\right) = 0, \quad \textit{i.e. } [\overline{Y}] = 0,$$

where $[\overline{Y}]$ refers to the operation

$$\left(\frac{\partial}{\partial y} - \frac{d}{dx}\frac{\partial}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial}{\partial y''} - \ldots\right)$$

upon $V - \lambda W$, regarding λ as a constant, which is the rule of Art. 1504.

3. *Consider the stationary value of* $\int_a^b \frac{\int_a^x y\,dx}{y'}\,dx$. *Comparison of the two cases.* [Ohm. Todhunter, *Hist.*, p. 35.]

Let $z = \int_a^x y\,dx$. Then $z' = y$, $z'' = y'$. We may either

(i) consider $\int_a^b \frac{z}{z''}\,dx$ unconditionally,

or (ii) $\int_a^b \frac{z}{y'}\,dx$, with condition $z' - y = 0$.

(i) Here $V = \frac{z}{z''}$, $X = 0$, $Z = \frac{1}{z''}$, $Z_{,} = 0$, $Z_{,,} = -\frac{z}{z''^2}$.

The equation $V = \overline{Z}_{,} z' + \overline{Z}_{,,} z'' + C$ gives $V = (Z_{,} - Z'_{,,})z' + Z_{,,}z'' + C$, *i.e.*

$$2\frac{z}{z''} = z'\frac{d}{dx}\left(\frac{z}{z''^2}\right) + C, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

a first integral of the equation to find z as a function of x.

(ii) Or make $[\overline{Y}] = 0$, $[\overline{Z}] = 0$, with condition $L \equiv z' - y = 0$,

$$\frac{d}{dx}\left(\frac{z}{y'^2}\right) - \lambda = 0, \quad \frac{1}{y'} - \frac{d\lambda}{dx} = 0, \quad z' - y = 0.$$

Eliminating y and λ, we have

$$\frac{1}{z''} = \frac{d^2}{dx^2}\left(\frac{z}{z''^2}\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

If (1) be differentiated to eliminate C, we find a result identical with (2), and equation (1) is a first integral of equation (2). The first method has therefore carried us one step onward in the integration, whilst the second has produced the original differential equation itself.

1516. If s (or t) denote the independent variable, and x, y, z, viz. the Cartesian or other coordinates, be the dependent variables, it will be desirable to alter our notation a little in conformity with such requirements.

We take the case of three dependent variables. It will make no difference in the investigation however many there may be. Accents will denote differentiations with regard to the independent variable.

Let $$V=\phi\left(\begin{matrix} x, x', x'', \dots \\ s, y, y', y'', \dots \\ z, z', z'', \dots \end{matrix}\right),$$

and we shall write

$$\frac{\partial V}{\partial s}=S,\quad \frac{\partial V}{\partial x}=X,\quad \frac{d^r}{ds^r}\left(\frac{\partial V}{\partial x^{(n)}}\right)=X_{(n)}^{(r)},\quad \frac{\partial^r}{\partial s^r}\left(\frac{\partial V}{\partial z^{(n)}}\right)=Z_{(n)}^{(r)},\text{ etc.}$$

$$\xi^{(r)}=\delta x^{(r)}-x^{(r+1)}\delta s,\quad \eta^{(r)}=\delta y^{(r)}-y^{(r+1)}\delta s,\quad \zeta^{(r)}=\delta z^{(r)}-z^{(r+1)}\delta s,$$

$$\overline{X}=X-X_{\prime}'+X_{\prime\prime}''-\dots,\quad \overline{X}_{\prime}=X_{\prime}-X_{\prime\prime}'+X_{\prime\prime\prime}''-\dots,\text{ etc.},$$

with similar meanings for $\overline{Y}$, $\overline{Y}_{\prime}$, etc., $\overline{Z}$, $\overline{Z}_{\prime}$, etc.

Then we have, to the first order,

$$\delta\int V\,ds=[V\delta s+(\overline{X}_{\prime}\xi+\overline{X}_{\prime\prime}\xi'+\dots)+(\overline{Y}_{\prime}\eta+\overline{Y}_{\prime\prime}\eta'+\dots)$$
$$+(\overline{Z}_{\prime}\zeta+\overline{Z}_{\prime\prime}\zeta'+\dots)]+\int(\overline{X}\xi+\overline{Y}\eta+\overline{Z}\zeta)\,ds$$
$$\equiv[H]+\int(\overline{X}\xi+\overline{Y}\eta+\overline{Z}\zeta)\,ds,\text{ say, as in earlier cases.}$$

1517. As before, if it be desired to discover the functional forms of x, y, z which will be required to give $\int V\,ds$ a stationary value, we have to make the above first order variation vanish.

There are two cases to consider, (i) when x, y, z are independent of each other; (ii) when some relation $L=0$, or some set of such relations exists between them.

1518. In Case (i), in the absence of any such relation, the arbitrary variations from point to point of the path, ξ, η, ζ, are independent of each other, and we have

$$\overline{X}=0,\quad \overline{Y}=0,\quad \overline{Z}=0,$$

three differential equations, whose orders are, in general, double the order of the highest respective differential coefficients contained in V, and whose solutions severally contain the same number of arbitrary constants as their order. Secondly, there are as many equations arising from $[H]=0$, by equating to zero the *independent* terminal variations therein contained, as there are independent terminal variations.

Also, as in Art. 1500 (i), if V does not contain s explicitly, so that $S=0$, we have

$$V=(\bar{X}_{,}x'+\bar{X}_{,,}x''+\dots)+(\bar{Y}_{,}y'+\bar{Y}_{,,}y''+\dots)$$
$$+(\bar{Z}_{,}z'+\bar{Z}_{,,}z''+\dots)+C.$$

Other special cases may arise. For example, if

$$V=\phi(x, y, z, x', y', z'),$$

the independent variable being absent, we have

$$V=X_{,}x'+Y_{,}y'+Z_{,}z'+C.$$

If $V=\phi(x', y', z', x'', y'', z'')$, we have

$$V=(X_{,}-X'_{,,})x'+X_{,,}x''+(Y_{,}-Y'_{,,})y'+Y_{,,}y''$$
$$+(Z_{,}-Z'_{,,})z'+Z_{,,}z''+C;$$

and also $X_{,}-X'_{,,}=C_1,\quad Y_{,}-Y'_{,,}=C_2,\quad Z_{,}-Z'_{,,}=C_3,$

viz. the solutions of $\bar{X}\equiv -X'_{,}+X''_{,,}=0$, etc.,

so that $V=C+C_1x'+C_2y'+C_3z'+X_{,,}x''+Y_{,,}y''+Z_{,,}z''$;

and so on with other cases.

1519. In Case (ii), when there is a connecting equation $L=0$, we make $\delta\int(V+\lambda L)\,ds=0$, according to Lagrange's rule, and consider λ a function of s only.

Then $\bar{X}+\lambda\frac{\partial L}{\partial x}-\frac{d}{ds}\left(\lambda\frac{\partial L}{\partial x'}\right)+\frac{d^2}{ds^2}\left(\lambda\frac{\partial L}{\partial x''}\right)-\dots=0,$

which, with the two similar equations in y and z and the connecting equation $L=0$, give four equations from which x, y, z, λ are to be determined as functions of s.

When L contains only x, y and z, so that $L=0$ is the equation of a surface on which the path lies, these equations reduce to

$$\bar{X}+\lambda L_x=0,\quad \bar{Y}+\lambda L_y=0,\quad \bar{Z}+\lambda L_z=0,$$

i.e. $\bar{X}/L_x=\bar{Y}/L_y=\bar{Z}/L_z$, with $L=0$.

These equations could be derived otherwise, as in Art. 1511; for we have

$$L_x\delta x+L_y\delta y+L_z\delta z=0 \quad\text{and}\quad L_x dx+L_y dy+L_z dz=0;$$

and, since $\xi=\delta x-x'\delta s,\quad \eta=\delta y-y'\delta s,\quad \zeta=\delta z-z'\delta s,$

we get $L_x\xi+L_y\eta+L_z\zeta=0,$

an equation which constitutes a linear relation amongst the otherwise arbitrary variations ξ, η, ζ, which involve the four variations $\delta s, \delta x, \delta y, \delta z$.

We also have $\overline{X}\xi+\overline{Y}\eta+\overline{Z}\zeta=0$. Let one of these variations be taken such that $\zeta=0$, then $\overline{X}/L_x=\overline{Y}/L_y$. Similarly taking another variation in which $\eta=0$, then $\overline{X}/L_x=\overline{Z}/L_z$. Thus we get

$$\overline{X}/L_x=\overline{Y}/L_y=\overline{Z}/L_z, \text{ with } L=0, \text{ as before.}$$

1520. When z and its differential coefficients are absent from V and L, we obtain over again the relations of Art. 1511, viz. $\overline{X}/L_x=\overline{Y}/L_y$ and $L=0$.

1521. In any case, where we are to make $\int V\,ds$ a maximum or a minimum and s is an arc of the path and x, y, z, Cartesian coordinates of a point upon it, we have the relation

$$L\equiv x'^2+y'^2+z'^2-1=0,$$

and we may make $\int\left\{V+\frac{\lambda}{2}(x'^2+y'^2+z'^2-1)\right\}ds$ an unconditional maximum or minimum. Here $\frac{1}{2}\lambda$ has been written instead of λ for later convenience. If V be a function of x, y, z alone, not containing s explicitly, we have

$$S=\frac{1}{2}\frac{d\lambda}{ds}(x'^2+y'^2+z'^2-1), \quad [X]=\frac{\partial V}{\partial x}, \quad [Y]=\frac{\partial V}{\partial y}, \quad [Z]=\frac{\partial V}{\partial z},$$

$$[X_{,}]=\lambda x', \quad [Y_{,}]=\lambda y', \quad [Z_{,}]=\lambda z', \quad [\overline{X}]=\frac{\partial V}{\partial x}-\frac{d}{ds}(\lambda x'),$$

$$[\overline{Y}]=\text{etc.}, \quad [\overline{Z}]=\text{etc.},$$

and $$[V]=[X_{,}]x'+[Y_{,}]y'+[Z_{,}]z'+C,$$

i.e. $$V+\frac{\lambda}{2}(x'^2+y'^2+z'^2-1)=\lambda(x'^2+y'^2+z'^2)+C,$$

i.e. $$V=\lambda+C. \quad\text{.............................}(1)$$

1522. Again the terminal equations give

$$[[V]\delta s+[\overline{X}_{,}]\xi+[\overline{Y}_{,}]\eta+[\overline{Z}_{,}]\zeta]=0,$$

i.e. $$\left[\left\{V+\frac{\lambda}{2}(x'^2+y'^2+z'^2-1)\right\}\delta s+\lambda x'(\delta x-x'\,\delta s)\right.$$
$$\left.+\lambda y'(\delta y-y'\,\delta s)+\lambda z'(\delta z-z'\,\delta s)\right]=0,$$

or $$[(V-\lambda)\,\delta s+\lambda\,(x'\,\delta x+y'\,\delta y+z'\,\delta z)]=0,$$

or $$[C\,\delta s+\lambda\,(x'\,\delta x++)]=0,$$

i.e. $$C\,(\delta s_1-\delta s_0)+\Big[\lambda\,(x'\,\delta x+y'\,\delta y+z'\,\delta z)\Big]_0^1=0,$$

and therefore $C(\delta s_1 - \delta s_0) = 0$ and $\Big[\lambda(x'\delta x + y'\delta y + z'\delta z)\Big]_0^1 = 0$, for the terminal variations of s are independent of the terminal variations of x, y, z.

In isoperimetric problems, *i.e.* those concerned with an arc of specific length, $\delta s_1 - \delta s_0$ vanishes; but in other cases δs_1 and δs_0 are not necessarily equal, and then $C=0$. Thus, for isoperimetric cases, $V=\lambda+C$, and the value of C is to be determined by the length of the arc; for non-isoperimetric cases with an undefined length of arc $C=0$ and $V=\lambda$.

In either case, provided λ be not such as to vanish at either terminal, we must have $x'\delta x + y'\delta y + z'\delta z = 0$ at each terminal. That is, if the terminals are to be on specific terminal curves the path must cut each orthogonally. But if the terminals be fixed points this expression will vanish identically by virtue of the vanishing of $\delta x, \delta y, \delta z$.

Since in non-isometric problems $V=\lambda$, we may write

$$\int\Big[V + \frac{\lambda}{2}(x'^2 + y'^2 + z'^2 - 1)\Big]ds \quad \text{as} \quad \frac{1}{2}\int V(x'^2 + y'^2 + z'^2 + 1)\,ds$$

at once. (See Williamson, *I.C.*, Art. 296.)

1523. If V be any function of x, y, z alone, and $\int V\,ds$ is to be made of stationary value for curves to be discovered lying upon a given surface $\phi(x, y, z) = 0$, and with fixed terminals or fixed terminal curves, we have $\delta\int V\,ds = 0$, and we may treat the variation *ab initio* as follows.

We have $\int(\delta V\,ds + V d\,\delta s) = 0$.

But $\delta V = V_x\delta x + V_y\delta y + V_z\delta z$, and $d\,\delta s = x'd\,\delta x + y'd\delta y + z'd\,\delta z$, so that

$$\begin{aligned}\delta\int V\,ds &= \int\{(V_x\delta x + V_y\delta y + V_z\delta z)\,ds + V(x'd\,\delta x + y'd\,\delta y + z'd\,\delta z)\}\\ &= [V(x'\delta x + y'\delta y + z'\delta z)]\\ &\quad + \int\Big\{\Big(V_x - \frac{d}{ds}Vx'\Big)\delta x + \Big(V_y - \frac{d}{ds}Vy'\Big)\delta y + \Big(V_z - \frac{d}{ds}Vz'\Big)\delta z\Big\}\,ds.\end{aligned}$$

So that we must have $[V(x'\delta x+y'\delta y+z'\delta z)]=0$, as the terminal condition and

$$\left(V_x-\frac{d}{ds}Vx'\right)\delta x+\left(V_y-\frac{d}{ds}Vy'\right)\delta y+\left(V_z-\frac{d}{ds}Vz'\right)\delta z=0$$

along the path.

We also have $\phi_x\delta x+\phi_y\delta y+\phi_z\delta z=0$, a linear connection between the otherwise arbitrary point to point variations $\delta x, \delta y, \delta z$. Hence

$$\left(V_x-\frac{d}{ds}Vx'-\lambda\phi_x\right)\delta x+\left(V_y-\frac{d}{ds}Vy'-\lambda\phi_y\right)\delta y$$
$$+\left(V_z-\frac{d}{ds}Vz'-\lambda\phi_z\right)\delta z=0.$$

Now, two of the variations are arbitrary, and λ is at our choice.

Take $\delta z=0$, and choose δx not equal to 0 and $\lambda=\dfrac{V_y-\frac{d}{ds}Vy'}{\phi_y}$.

Then it follows that $V_x-\frac{d}{ds}Vx'-\lambda\phi_x=0$; and similarly we may show, by taking $\delta x=0$, that $V_z-\frac{d}{ds}Vz'-\lambda\phi_z=0$.

Thus $$\frac{V_x-\frac{d}{ds}(Vx')}{\phi_x}=\frac{V_y-\frac{d}{ds}(Vy')}{\phi_y}=\frac{V_z-\frac{d}{ds}(Vz')}{\phi_z}.$$

The terminal condition $[V(x'\delta x+y'\delta y+z'\delta z)]=0$ shows that, provided V be not zero at the terminals, *the path must cut each of the terminal curves orthogonally.*

IMPORTANT APPLICATIONS.

1524. GEODESICS.

To find the shortest line, or geodesic, on a given surface $\phi(x, y, z)=0$, from one given terminal curve to another drawn upon the surface.

Here $u=\int ds$, *i.e.* $V=\sqrt{x'^2+y'^2+z'^2}$.

Then $$X=0,\quad X_{\prime}=\frac{x'}{\sqrt{x'^2+y'^2+z'^2}},$$

$$\overline{X}=X-\frac{d}{ds}X_{\prime}=-\frac{d}{ds}x'=-x'',\quad \overline{Y}=-y'',\quad \overline{Z}=-z''.$$

Thus, by Art. 1519, $x''/\phi_x = y''\phi_y = z''/\phi_x$, *i.e.* the osculating plane at each point of the curve must contain the normal to the surface at that point.

The terminal condition is $[V\delta s + \overline{X}_{\prime}\xi + \overline{Y}_{\prime}\eta + \overline{Z}_{\prime}\zeta] = 0$,

i.e. $\qquad [\delta s + x'(\delta x - x'\delta s) + y'(\delta y - y'\delta s) + z'(\delta z - z'\delta s)] = 0$,

i.e. $\qquad [x'\delta x + y'\delta y + z'\delta z] = 0$.

Now fix one end, then $x'\delta x + y'\delta y + z'\delta z = 0$ at the other end, so that the line sought must cut the terminal curve at that end orthogonally. Similarly for the other end of the path. Thus the path must be such that

(1) the osculating plane at each point contains the normal to the surface at that point;

(2) it must cut both terminal curves orthogonally.

1525. We might, without quoting the general theorem of Art. 1519, proceed as follows, a course which is usually preferable.

Since we are to make $\delta\int\sqrt{dx^2+dy^2+dz^2} = 0$, we have

$$\int\frac{dx\,d\,\delta x + dy\,d\,\delta y + dz\,d\,\delta z}{ds} = 0\,;$$

$$\therefore\ [x'\,\delta x + y'\,\delta y + z'\,\delta z] - \int(x''\,\delta x + y''\,\delta y + z''\,\delta z)\,ds = 0\,;$$

and along the path we have

$$x''\,\delta x + y''\,\delta y + z''\,\delta z = 0, \text{ with condition } \phi_x\delta x + \phi_y\delta y + \phi_z\delta z = 0,$$

i.e $\qquad (x'' - \lambda\phi_x)\,\delta x + (y'' - \lambda\phi_y)\,\delta y + (z'' - \lambda\phi_z)\,\delta z = 0.$

Now of the three δx, δy, δz, two are independent, say δy and δz.

Let $\delta z = 0$, and take $\delta y \neq 0$; λ is at our choice; take it $= x''/\phi_x$. Then $y'' = \lambda\phi_y$. Thus $x''/\phi_x = y''/\phi_y$, and similarly $= z''/\phi_z$.

We also have the terminal condition $x'\,\delta x + y'\,\delta y + z'\,\delta z = 0$ at each end, and the path cuts the terminal curves orthogonally.

1526. **Geodesic on a Surface of Revolution.**

Let the surface be, say, $x^2 + y^2 = f(z)$, the z-axis being the axis of revolution. Then $x''/x = y''/y$, *i.e.* $xy'' - yx'' = 0$, or $xy' - yx' = \text{const.} = h$, say. Referring to cylindrical coordinates (ρ, ϕ, z), $\rho^2\phi' = h$, *i.e.* $\rho\sin\chi = h$, where χ is the angle between the path and a meridian at any point of the curve. This is the leading property of such geodesics.

1527. **Geodesics on a Quadric.**

For geodesics upon an ellipsoid we have the relation $pd = \text{const.}$, where p is the perpendicular on the tangent plane to the ellipsoid at any point on the curve and d is the semi-diameter parallel to the tangent to the curve at that point. For proof of this and for the general properties of geodesics on a quadric, see Smith, *Solid Geom.*, ch. xii.

1528. ***Required the nature of the projection upon the z-plane of geodesics upon the helicoidal surface*** $z=a\tan^{-1}y/x$.

Here $\phi=x\sin z/a-y\cos z/a=0,\quad \phi_x=\sin z/a,\quad \phi_y=-\cos z/a.$

The geodesic equations give $x''/\sin\dfrac{z}{a}=y''\Big/\left(-\cos\dfrac{z}{a}\right)$, *i.e.* $xx''+yy''=0$;

changing to cylindricals $x=\rho\cos\theta$, $y=\rho\sin\theta$, $z=a\theta$, $ds^2=d\rho^2+\rho^2\,d\theta^2+dz^2$. Then indicating differentiations with regard to θ by suffixes, and those with regard to s by accents, $s_1{}^2=\rho_1{}^2+\rho^2+a^2$, *i.e.* $s_1s_2=\rho_1\rho_2+\rho\rho_1$.

Now $\qquad \rho^2=x^2+y^2,\quad \rho\rho'=xx'+yy'.$

Hence $\qquad \rho\rho''+\rho'^2=xx''+yy''+x'^2+y'^2=x'^2+y'^2=\rho'^2+\rho^2\theta'^2$;

$$\therefore\ \rho\rho''=\rho^2\theta'^2 \text{ and } \frac{d^2\rho}{ds^2}\cdot\left(\frac{ds}{d\theta}\right)^2=\rho,\quad i.e.\ \frac{d}{d\theta}\left(\frac{d\rho}{ds}\right)=\frac{\rho}{s_1} \text{ or } \frac{d}{d\theta}\left(\frac{\rho_1}{s_1}\right)=\frac{\rho}{s_1};$$

whence $\qquad (\rho_2s_1-\rho_1s_2)/s_1{}^2=\rho/s_1,\quad$ *i.e.* $\rho_2s_1{}^2-\rho_1s_1s_2=\rho s_1{}^2$,

i.e. $(\rho_2-\rho)(\rho_1{}^2+\rho^2+a^2)=\rho_1(\rho_1\rho_2+\rho\rho_1)$ or $\rho_2(\rho^2+a^2)-2\rho\rho_1{}^2=\rho(\rho^2+a^2)$.

Let $\rho=a\cot\chi$, then $\rho_1=-a\operatorname{cosec}^2\chi\dfrac{d\chi}{d\theta}$; $\quad\therefore\ \dfrac{d}{d\theta}\left(\dfrac{d\chi}{d\theta}\right)=-\sin\chi\cos\chi$;

$$\therefore\ \left(\frac{d\chi}{d\theta}\right)^2=-\sin^2\chi+\frac{1}{k^2},\quad \text{where } \frac{1}{k^2} \text{ is a constant} >1;$$

$\therefore\ \dfrac{d\theta}{k}=\dfrac{d\chi}{\sqrt{1-k^2\sin^2\chi}}$ and $\chi=\operatorname{am}\left(\dfrac{\theta}{k}+\alpha\right)$, where α is a second arbitrary constant. Hence the projection of the geodesics on the z-plane has an equation of the form $r=a\operatorname{ctn}\left(\dfrac{\theta}{k}+\alpha\right)$, mod. k, k and α being constants depending upon the position of the terminals.

The reader will have no difficulty in showing that if ϕ be the angle which the tangent at any point of the geodesic makes with the generator at this point, and ψ the angle the normal to the surface makes with the axis of the helicoid, then $\sin\phi=k\sin\psi$; and hence that if $A_1A_2A_3\ldots$ be any closed geodesic polygon drawn upon the surface, and ϕ_r, ϕ_r' be the angles which A_rA_{r-1}, A_rA_{r+1} make with the generator through A_r, then $\Pi\sin\phi_r=\Pi\sin\phi_r'$.

1529. *Suppose we are to obtain the stationary value of*

$$\int\sqrt{E+2Fy'+Gy'^2}\,dx,$$

where E, F, G are known functions of the variables x and y.

Here $\qquad Y=\dfrac{E_y+2F_yy'+G_yy'^2}{2V},\quad Y_{\prime}=\dfrac{F+Gy'}{V},$

where suffixes denote partial differentiations.

The differential equation to be satisfied is $\overline{Y} \equiv Y - Y'_{\prime} = 0$,

i.e. $$\frac{E_y + 2F_y y' + G_y y'^2}{2V} = \frac{d}{dx}\frac{F + Gy'}{V}.$$

After differentiation and considerable reduction, this leads to an equation

$$A + By' + Cy'^2 + Dy'^3 + 2(F^2 - EG)y'' = 0, \quad \text{..........(1)}$$

where $A = EE_y - 2EF_x + FE_x$, $B = 3FE_y - 2EG_x - 2FF_x + GE_x$,

$C = -3FG_x + 2GE_y + 2FF_y - EG_y$, $D = -GG_x + 2GF_y - FG_y$,

for the terms in y'^4, $y'y''$, y'^2y'' all cancel out.

The equation (1) is incapable of *general* solution, but many cases arise in which at least a first integration may be effected, and sometimes the complete integration.

1530. (i) For instance, if E, F and G be constants, $A = B = C = D = 0$, and the solution is that of $y'' = 0$, *i.e.* a straight line.

(ii) If $E = G = L - M$ where L is a function of x alone and M a function of y alone, and if $F = 0$,

$$A = (L - M)(-M_y), \quad B = -(L - M)L_x,$$

$$C = (L - M)(-M_y), \quad D = -(L - M)L_x,$$

and equation (1) becomes

$$2(L - M)y'' + (1 + y'^2)(M_y + y'L_x) = 0$$

or $$\frac{2y'y''}{1 + y'^2} - \frac{L_x - M_y y'}{L - M} + \frac{L_x(1 + y'^2)}{L - M} = 0,$$

i.e. $$\frac{d}{dx}[\log(1 + y'^2) - \log(L - M)] + L_x \frac{1 + y'^2}{L - M} = 0;$$

or putting $\dfrac{1 + y'^2}{L - M} = z$, $\dfrac{1}{z}\dfrac{d}{dx}\log z + L_x = 0$, whence $\dfrac{1}{z^2}\dfrac{dz}{dx} + L_x = 0$.

Hence a first integral is $\dfrac{L - M}{1 + y'^2} - L = -\lambda$, *i.e.* $y'^2 = \dfrac{M - \lambda}{\lambda - L}$,

i.e. $$\int \frac{dx}{\sqrt{\lambda - L}} = \int \frac{dy}{\sqrt{M - \lambda}} + \text{const., a second integral,}$$

for by supposition L is a function of x alone and M a function of y alone, so that the variables are "separable" in such cases.

1531. The case of Art. 1529 is an important one, for it will be remembered that if the coordinates of a point upon a surface be expressed in terms of two parameters u and v, the element of arc may be expressed in the form $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$.

Hence the determination of a geodesic upon the surface depends upon the possibility of integrating the differential equation (1).

1532. The direct investigation of the geodesic may be sometimes effected by a transformation. For example, if the square of the linear element on a surface be given by $ds^2 = \dfrac{(1-v^2)\,du^2 + (1-u^2)\,dv^2 + 2uv\,du\,dv}{1-u^2-v^2}$, let us take a third variable w such that $u^2+v^2+w^2=1$, whence

$$u\,du + v\,dv + w\,dw = 0.$$

Then $\quad ds^2 = \{(u^2+w^2)\,du^2 + (v^2+w^2)\,dv^2 + 2uv\,du\,dv\}/w^2$

$$= \{(u\,du+v\,dv)^2 + w^2(du^2+dv^2)\}/w^2 = du^2+dv^2+dw^2,$$

so $s = \int \sqrt{du^2+dv^2+dw^2}$, with condition $u^2+v^2+w^2=1$.

That is, the arc of the curve on the original surface is the same length as the corresponding arc of a corresponding curve on the unit sphere in a system of rectangular coordinates u, v, w. And the geodesics on the sphere are given by the great circles, *i.e.* by equations of the form $au+bv+cw+0$; hence the geodesics on the original surface are given by $au+bv+c\sqrt{1-u^2-v^2}=0$, where a, b, c are constants.

1533. **Principle of Least Action.**

Suppose a particle of mass m to be in motion under the action of any conservative system of forces and either to be moving freely or under compulsion to remain on a smooth surface from any one point to any other point. Then, if v be the velocity at any time t, and ds an element of the path, we shall show that the integral $m\int v\,ds$ *has a stationary value.*

The quantity A defined as $m\int v\,ds$ is called the Action, or the Characteristic Function, by Sir W. R. Hamilton, and the principle is known as the Principle of Least Action.

1534. If X, Y, Z be the force components per unit mass, R the normal pressure exerted by the surface, if any pressure exist, and λ, μ, ν the direction cosines of the normal, the ordinary equations of motion are

$$\ddot{x} = X + R\lambda, \quad \ddot{y} = Y + R\mu, \quad \ddot{z} = Z + R\nu,$$

and the energy equation is

$$m\frac{v^2}{2} = m\int (X\,dx + Y\,dy + Z\,dz) = m\chi(x, y, z) \text{ say,}$$

for the expression $X\,dx + Y\,dy + Z\,dz$ satisfies the condition of integrability, since the forces form a conservative system, *i.e.* are such as occur in nature.

Hence, we have $\qquad v\,\delta v = X\,\delta x + Y\,\delta y + Z\,\delta z.$

But we also have $ds^2 = dx^2 + dy^2 + dz^2$, so that $\dot{s}\, d\delta s = \dot{x}\, d\delta x + \dot{y}\, d\delta y + \dot{z}\, d\delta z$, and the variation in A, *i.e.* $\delta A = m\delta \int v\, ds = m \int (\delta v\, ds + v\, d\delta s)$

$$= m \int \{(X\,\delta x + Y\,\delta y + Z\,\delta z)\, dt + \dot{x}\, d\delta x + \dot{y}\, d\delta y + \dot{z}\, d\delta z\}$$

$$= m[\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\, dz] + m \int \{(X - \ddot{x})\,\delta x + (Y - \ddot{y})\,\delta y + (Z - \ddot{z})\,\delta z\}\, dt$$

$$= m[\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\,\delta z] - m \int R(\lambda\,\delta x + \mu\,\delta y + \nu\,\delta z)\, dt,$$

and since the direction defined by λ, μ, ν, *i.e.* the normal to the surface, is necessarily perpendicular to any displacement $\delta x, \delta y, \delta z$ on the surface, $\lambda\,\delta x + \mu\,\delta y + \nu\,\delta z$ vanishes, as also does each of the terminal values of $\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\,\delta z$.

So that the variation of A is zero and the "action" has a stationary value. Conversely, if we assume that if $\int v\, ds$ has a stationary value, we can establish the general equations of motion of the particle.

1535. It follows of course that if X, Y, Z be all zero, *i.e.* if the particle be in motion on a smooth surface under the action of no forces except those due to the constraint of the surface, then v is constant, as shown by the energy equation, and $\int v\, ds$ being of stationary value, so also is $\int ds$. That is, the particle searches out for itself and travels along a geodesic on the surface. (See Tait and Steele, *Dyn. of a Particle*, Art. 233, also Routh, *Dyn. of a Particle*.)

1536. Path of a Ray of Light in a Heterogeneous Medium.

When a ray of light travels in a medium of variable refractive index μ from one point to another, it does so in such a manner as to make $\int \mu\, ds$ a minimum. It is required to deduce the equations of the path of the ray.

This case is similar to the one just discussed.

We have $\quad \delta \int \mu\, ds = 0, \quad$ *i.e.* $\int (\delta\mu\, ds + \mu\, d\delta s) = 0,$

and $\quad ds\, d\delta s = dx\, d\delta x + dy\, d\delta y + dz\, d\delta z\,;$

$$\therefore \int \{\delta\mu\, ds + \mu(x'\, d\delta x + y'\, d\delta y + z'\, d\delta z)\} = 0,$$

and $\quad \delta\mu = \mu_x \delta x + \mu_y \delta y + \mu_z \delta z.$

Hence $\quad [\mu x'\,\delta x + \mu y'\,\delta y + \mu z'\,\delta z]$

$$+ \int \left[\left\{ \mu_x - \frac{d}{ds}\left(\mu \frac{dx}{ds}\right) \right\} \delta x + \left\{ \mu_y - \frac{d}{ds}\left(\mu \frac{dy}{ds}\right) \right\} \delta y + \left\{ \mu_z - \frac{d}{ds}\left(\mu \frac{dz}{ds}\right) \right\} \delta z \right] ds = 0\,;$$

and since the ray is to pass from one definite point to another, the integrated portion vanishes at each terminal, and the variations δx,

δy, δz under the integral sign being arbitrary from point to point, we must have also

$$\frac{\partial\mu}{\partial x}=\frac{d}{ds}\left(\mu\frac{dx}{ds}\right),\quad \frac{\partial\mu}{\partial y}=\frac{d}{ds}\left(\mu\frac{dy}{ds}\right),\quad \frac{\partial\mu}{\partial z}=\frac{d}{ds}\left(\mu\frac{dz}{ds}\right),$$

which form the differential equations of the path of the ray.

1537. **Brachistochronism. The General Problem.**

A particle is in motion under the action of a given conservative system of forces. It is required to find the path along which it must be constrained to move so as to accomplish that path from one given point to another, or from one given surface to another, in the shortest time. Such constrained paths are called Brachistochrones. The case of brachistochronism under the action of gravity has already been considered.

Let $m\phi(x, y, z)$ be the potential energy of the force system, m being the mass of the particle.

Then the energy equation gives $\frac{1}{2}v^2+\phi(x, y, z)=\text{const.}$

The force-components per unit mass are $-\phi_x$, $-\phi_y$, $-\phi_z$, being the rates of decrease of potential energy. By varying v, we have

$$v\,\delta v+\phi_x\delta x+\phi_y\delta y+\phi_z\delta z=0.$$

Also $ds\,d\delta s=dx\,d\delta x+dy\,d\delta y+dz\,d\delta z$, *i.e.* $d\delta s=x'd\delta x+y'd\delta y+z'd\delta z$.

Now we are to make $t\equiv\int\frac{ds}{v}$ a minimum.

So $$\delta t=\delta\int\frac{ds}{v}=\int\left(\frac{d\delta s}{v}-\frac{1}{v^2}ds\,\delta v\right)=0.$$

Therefore $$\int\left\{\frac{1}{v}(x'd\delta x++)\right\}+\int\left\{\frac{1}{v^3}(\phi_x\delta x++)\right\}ds=0,$$

i.e. $$\left[\frac{x'\delta x++}{v}\right]+\int\left[\left\{\frac{\phi_x}{v^3}-\frac{d}{ds}\left(\frac{x'}{v}\right)\right\}\delta x++\right]ds=0,$$

and δx, δy, δz are arbitrary all along the path and independent of each other, and of the variations at the terminals. Hence

$$\left[\frac{x'\delta x+y'\delta y+z'\delta z}{v}\right]=0\quad\text{and}\quad \frac{d}{ds}\left(\frac{x'}{v}\right)=\frac{\phi_x}{v^3},\quad \frac{d}{ds}\left(\frac{y'}{v}\right)=\frac{\phi_y}{v^3},\quad \frac{d}{ds}\left(\frac{z'}{v}\right)=\frac{\phi_z}{v^3}.$$

1538. **The Terminal Conditions.**

If the terminals be fixed points, the expression in square brackets vanishes identically at each end of the path.

If the path start from a fixed point (x_0, y_0, z_0) and terminate at the surface $F(x, y, z)=0$, then δx, δy, δz vanish at the starting point, and provided the velocity be not infinite at the other terminal $x'\delta x+y'\delta y+z'\delta z$ must vanish there; that is, the path must cut the surface $F(x, y, z)=0$ orthogonally, for the only admissible variations δx, δy, δz at this end are such as lie on the surface.

If the path start from a point x_0, y_0, z_0, which is only defined as lying upon a surface $F_0(x, y, z)=0$, a similar result will hold, provided that the whole energy of the system be a given quantity, and that $F_0=0$ be an equipotential surface of the force system. If the surface $F_0=0$ were not an equipotential surface, terms depending on δx_0, δy_0, δz_0 would make their appearance in the integral, and such terms if existent would have to be included with the rest of the terminal terms.

If the motion terminate at a given curve instead of at a given surface, the terminal conditions may be discussed in a similar manner.

1539. **The Normal Pressure in the Case of Brachistochronous Description.**

From the general equations $\frac{d}{ds}\left(\frac{1}{v}\frac{dx}{ds}\right)=\frac{\phi_x}{v^3}$, etc., which may be written

$$v^2x''-vv'x'-\phi_x=0, \text{ etc.},$$

we have, by eliminating v^2 and vv',

$$\begin{vmatrix} x'', & x', & \phi_x \\ y'', & y', & \phi_y \\ z'', & z', & \phi_z \end{vmatrix}=0,$$

so that the resultant force at any point lies in the osculating plane of the curve.

Moreover, multiplying the equations $v^2x''-vv'x'-\phi_x=0$, etc, by $\rho x''$, $\rho y''$, $\rho z''$ respectively, ρ being the radius of absolute curvature, we have by addition $v^2/\rho=\phi_x\rho x''+\phi_y\rho y''+\phi_z\rho z''=-N$, where N is the normal force component.

If, however, R be the pressure per unit mass upon the curve, the normal resolution gives the equation $v^2/\rho=N+R$.

Hence $R=-2N$. That is, the pressure upon the curve is equal to twice the normal component of the forces, and acts in the opposite direction.

Now for a free path under a conservative system of forces for which the components in the direction of the tangent and principal normal are T and N', there being no component in the direction of the binormal, we have $\frac{v\,dv}{ds}=T$ and $\frac{v^2}{\rho}=N'$, whilst for the same path to be brachistochronous under frictionless constraint under the action of a corresponding set of forces whose components are T, N, 0, we have $\frac{v\,dv}{ds}=T$ and $\frac{v^2}{\rho}=-N$ (*i.e.* $=N+R$ where $R=-2N$).

1540. Hence we have Townsend's theorem : "If for the same velocity of description any curve, plane or twisted, be at once a free path for one conservative system of forces and a brachistochronous path under frictionless constraint for another conservative system of forces, the resultants of the two force systems must at every point of the curve be reflexions of each other as regards both magnitude and direction with respect to the current tangent at the point."

1541. The principal cases are :

(*a*) When the motion is under a single force in a given direction.

(*b*) When the force tends to or from a fixed point.

1542. Case (a). Force in a Given Direction.

Take the y-axis parallel to this direction. Let m be the mass of the particle, $mF(y)$ the potential energy. The force to increase y, being the rate of decrease of potential energy, is $-mF'(y)$. The pressure on the curve is $R \equiv 2mF'(y)\cos\psi$, ψ being the inclination of the tangent to the x-axis.

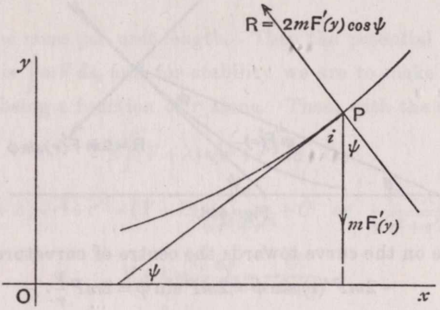

Fig. 447.

Let $y=a$ be the line of zero velocity; then we have $\frac{1}{2}v^2+F(y)=F(a)$, and $v^2/\rho=F'(y)\cos\psi$.

Hence $$\frac{v^2}{\rho\cos\psi}=F'(y)=-\frac{v\,dv}{dy},$$

i.e. $$\frac{1}{v}\frac{dv}{ds}=-\frac{dy}{ds}\cdot\frac{1}{\rho\cos\psi}=-\tan\psi\frac{d\psi}{ds},$$

whence $v=u\cos\psi$, where u is the value of v when $\psi=0$.

Also the y-ψ equation of the brachistochrone is $\frac{1}{2}u^2\cos^2\psi=F(a)-F(y)$. It is convenient to use the angle i, the angle between the ordinate and the current tangent, in place of ψ, and $\iota=\frac{\pi}{2}-\psi$.

Then the law of force necessary for brachistochronism is given by $P\equiv\frac{u^2}{2}\frac{d}{dy}(\sin^2\iota)$, per unit mass, repulsive from the x-axis, with a line of zero velocity found by the vanishing of i. Also the pressure upon the curve is $R=2mF'(y)\cos\psi=-2mP\cos\psi$ towards the centre of curvature.

1543. Case (b). Central Force.

Take the origin at the centre of force. Let $mF(r)$ be the potential energy. The radial force from the origin is $-mF'(r)$ and $R=2mF'(r)\sin\phi$, where ϕ is the angle between the tangent and the radius vector. Let a be the radius of the circle of zero velocity.

Then $\frac{1}{2}v^2+F(r)=F(a)$ and $v^2/\rho=-F'(r)\sin\phi$.

Hence $$\frac{v^2}{\rho\sin\phi}=-F'(r)=\frac{v\,dv}{dr};\quad i.e.\ \frac{1}{v}\frac{dv}{dr}=\frac{dp}{r\,dr}\cdot\frac{r}{p}=\frac{1}{p}\frac{dp}{dr}.$$

Therefore $v/p = \text{const.} = h$, say. Whence the pedal equation of the brachistochrone is $\frac{1}{2}h^2p^2 + F(r) = F(a)$, and the law of force is $P = \frac{h^2}{2}\frac{dp^2}{dr}$, repulsive from the origin, with a circle of zero velocity whose radius is to be obtained by the vanishing of p.

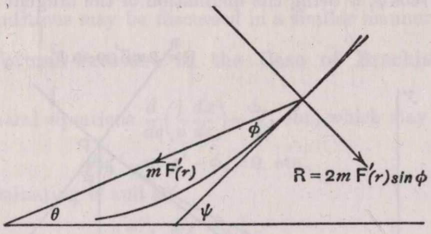

Fig. 448.

The pressure on the curve towards the centre of curvature is

$$-2mF'(r)\sin\phi = 2mP\sin\phi = 2mP\frac{p}{r}.$$

1544. Comparison of Analogous Results.

It is worth while for the student to note that

(a) For parallel forces:

(i) for a free path $\int v\,ds = \text{min.}$, $v\cos\psi = u$ (a constant);

(ii) for brachistochrone $\int \frac{ds}{v} = \text{min.}$, $v/\cos\psi = u$.

(b) For central forces:

(i) for free path $\int v\,ds = \text{min.}$, $vp = h$ (constant);

(ii) for brachistochrone $\int \frac{ds}{v} = \text{min.}$, $v/p = h$.

Compare the following laws of central force for various circumstances:

(a) Particle in free motion $P = \frac{h^2}{p^3}\frac{dp}{dr}$, $pv = h$.

(b) Particle in brachistochronous motion $P = h^2p\frac{dp}{dr}$, $v/p = h$.

(c) Equilibrium of inextensible string $P = \frac{h}{p^2}\frac{dp}{dr}$, $Tp = h$.

(d) Equilibrium of extensible string $P = \frac{h}{p^2}\frac{dp}{dr} + \lambda\frac{h^2}{p^3}\frac{dp}{dr}$, $Tp = h$.

1545. Energy Condition for an Equilibrating System.

If V be the potential energy of a field of force in which any system of material particles has assumed a position of equilibrium, it is known that the configurations of stability and instability are those of minimum or maximum values of V.

Cases in which a stationary value of V occurs without a true maximum or minimum give neutral equilibrium, in which there may be stability for some displacements, instability for others. The Calculus of Variations supplies a very powerful instrument for the discussion of such problems.

1546. Ex. *An inelastic string of uniform density and length l is attached to two fixed points A and B. Find the condition that it disposes itself in a curve of specified shape under the action of a central force in a field of potential V.*

Let m be the mass per unit length. Then the potential energy of the whole string is $\int mV\,ds$, and for stability we are to make $\int (V+\lambda)\,ds$ a minimum, V being a function of r alone. Then, with the usual notation of polars,

$$\delta\int (V+\lambda)\sqrt{r^2+r'^2}\,d\theta=0;$$

$$\therefore\ (V+\lambda)\sqrt{r^2+r'^2}=(V+\lambda)\frac{r'^2}{\sqrt{r^2+r'^2}}+C \quad\text{or}\quad \frac{V+\lambda}{\sqrt{r^2+r'^2}}=\frac{C}{r^2}.$$

Hence $$V+\lambda=\frac{C}{r^2}\frac{ds}{d\theta}=\frac{C}{r\sin\phi},$$

ϕ being the angle between the tangent and the radius vector, *i.e.*

$$V+\lambda=\frac{C}{p}, \quad\text{.................................}(1)$$

C being a constant.

This gives the law of potential of the field of force.

Thus P (viz. the repulsive force from the pole) $=-\dfrac{dV}{dr}=\dfrac{C}{p^2}\dfrac{dp}{dr}$. ....(2)

V being supposed a known function of r, we now have a relation from (1) in terms of r, θ, λ, C, and another constant which will be introduced when we have integrated equation (1) to get that relation into the r, θ form. Two of the equations to determine the three constants will be obtained by making the curve pass through the terminal points; the other is provided by making

$$\int_A^B \sqrt{r^2+r'^2}\,d\theta=l.$$

If T be the tension, a resolution along the normal gives

$$\frac{T\,ds}{\rho}=P\,ds\sin\phi=P\,ds\,.\,\frac{p}{r},$$

i.e. $$Tp=P\,.\,\frac{p^2}{r}\frac{r\,dr}{dp}=C, \quad \textit{i.e. } T=V+\lambda.$$

That Tp is constant is also obvious by taking moments about the centre of force for any portion of the string. (See Art. 1544.)

Taking the more general case of a string of length l, attached to two given points A, B, and of variable line-density ρ, which is a function of s, the arcual distance of any point from A, and constrained to lie upon a given smooth surface f(x, y, z)=0, and in a field of force of which the potential is V, now a function of x, y, z, we are to make

$$u\equiv\int[\rho V+\lambda f(x,y,z)+\tfrac{1}{2}\mu(x'^2+y'^2+z'^2-1)]\,ds,$$

a minimum, λ and μ being functions of s alone, to be determined so that $x'^2+y'^2+z'^2=1$ and that $f(x, y, z)=0$.

The terminals being fixed, we vary x, y, z alone, keeping s constant.

Then $\delta u=\int\left[\rho(V_x\delta x++)+\lambda(f_x\delta x++)+\mu\left(x'\frac{d}{ds}\delta x++\right)\right]ds.$

The terms of the third group may be integrated by parts.

$$\int\left(\mu x'\frac{d}{ds}\delta x\right)ds=[\mu x'\,\delta x]-\int\left\{\frac{d}{ds}(\mu x')\,\delta x\right\}ds, \text{ etc.}$$

Hence, for a minimum, we have

$$\rho V_x+\lambda f_x-\frac{d}{ds}(\mu x')=0,$$

with two similar equations.

These three equations, combined with $x'^2++=1$ and $f(x, y, z)=0$, are sufficient to determine λ, μ, x, y, z in terms of s.

PROBLEMS.

1. Given that (x_1, y_1), (x_2, y_2) are two points movable in a plane, and such that their distance apart is always equal to a definite constant a, what must be the circumstances of the motion in order that we shall always have

$$x_1\delta x_1+x_2\delta x_2+y_1\delta y_1+y_2\delta y_2=0\,?$$

[De Morgan, *D.C.*, p. 455.]

2. Prove that to the first order the variation of the integral $\int f(x, y, p)\,dx$, with constant limits, is $\int\omega\left\{\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial p}\right)\right\}dx$, where

$$\omega\equiv\delta y-p\,\delta x \quad\text{and}\quad p=\frac{dy}{dx}.$$

Determine a curve joining the origin to the point $(a, 1)$ for which the integral $\int(p^2+n^2y^2)\,dx$ has a minimum value.

[Math. Trip., 1896.]

3. Prove that the shortest time path between two curves which lie in one plane when the velocity varies as the distance from a line in that plane, is the arc of a circle cutting the curves orthogonally, and having its centre on the line. [Colleges γ, 1893.]

4. Find the relation between y and p in a curve which makes $\int y^n\sqrt{1+p^2}\,dx$ a maximum. Obtain the polar equation of the curve whose pole will generate this by rolling on a straight line.

[Colleges, 1877.]

5. A particle is moving under the action of a force perpendicular to and proportional to the distance from the line of zero velocity. Show that the brachistochrone is a circle. [TOWNSEND.]

6. Find the law of force parallel to the y-axis for which each of the following curves is brachistochronous, stating in each case the line of zero velocity and the pressure upon the curve:

Curve.	x-axis.	Curve.	x-axis.
1. Circle,	diameter.	2. Parabola,	directrix.
3. Parabola,	axis.	4. Catenary,	directrix.
5. Tractrix,	directrix.	6. Evolute of Parabola,	axis.
7. Evolute of Catenary,	directrix.	8. Four-cusped hypocycloid,	line of opposite cusps.
9. Rect. Hyp.,	asymptote.	10. Bifocal conic,	axis.

[TOWNSEND.]

7. Find the law of central force for which each of the following curves is brachistochronous, stating whether the force is attractive or repulsive, the radius of the circle of zero velocity, and the pressure on the curve in each case:

Curve.	Pole.	Curve.	Pole.
1. Parabola,	focus.	2. Equiang. Spiral,	pole.
3. Cardioide,	pole.	4. Circle,	point on circumf.
5. Lemiscate of Bernoulli,	node.	6. Rect. Hyp.,	centre.
7. $r^n = a^n \cos n\theta$,	pole.	8. Invol. of Circle,	centre.
9. Epi- or hypocycloid,	cent. of fixed circle.	10. Reciprocal Spiral,	pole.
11. Central Conic,	centre.	12. Central Conic,	focus.

[TOWNSEND.]

8. Show that the curve of quickest descent under gravity from a given point to a given vertical straight line is a complete semicycloid with cusp at the given point.

9. Determine the minimum value of $\int_0^1 \left(\frac{dy}{dx}\right)^2 dx$, having given that

$$y_0 = 1 \quad \text{and} \quad \int_0^1 \frac{y}{y_1}\, dx = -1,$$

where y_0, y_1 are the values of y at the lower and upper limits respectively, and y_1 is subject to variation.

[ST. JOHN'S, 1883; TODHUNTER, *Hist. of Calc. Var.*]

10. Find the equation of a curve such that the area between it and the x-axis has a given value, whilst the area of the surface of revolution, bounded by it when revolving about the x-axis, is a minimum. [OXF. II. P., 1880.]

11. A piece of string of given length in the plane of the curve $ax^2 = y^3$, has its two ends movable on the two branches of the curve; find the form of the string when the area between the string and the curve is a maximum, and when that is the case prove that the string at each of its ends is at right angles to the curve.

[ST. JOHN'S, 1889.]

12. A surface of revolution has a given area, and its generating curve intersects the axis in given points; determine the form of the surface so that its volume may be greatest. [γ, 1899.]

13. Show how to connect two fixed points by a curve of given length, so that the area bounded by the curve, the ordinates of the fixed points and the axis of abscissae shall be a minimum.

[MATH. TRIP., 1887.]

14. Find the curve in which at every point

$$\left\{y + (m - x)\frac{dy}{dx}\right\}\left\{y + (n - x)\frac{dy}{dx}\right\}$$

is a maximum or a minimum. Interpret this problem geometrically.

[LACROIX, *Calc. Diff.*, II., p. 689.]

15. Prove by means of the Calculus of Variations that the minimum value of $\int_{x_0}^{x_1}(a-x)^2\left(\frac{dy}{dx}\right)^2 dx$ is $(y_1 - y_0)^2(a - x_1)(a - x_0)/(x_1 - x_0)$, where y_0, y_1 are the values of y corresponding respectively to the initial and final values of x, and supposing that $\frac{dy}{dx}$ does not become infinite between the limits.

[OXF. II. P., 1885.]

16. Find what functions of x, satisfying the conditions $y = 0$, when $x = 0$ and when $x = l$, make $\int_0^l \left(\frac{dy}{dx}\right)^2 dx$ stationary in value when $\int_0^l y^2 dx$ is given. [MATH. TRIP., 1876.]

17. Show that the equation in polar coordinates to the plane curve of given length, for which $\int \frac{ds}{r}$ is a maximum or minimum, is of one of the forms

$$\frac{a}{r} = \sqrt{1 - m^2} - \cos m(\theta - \alpha), \quad \frac{a}{r} = \cosh m(\theta - \alpha) - \sqrt{1 + m^2}.$$

[OXF. II. P., 1890.]

18. A lamina of given mass is symmetrical with respect to an axis, and its density at any point varies as the square of the abscissa measured from one end of its axis; if the attraction upon a particle at that point of the axis be a maximum, prove that the lamina is bounded by the oval $r^2=\sqrt{\dfrac{32m}{3\pi\sigma}}\cos\theta$, where m is the given mass and σ the density at unit distance along the axis, assuming the law of attraction to be that of the inverse square of the distance.

[MATH. TRIP., 1875.]

19. A curve passing through the point whose polar coordinates are a, $a\cos^{-1}e$, is such that $\int\{2r^{-1}-a^{-1}\}^{\frac{1}{2}}ds$, taken along the arc of the curve between the initial line and the given point, is a minimum. Prove that, provided that $2r^{-1}-a^{-1}$ is always finite and greater than zero, the required curve cuts the initial line at right angles in two points, the sum of whose distances from the origin is $2a$; and find the equation of the curve. [OXF. II. P., 1903.]

Interpret the result dynamically.

20. If $\int\sqrt{\lambda+\mu p^2}\,dx$ has a maximum or minimum, and λ, μ are independent of p and of any higher differential coefficients, and the differential equation resulting is satisfied by $y=ax+b$ for all constant values of a and b, prove that λ and μ must be mere constants. [OXF. II. P., 1918.]

21. A swimmer who can swim at a given rate v starts from the bank of a wide straight river, and the strength of the current varies directly as the distance from the bank. He wishes to get as far down the river as he can in a given time T. Show that he must start from the bank at an angle whose tangent is proportional to T. Show also that the tangents of the angles his direction of swimming makes with the bank at equal intervals of time are in arithmetical progression, and that at the end of the time T he is swimming directly down stream. If the x-axis be taken along the river bank, μy the velocity of the stream and α his initial angle with the bank, show that he is ultimately swimming at a distance $2v\sec^2\dfrac{\alpha}{2}\Big/\mu\cos\alpha$ from the bank.

22. An oval curve of given length rolls on a straight line; find its form when the area traced out in one revolution by a given

point on the plane of the curve is a minimum, the boundaries of the area being the curve traced out by the moving point, the given straight line and two perpendiculars upon it from the extremities of the curve. [MATH. TRIP., 1870.]

23. If the velocity of a carriage along a road be proportional to the cube of the cosine of the inclination of the road to the horizon, determine the path of quickest ascent from the bottom to the top of a hemispherical hill, and show that it consists of the spherical curve described by a point of a great circle which rolls on a small circle described about the pole with a radius $\pi/6$, together with an arc of a great circle. How is the discontinuity introduced into this problem? [MATH. TRIP., 1873.]

24. If $r^2 = x^2 + y^2$ and $ds^2 = dx^2 + dy^2$, prove, assuming such results of theory as may be convenient, that the curves along which from point to point $\int r\,ds$ is a maximum or minimum are rectangular hyperbolae. [OXF. II. P., 1886.]

25. Find the curve of given length joining two fixed points, which is such that the distance of the centroid of the arc from the chord connecting the two points may be the greatest possible. [OXF. II. P., 1887.]

26. A variable curve of given length $\pi a\sqrt{2}/4$ has one extremity at a fixed point $(3a, a)$ and the other on a fixed line $x = 2a$. Show that when the area enclosed by the curve, the axis of x and the lines $x = 2a$, $x = 3a$, is a maximum the curve is one-eighth of a circle. [OXF. II. P., 1888.]

27. On the surface of an ellipsoid a curve is drawn which intersects at a constant angle all the geodesics passing through a given umbilic. Prove that its total length from umbilic to umbilic is $l \sec \alpha$, where l is the geodesic distance between that umbilic and the opposite one. [MATH. TRIP. I., 1888.]

28. Find the form of the function p, in order that $\int\left(p + \frac{d^2p}{d\psi^2}\right) p\,d\psi$ may be a maximum, subject to the condition that $\int\left(p + \frac{d^2p}{d\psi^2}\right) d\psi$ is constant, and interpret the result geometrically. [OXF. II. P., 1889.]

29. A man swims from a point on the bank of a straight river to a point in mid-stream, with a constant velocity relative to the water.

Prove that, in order that the passage may occupy the shortest time, his actual course must be straight if the strength of the current is constant, but that if the strength of the current is proportional to the distance from the bank the path must have for its equation

$$y = c\sqrt{(cb+x)^2 - b^2} - \frac{cb}{2}\sqrt{c^2-1} - \frac{(cb+x)\sqrt{(cb+x)^2-b^2}}{2b}$$
$$+\frac{b}{2}\cosh^{-1}\frac{cb+x}{b} - \frac{b}{2}\cosh^{-1}c,$$

where the starting point is the origin, the bank is the axis of y, b the distance from the bank where the velocity of the stream is equal to that of the man relative to the water, and c is a constant. How is c obtained? [COLLEGES, 1896.]

30. Apply the principle of energy to determine the equation of equilibrium of an inextensible string under the action of a central force, its ends being fixed. [ST. JOHN'S, 1881.]

31. A heavy particle moves on the surface of a smooth circular cone with a vertical axis and vertex upwards. Find the brachistochrone from a fixed point on the surface to a fixed generating line. [ST. JOHN'S, 1881.]

32. Show that the curve, such that $\int r^n\,ds$ between two fixed points in the plane of the curve may be a minimum, is $r^{n+1} = a^{n+1}\sec(n+1)\theta$. [TRIN. COLL., 1881.]

33. A man walks up a uniform incline from a given point to reach a given height. His velocity varies as the sine of the angle between his path and the line of greatest slope on the incline. If he exhausts himself at a rate proportional to the product of the whole height ascended, and the square of the cosine of the inclination of his path to the line of greatest slope, show that he will get to the required height with least exertion along a curve whose equation is

$$y^3 = ax^2.$$ [ST. JOHN'S COLL., 1883.]

34. Prove that the minimum value of $\int (xy\,dx\,dy)^{\frac{1}{2}}$ between the limits $x=a$, $y=b$ and $x=a'$, $y=b'$ is equal to $\frac{1}{2}(a'^2-a^2)^{\frac{1}{2}}(b'^2-b^2)^{\frac{1}{2}}$.

35. A curve is drawn on the surface $x(y+z)=a^2$ such that $\int\frac{ds}{x^2}$ is a maximum or a minimum; prove that $\left(\frac{ds}{dx}\right)^2 = \frac{c^4}{x^4}\,\frac{2x^4+a^4}{2c^4-x^4}$, c being an arbitrary constant. [ST. JOHN'S COLL., 1882.]

36. Show that the surface, whose superficial area is given and which encloses the greatest possible volume between itself and a given plane, has the sum of its curvatures at every point constant.

[MATH. TRIP., 1888.]

37. Geodesics are drawn upon the surface formed by the revolution of the curve $x=2a\sec u$, $y=a(\sec u\tan u-\cosh^{-1}\sec u)$ about the y-axis. Show that the projections of these geodesics upon a plane perpendicular to the axis of revolution are of the forms of the inverses with regard to the origin of a certain Cotes's spiral.

38. Show that if S, H be two fixed points at distance apart $2a$, and O the mid-point of SH, the law of repulsive force from O under which the curve $SP\,.\,HP=c^2$ can be described in a brachistochronous manner is one varying as $(OP^4+d^4)(3OP^4-d^4)/OP^3$ where $a^4+d^4=c^4$. Show also that the normal pressure upon the curve varies as

$$(OP^4+d^4)^2(3OP^4-d^4)/OP^5.$$

39. Find the variation, to the first order, of the integral

$$\int f(x, y, z)\,ds$$

taken along an arc of a curve traced on a surface $\phi(x, y, z)=0$ between two given points of the surface; and show that if the integral have a maximum or minimum value the curve is found from the differential equations

$$\left[\frac{d}{ds}\left(V\frac{dx}{ds}\right)-\frac{\partial V}{\partial x}\right]\Big/\frac{\partial\phi}{\partial x}=\left[\frac{d}{ds}\left(V\frac{dy}{ds}\right)-\frac{\partial V}{\partial y}\right]\Big/\frac{\partial\phi}{\partial y}=\left[\frac{d}{ds}\left(V\frac{dz}{ds}\right)-\frac{\partial V}{\partial z}\right]\Big/\frac{\partial\phi}{\partial z}.$$

The line joining the centre of curvature at any point P of the above curve to the centre of curvature of the corresponding normal section of the surface meets the tangent plane at P in G; GT is perpendicular to GP, and PT is the tangent at P to that curve of the family $\phi=0$, $V=\text{const.}$ which passes through P. Show that $V\Big/\dfrac{dV}{ds}=GT$.

[MATH. TRIP., 1897.]

40. A heavy particle moves on a smooth surface of revolution $z=f(\sqrt{x^2+y^2})$, the axis of which is vertical and vertex upwards. Find the brachistochrone from a fixed point on the surface at a depth c below the vertex to a given meridian, and prove that the brachistochrone cuts the given meridian at right angles, and that the area swept over by the radius vector on a horizontal plane is proportional to the Action. If the brachistochrone be from the

fixed point to the curve defined by the equations $z=f(\sqrt{x^2+y^2})$, $y+z=2c$, prove that, if r and θ be cylindrical coordinates, the lower end of the brachistochrone is given by the equations

$$r\sin\theta+f(r)=2c,$$

$$[\sin\theta+f'(r)]^2=\cos^2\theta[1+\{f'(r)\}^2]\Big/\left[\frac{r^2}{m^2\{f(r)-c\}}-1\right].$$

[ST. JOHN'S COLL., 1884.]

41. Show that $\phi(x)\dfrac{d^n\psi(x)}{dx^n}-(-1)^n\psi(x)\dfrac{d^n\phi(x)}{dx^n}$ is an exact differential.

42. Show that the conditions that $\iint V\,dx\,dx$ is integrable *per se*, where $V=\phi\{x, y, y', \dots y^{(n)}\}$, are

$$\frac{\partial V}{\partial y}-\frac{d}{dx}\frac{\partial V}{\partial y'}+\frac{d^2}{dx^2}\frac{\partial V}{\partial y''}-\frac{d^3}{dx^3}\frac{\partial V}{\partial y'''}+\dots=0$$

and
$$\frac{\partial V}{\partial y'}-2\frac{d}{dx}\frac{\partial V}{\partial y''}+3\frac{d^2}{dx^2}\frac{\partial V}{\partial y'''}-\dots=0.$$

[TODHUNTER, *I.C.*, p. 369.]

43. Show that the conditions that $\iiint V\,dx\,dx\,dx$ is integrable *per se* are those of Question 42, together with

$$1\,.\,2\frac{\partial V}{\partial y''}-2\,.\,3\frac{d}{dx}\frac{\partial V}{\partial y'''}+3\,.\,4\frac{d^2}{dx^2}\frac{\partial V}{\partial y^{(\text{iv})}}-4\,.\,5\frac{d^3}{dx^3}\frac{\partial V}{\partial y^{(\text{v})}}+\dots=0,$$

and generally, that V is integrable n times *per se*, provided that each of the functions $V, xV, x^2V, \dots x^{n-1}V$ be so integrable *once*.

[TODHUNTER, *I.C.*, p. 369.]

44. Show how to find the relation between x and y which will make the expression $\int_{x_0}^{x_1} f(x, y, x_1, y_1, x_0, y_0, p, p_1, p_0)\,dx$ a maximum or a minimum, it being given that x_1, y_1 are connected by an equation, and that x_0, y_0 are also connected by an equation.

A curve of given length l is drawn in the plane x, y so that one end is on the axis of the parabola $x^2=4ay$ and the other end on the arc of the parabola. If the figure revolves round the tangent at the vertex of the parabola, show that when the surface generated by the curve is the greatest possible the form of the curve is that of a portion of the catenary

$$l\cosh\frac{2a}{l}+a\,\text{cosech}\,\frac{2a}{l}-y\sinh\frac{2a}{l}=l\cosh\left(\frac{x}{l}\sinh\frac{2a}{l}\right).$$

[MATH. TRIP., 1886.]

45. It is required to find a smooth guiding curve for a particle moving under gravity from rest, such that the *horizontal* space described in time t is the greatest possible. Show that the curve must be a cycloid, and that the space is gt^2/π.

[MATH. TRIP. II., 1914.]

46. Uniform elastic wire is held bent by proper forces between two points A and B so that the area between the wire and AB being given, the work expended in bending the wire may be the least possible. Show that the curvature at any point varies as $r^2 - a^2$, where $AB = 2a$ and r is the distance of the point from the middle point of AB. Show also that if the wire be bent completely round to satisfy the same conditions, the form of the wire will be given by $r^3 = c^3 \cos 3\theta$. [MATH. TRIP., 1878.]

[It may be assumed that the work done in bending the wire is measured by $\frac{1}{2}\int \frac{a^2}{\rho^2} ds$.]

47. A right cone is capable of revolving freely round its axis, which is vertical. A groove is to be cut in the surface of the cone such that a particle of mass m sliding down the groove without initial velocity from a given point may in the shortest time reach a given point in the horizontal plane through the base of the cone; show that the differential equation of the particle's path projected on the horizontal plane is

$$\left(\frac{dr}{d\theta}\right)^2 = r^2\left(\frac{r^2}{k^2}+1\right)\left\{\frac{r^2(r^2+k^2)}{k^2(r-r_0)c} - 1\right\}\sin^2\alpha,$$

where α is the semi-vertical angle of the cone and mk^2 its moment of inertia about its axis. [MATH. TRIP. III., 1885.]

48. A curve is drawn to touch two fixed straight lines at the fixed points P and Q. The area included by its pedal with respect to a fixed point O and the perpendiculars from O to the fixed tangents is a minimum, whilst the area included between the curve and the straight lines OP, OQ is constant. Show that the curve is part of an epi- or hypo-cycloid.

49. If a point move in a plane with velocity always proportional to the curvature of its path, show that the brachistochrone of continuous curvature between any two given points is a complete cycloid.

Prove that in the ordinary gravitation brachistochrone (which is also a cycloid), the velocity is inversely as the curvature of the path, and state the connexion between the two results.

[MATH. TRIP., 1875.]

50. Prove that the curve of a uniform chain of given length joining two fixed points is given by an equation of the form $y = b \operatorname{sn} K\dfrac{x}{a}$, when the moment of inertia of the chain about a given fixed line, in a plane with the two given points, is a maximum; and by an equation of the form $y \operatorname{cn} K\dfrac{x}{a} = b$, when the moment of inertia is a minimum, the given straight line being taken as the x-axis. [MATH. TRIP. III., 1884.]

51. Use the method of the Calculus of Variations to show that the general equation of the geodesics on a right circular cone, whose equation in polar coordinates is $\theta = \alpha$, is $r \cos\{(\phi - \beta)\sin\alpha\} = a$, where β and a are arbitrary constants. [OXF. II. P., 1914.]

52. Prove that the polar equation of the projection of a geodesic on a catenoid formed by the revolution of a catenary about its directrix upon a plane perpendicular to the directrix is of one of the forms

$$r \operatorname{sn}\left(\frac{\theta}{k}, k\right) = \text{const}, \quad r \operatorname{sn}\theta = \text{const.}, \quad r \tanh\theta = \text{const.},$$

and distinguish the cases.

[MATH. TRIP. III. 1884, II. 1913; GREENHILL, *E.F.*, p. 96.]

53. Prove that if, from any point of a surface, geodesic lines of equal length be drawn in all directions, the curve which is the locus of their extremities cuts all the geodesics at right angles

54. Prove that on the surface of revolution determined by the equations

$$x = ak\cos\omega\cos\phi, \quad y = ak\cos\omega\sin\phi, \quad z = a\int_0^\omega \sqrt{1 - k^2\sin^2\omega}\, d\omega,$$

the equation of a geodesic line is $\tan\omega = A \sin k(\phi + \beta)$.

Prove also that the locus of the extremities of geodesic lines of length $\frac{1}{2}\pi a$ drawn from the point at which $\omega = \Omega$ and $\phi = 0$ is

$$\cos k\phi + \tan\omega\tan\Omega = 0.$$

[MATH. TRIP., 1896.]

55. Prove that the projection of a geodesic on a surface of revolution on a plane perpendicular to the axis is in polar coordinates $r^{-2} = \alpha^{-2}\operatorname{cn}^2\mu\theta + \beta^{-2}\operatorname{sn}^2\mu\theta$, if the meridian curve of the surface is the roulette of the focus of an ellipse rolling upon the axis, α and β denoting the greatest and least values of the focal distances.

Show that if the geodesic cuts the meridian plane at its maximum distance at an angle γ, then

$$\mu = \beta \cot \gamma / (\alpha + \beta), \quad \beta^2 k^2 = (\alpha^2 - \beta^2) \tan^2 \gamma.$$

[MATH. TRIP. III., 1885.]

56. The line element of a certain surface is expressed in terms of parameters u and v by the equation

$$ds^2 = \{(du)^2 + (dv)^2 - (u\,dv - v\,du)^2\} / (1 - u^2 - v^2)^2.$$

Prove that the equations of the geodesics on the surface are of the form $au + bv + c = 0$, where a, b, c are constants.

[MATH. TRIP. II., 1920.]

57. Prove that a surface for which

$$ds^2 = \{dx^2 + dy^2 - (x\,dy - y\,dx)^2\} / (1 - x^2 - y^2)^2$$

has its geodesics represented by straight lines on the plane of x-y and its geodesic circles by conics having double contact with $x^2 + y^2 - 1 = 0$, and the geodesic distance ρ between (x_0, y_0) and (x, y) being given by

$$(1 - x_0^2 - y_0^2)(1 - x^2 - y^2) \cosh^2 \rho = (1 - xx_0 - yy_0)^2.$$

Prove also that the specific curvature is constant and equal to -1.

[MATH. TRIP. II., 1919.]

58. Show that the conditions that the parametric curves may be geodesics on the surface of which the line element is given by $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ are respectively that $(E\,du + F\,dv)/\sqrt{E}$ and $(F\,du + G\,dv)/\sqrt{G}$ must be complete differentials. Show also that if these conditions be satisfied, the specific curvature at a point of the surface is $\dfrac{1}{V} \dfrac{\partial^2 \omega}{\partial u\, dv}$, where $V^2 = EG - F^2$ and ω is the angle between the parametric curves at the point.

[MATH. TRIP. II., 1919.]