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A DEMONSTRATION OF SIR W. R. HAMILTON'S THEOREM OF THE ISOCHRONISM OF THE CIRCULAR HODOGRAPH.

[From the *Philosophical Magazine*, vol. XIV. (1857), pp. 427—430.]

IMAGINE a body moving *in plano* under the action of a central force, and let h denote, as usual, the double of the area described in a unit of time; let P be any point of the orbit, then measuring off, on the perpendicular let fall from the centre of force O on the tangent at P to the orbit, a distance OQ equal or proportional to h into the reciprocal of the perpendicular on the tangent, the locus of Q is the hodograph, and the points P, Q are corresponding points of the orbit and hodograph.

It is easy to see that the hodograph is the polar reciprocal of the orbit with respect to a circle having O for its centre, and having its radius equal or proportional to $\sqrt{h}$. And it follows at once that Q is the pole, with respect to this circle, of the tangent at P to the orbit.

In the particular case where the force varies inversely as the square of the distance, the hodograph is a circle. And if we consider two elliptic orbits described about the same centre, under the action of the same central force, and such that the major axes are equal, then (as will be presently seen) the common chord or radical axis of the two hodographs passes through the centre of force.

Imagine an orthotomic circle of the two hodographs (the centre of this circle is of course on the common chord or radical axis of the two hodographs), and consider the arcs intercepted on the two hodographs respectively by the orthotomic circle; then the theorem of the isochronism of the circular hodograph is as follows, viz. the times of hodographic description of the intercepted arcs are equal; in other words, the times of description in the orbits, of the arcs which correspond to the intercepted arcs of the hodographs, are equal. It was remarked by Sir W. R. Hamilton, that the theorem is in fact equivalent to Lambert's theorem, that the time depends only on the chord of the described arc and the sum of the two radius vectors. And this remark suggests a mode of investigation of the theorem. Consider the intercepted arc of one of the hodographs: the tangents to the hodograph at the extremities of this arc are radii of the orthotomic circle; i.e. the corresponding arc of the orbit is the arc cut off by the polar (in respect to the directrix circle by which the hodograph is determined) of the centre of the orthotomic circle; the portion of this polar intercepted by the orbit is the elliptic chord, and this elliptic chord and the sum of the radius vectors at the two extremities of the elliptic chord determine the time of description of the arc; and the values of these quantities, viz. the elliptic chord and the sum of the radius vectors, must be the same in each orbit.

The analytical investigation is not difficult. I take as the equation of the first orbit,

$$r = \frac{a(1-e^2)}{1 + e\cos(\theta - \varpi)};$$

then the polar of the orbit with respect to a directrix circle $r = c$ is

$$r^2 - \frac{c^2 e}{a(1-e^2)} r\cos(\theta - \varpi) - \frac{c^4}{a^2(1-e^2)} = 0,$$

and putting $c^2 = k\sqrt{k}\sqrt{a}(1-e^2)$ (where k is a constant quantity, i.e. it is the same in each orbit), the equation becomes

$$r^2 - \frac{ek\sqrt{k}}{\sqrt{a(1-e^2)}} r\cos(\theta - \varpi) - \frac{k^3}{a} = 0.$$

But since a is supposed to be the same in each orbit, we may for greater simplicity write $k^3 = m^2 a$; it will be convenient also to put $e = \sin\kappa$; we have then

$$r = \frac{a\cos^2\kappa}{1 + \sin\kappa\cos(\theta - \varpi)}$$

for the equation of the orbit, $r^2 = ma\cos\kappa$ for the equation of the directrix circle, and

$$r^2 - r\,.\,m\tan\kappa\cos(\theta - \varpi) - m^2 = 0$$

for the equation of the hodograph.

We have in like manner

$$r = \frac{a\cos^2\kappa'}{1 + \sin\kappa'\cos(\theta - \varpi')}$$

for the equation of the second orbit, $r^2 = ma\cos\kappa'$ for the equation of the corresponding directrix circle, and

$$r^2 - r\,.\,m\tan\kappa'\cos(\theta - \varpi') - m^2 = 0$$

for that of the hodograph.

The equations of the two hodographs give at once

$$\tan\kappa\cos(\theta-\varpi)-\tan\kappa'\cos(\theta-\varpi')=0$$

for the equation of the common chord or radical axis of the two hodographs,—an equation which shows that, as already noticed, the common chord passes through the origin or centre of force. This equation gives $\theta=\alpha$ if

$$\tan\kappa\cos(\alpha-\varpi)-\tan\kappa'\cos(\alpha-\varpi')=0\,;$$

i.e. α is a quantity such that the expressions $\tan\kappa\cos(\alpha-\varpi)$ and $\tan\kappa'\cos(\alpha-\varpi')$, which correspond to each other in the two orbits, are equal. We may take R, α as the polar coordinates of the centre of the orthotomic circle (where R is arbitrary); the equation of the polar of this point with respect to the directrix circle $r^2=ma\cos\kappa$, is then at once seen to be

$$r\cos(\theta-\alpha)=\frac{ma\cos\kappa}{R},$$

which is the equation of the line cutting off the arc of the elliptic orbit

$$r=\frac{a\cos^2\kappa}{1+\sin\kappa\sin(\theta-\varpi)}.$$

Writing $\theta-\varpi=\theta-\alpha+(\alpha-\varpi)$, the two equations give

$$\cos(\theta-\alpha)=\frac{A}{r},$$

$$\sin(\theta-\alpha)=\frac{B}{r}+C,$$

if for shortness

$$A=\frac{ma\cos\kappa}{R},$$

$$B=\frac{ma\cos\kappa}{R}\,\frac{\cos(\alpha-\varpi)}{\sin(\alpha-\varpi)}-\frac{a\cos^2\kappa}{\sin\kappa\sin(\alpha-\varpi)},$$

$$C=\frac{1}{\sin\kappa\sin(\alpha-\varpi)};$$

we have therefore

$$\frac{A^2+B^2}{r^2}+\frac{2BC}{r}+C^2=1,$$

or, what is the same thing,

$$(1-C^2)\,r^2-2BCr-(A^2+B^2)=0\,;$$

and thence, if r', r'' are the two values of r,

$$r'+r''=\frac{2BC}{1-C^2},$$

$$r'r''=-\frac{A^2+B^2}{1-C^2}.$$

Let θ', θ'' be the corresponding values of θ, we have

$$\theta' - \theta'' = \theta' - \alpha - (\theta'' - \alpha),$$

and thence

$$\cos(\theta' - \theta'') = \frac{A}{r'} \cdot \frac{A}{r''} + \left(\frac{B}{r'} + C\right)\left(\frac{B}{r''} + C\right), \quad = \frac{A^2 + B^2}{r'r''} + BC\left(\frac{1}{r'} + \frac{1}{r''}\right) + C^2,$$

or adding unity to each side, multiplying by $r'r''$, and on the right-hand side substituting for $r' + r''$, $r'r''$ their values

$$r'r''\left(1 + \cos(\theta' - \theta'')\right) = -\frac{2C^2A^2}{1 - C^2};$$

the square of the chord is $r^2 + r'^2 - 2rr'\cos(\theta' - \theta'')$, or, what is the same thing, $(r' + r'')^2 - 2r'r''\left(1 + \cos(\theta' - \theta'')\right)$; hence to prove the theorem, it is only necessary to show that $r' + r''$ and $r'r''\left(1 + \cos(\theta' - \theta'')\right)$ have the same values in each orbit, that is, that $\dfrac{2BC}{1 - C^2}$ and $-\dfrac{2C^2A^2}{1 - C^2}$ have the same values in each orbit. But observing that

$$1 - \sin^2\kappa \sin^2(\alpha - \varpi) = \cos^2\kappa + \sin^2\kappa\cos^2(\alpha - \varpi) = \cos^2\kappa\,\{1 + \tan^2\kappa\cos^2(\alpha - \varpi)\},$$

the values of these expressions are respectively

$$-\frac{2a}{R}\,\frac{\left(m\tan\kappa\cos(\alpha - \varpi) - R\right)}{1 + \tan^2\kappa\cos^2(\alpha - \varpi)},$$

$$\frac{2a^2}{R^2}\,\frac{m^2}{1 + \tan^2\kappa\cos^2(\alpha - \varpi)},$$

which contain only the quantities m, a, R, $\tan\kappa\cos(\alpha - \varpi)$, which are the same for each orbit, and the theorem is therefore proved, viz. it is made to depend on Lambert's theorem. I may remark, that a geometrical demonstration which does not assume Lambert's theorem is given by Mr Droop in his paper "On the Isochronism of the Circular Hodograph," *Quarterly Mathematical Journal*, vol. I. [1857] pp. 374—378, where the dependence of the theorem on Lambert's theorem is also shown.

By what precedes, the theorem may be stated in a geometrical form as follows:—"Imagine two ellipses having a common focus, and their major axes equal; describe about the focus two directrix circles having their radii proportional to the square roots of the minor axes of the ellipses respectively; the polar reciprocal of each ellipse in respect to its own directrix circle will be a circle (the hodograph), and the common chord or radical axis of the two hodographs will pass through the focus. Consider any point on the common chord, and take the polar with respect to each directrix circle; such polar will cut off an arc of the corresponding ellipse; and then, *theorem*, the elliptic chord, and the sum of the radius vectors through the two extremities of the chord, will be respectively the same for each ellipse."

2, *Stone Buildings, W.C., June* 24, 1857.