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NOTE ON THE EQUIPOTENTIAL CURVE $\frac{m}{r}+\frac{m'}{r'}=C$.

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THE equation $\frac{m}{r}+\frac{m'}{r'}=C$, where m, m', C are constants, and r, r' are the distances of a point P of the locus from two given points M, M' respectively, expresses that the potential of the attracting or repelling masses m, m' has a constant value at all points of the locus. The locus is obviously a surface of revolution, having the line through the points M, M' for its axis; and instead of the surface, we may consider the section by a plane through the axis, or what is the same thing, we may consider r, r' as the distances *in plano* of a point P of the curve from the given points M, M': such curve may be termed the equipotential curve. I propose in the present Note to investigate in a general manner, and without entering into any analytical detail, the general form of the curve corresponding to different values of the quantity C.

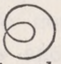
It is proper to remark, that the curve is not altered by changing the signs of each or any of the quantities m, m', C (in fact, analytically the distances r, r' are essentially ambiguous in sign), so that we may without loss of generality consider m, m', C as all of them positive. The different branches of the complete analytical or geometrical curve have distinct mechanical significancies; thus, r, r' being positive, $\frac{m}{r}+\frac{m'}{r}=C$ is the curve for which the potential of the attracting masses m, m' is equal to C; but $\frac{m}{r}-\frac{m'}{r'}=C$ is the curve for which the attracting mass m, and the repulsive mass m', have the potential C; but this is a distinction to which I do not attend. I write for homogeneity $\frac{k}{a}$ instead of C, where a is the distance between the points M, M'; the equation thus becomes

$$\frac{m}{r}+\frac{m'}{r'}=\frac{k}{a},$$

where a is a positive distance, m, m', k may be considered as positive abstract numbers. The curve is obviously a curve of the eighth order. When k is large in comparison with m, m', then since r, r' cannot be both of them small in comparison of a (for if one be small, the other will be nearly equal to a), it is clear that one of these distances, for instance r, will be small, and the other, r', nearly equal to a. We in fact have $\left(\text{neglecting in the first instance } \frac{m'}{r'} \text{ in comparison with } \frac{m}{r}\right)$ $\frac{m}{r}=\frac{k}{a}$, or more accurately, $\frac{m}{r}=\frac{k\pm m'}{a}$, i.e. $r=\frac{m}{k\pm m'}$, which shows that a part of the curve consists of two ovals, which are approximately concentric circles, radii $\frac{m'}{k\pm m'}a$, about the point M as centre. In like manner a part of the curve consists of two ovals, which are approximately concentric circles, radii $\frac{m'}{k\pm m}a$, about the point M' as centre. I denote by A, B, the two ovals about M, viz. A is the exterior, and B the interior oval; and in like manner by A', B' the two ovals about M', viz. A' is the exterior, and B' the interior oval. The distances *inter se* of the ovals A and B, or of the ovals A' and B', are small in comparison with the radii of these ovals respectively; and if, to fix the ideas, m' be greater than M, then the ovals A', B' are greater than the ovals A, B.

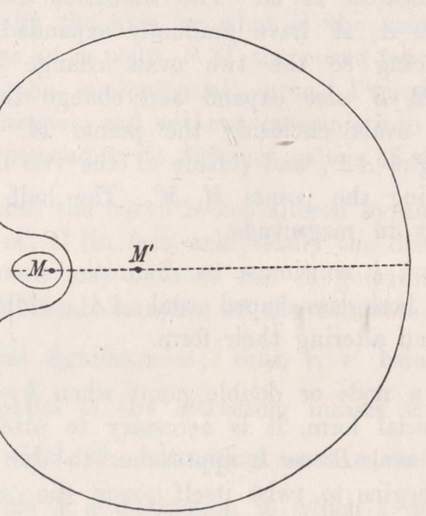
It is easy to see that the curve will have a node or double point on the axis if $k=(\sqrt{m'}\pm\sqrt{m})^2$; and we must first consider the case $k=(\sqrt{m'}+\sqrt{m})^2$. The node lies between the points M, M', and its distances from these points are respectively as $\sqrt{m} : \sqrt{m'}$, that is, it is nearest to M. The transition from the original form is very obvious; the exterior ovals A, A' have gradually expanded until they come in contact, and at the instant of doing so the two ovals change themselves into a figure of eight, AA'. The ovals B, B' also expand and change their form, but they preserve the general character of ovals enclosing the points M, M' respectively. The curve consists of a figure of eight AA', and (inside of the two divisions thereof respectively) of the ovals B, B' enclosing the points M, M'. The half of the curve nearest to M' is, as before, preponderant in magnitude.

The next change when k continues to diminish is an obvious one: the figure of eight opens out into an hourglass-shaped oval AA', while the ovals B, B' continue increasing in magnitude and altering their form.

There will be again a node or double point when $k=(\sqrt{m'}-\sqrt{m})^2$; but to explain the transition to this special form, it is necessary to attend more particularly to the change of form in the oval B' as k approaches to the value in question, viz. this oval lengthens out and begins to twist itself round the oval B; and when k' becomes $=(\sqrt{m'}-\sqrt{m})^2$, then the oval B' has completely encircled B, the two extremities of B' meeting together at the double point, which is a point beyond M (i.e. on the other side to M'), such that its distances from M, M' are in the ratio of $\sqrt{m} : \sqrt{m'}$. And at the instant of contact there is, as in the former case, a modification of the form of the portions which come into contact, so that the node is an ordinary double point. The oval B' has, in fact, become what may be termed a re-entrant figure of eight, , the small part of which encloses the oval B which encloses the point M, while the large part encloses the point M'. The curve consists of the exterior oval AA' (which has probably lost wholly or partially its hourglass form, and is more nearly an ordinary oval), of the re-entrant figure of eight, B', and of the enclosed oval B.

As k continues to diminish, the re-entrant figure of eight, B', breaks up into two detached ovals lB', mB', the larger of which, lB', encloses the other one and also the point M'; while the smaller one, mB', does not enclose M', but encloses the oval B which encloses M; the curve consists of the exterior oval AA', the ovals lB' and mB' which have arisen out of the oval B', and the oval B. As k further decreases, the ovals AA' and lB' continually increase in magnitude, and the ovals mB' and B approximate more and more nearly together; and at length, when k becomes $=0$, the ovals AA' and lB' disappear at infinity, while the ovals mB' and B unite themselves into a circle enclosing M, but not enclosing M': the equation of this circle is, in fact, $\frac{m}{r}+\frac{m'}{r'}=0$; or what is the same thing, $r^2=\frac{m^2}{m'^2}r'^2$, and the points M, M' have, in relation to this circle, the well-known relation that each is the image of the other.

The preceding description is, I think, intelligible without the assistance of a series of figures illustrating the different forms of the curve, but there is no difficulty in actually tracing the curve for any particular values of the constant parameters. Thus



(taking the distance MM' for unity) suppose that the equation of the curve is $\frac{1}{r}+\frac{4}{r'}=1{\cdot}2$; (the value 1·2 was selected as a value not far from that for which the oval B' becomes a re-entrant figure of eight, though the change of form is so rapid that this value shows only the incipient tendency of the oval B' to take the form in question). The form of the portion of the curve consisting of the two ovals B, B' will be that shown by the figure, which was constructed by points on a double scale with some accuracy.

The case $m = m'$ is an exception, and must be considered separately: the curve is here in all its changes symmetrical about a perpendicular to the axis midway between the two centres M, M'. The curve in the first instance, i.e. when k is greater than $(\sqrt{m} + \sqrt{m})^2, = 4m$, consists of the two ovals B, A about M, and the two ovals B', A' about M'. As k decreases to $4m$, the two ovals A, A' gradually increase in magnitude, and at length come together, as before, into a figure of eight, AA'; and as k continues to diminish, the figure of eight opens out into an hourglass form AA', which continues increasing in magnitude, and degenerating into the form of an oval. The interior ovals B, B' approach more and more nearly together, lengthen out in the direction perpendicular to the axis, and present to each other a more and more flattened portion. The second value,

$$k = (\sqrt{m'} - \sqrt{m})^2,$$

which in the general case gives a node, in the present case only arises when $k = 0$; and there is not then any node, but the curve degenerates in a similar manner to what happens for $k = 0$ in the general case; viz. the oval AA' disappears at infinity, while the ovals B, B' coalesce together (their outer parts disappearing at infinity) into a pair of lines coincident with the perpendicular to the axis midway between the two centres.

2, *Stone Buildings*, *May* 31, 1857.