# 160.

## ON A THEOREM RELATING TO RECIPROCAL TRIANGLES.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 7—10.]

THE following theorem is, I assume, known; but the analytical demonstration of it depends upon a formula in determinants which is not without interest. The theorem referred to may be thus stated:

"A triangle and its reciprocal are in perspective;" where by the reciprocal of a triangle is meant the triangle the sides of which are the polars of the angles of the first-mentioned triangle with respect to a conic; and triangles are in perspective when the three lines forming the corresponding angles meet in a point, or what is the same thing, when the three points of intersection of the corresponding sides lie in a line.

Let the equation of the conic be

$$x^2+y^2+z^2=0,$$

and take $(\alpha, \beta, \gamma)$, $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ for the coordinates of the angles of the triangle, then if $K$ be the determinant, and $(A, B, C)$ $(A', B', C')$ $(A'', B'', C'')$ the inverse system, i.e. if

$$\begin{aligned} KA &= (\beta'\gamma''-\beta''\gamma'), & KB &= \gamma'\alpha''-\gamma''\alpha', & KC &= \alpha'\beta''-\alpha''\beta', \\ KA' &= (\beta''\gamma-\beta\gamma''), & KB' &= \gamma''\alpha-\gamma\alpha'', & KC' &= \alpha''\beta-\alpha\beta'', \\ KA'' &= (\beta\gamma'-\beta'\gamma), & KB'' &= \gamma\alpha'-\gamma'\alpha, & KC'' &= \alpha\beta'-\alpha'\beta, \end{aligned}$$

equations which may be represented in the notation of matrices by the single equation

$$\left|\begin{matrix} \alpha, & \beta, & \gamma' \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{matrix}\right|^{-1} = \left|\begin{matrix} A, & A', & A'' \\ B, & B', & B'' \\ C, & C', & C'' \end{matrix}\right|,$$

then the equations of the sides of the triangle are

$$A\,x+B\,y+C\,z=0,$$
$$A'\,x+B'\,y+C'\,z=0,$$
$$A''x+B''y+C''z=0,$$

and the coordinates of the angles of the reciprocal triangle may be taken to be $(A, B, C)$ $(A', B', C')$ $(A'', B'', C'')$; the equations of the lines joining the corresponding angles of the two triangles are therefore

$$(B\gamma-C\beta)\,x+(C\alpha-A\gamma)\,y+(A\beta-B\alpha)\,z=0,$$
$$(B'\gamma'-C'\beta')\,x+(C'\alpha'-A'\gamma')\,y+(A'\beta'-B'\alpha')\,z=0,$$
$$(B''\gamma''-C''\beta'')\,x+(C''\alpha''-A''\gamma'')\,y+(A''\beta''-B''\alpha'')\,z=0\,;$$

the condition that these lines may meet in a point is therefore

$$\begin{vmatrix} B\,\gamma-C\,\beta\,, & C\,\alpha-A\,\gamma\,, & A\,\beta-B\,\alpha \\ B'\gamma'-C'\beta'\,, & C'\alpha'-A'\gamma', & A'\beta'-B'\alpha' \\ B''\gamma''-C''\beta'', & C''\alpha''-A''\gamma'', & A''\beta''-B''\alpha'' \end{vmatrix}=0,$$

an equation which is satisfied identically when $A, B, C$; $A', B', C'$; $A'', B'', C''$ are replaced by their values. To prove this I transform the different quantities which enter into the determinant as follows: putting

$$F=\alpha'\alpha''+\beta'\beta''+\gamma'\gamma'',$$
$$G=\alpha''\alpha+\beta''\beta+\gamma''\gamma,$$
$$H=\alpha\,\alpha'+\beta\,\beta'+\gamma\,\gamma'\,;$$

we have

$$\begin{aligned} K\,(B\gamma-C\beta)&=\gamma\,(\gamma'\alpha''-\gamma''\alpha')-\beta\,(\alpha\beta''-\alpha''\beta') \\ &=\alpha''\,(\beta\beta'+\gamma\gamma')-\alpha'\,(\beta\beta''+\gamma\gamma'') \\ &=\alpha''\,(\alpha\alpha'+\beta\beta'+\gamma\gamma')-\alpha'\,(\alpha\alpha''+\beta\beta''+\gamma\gamma'') \\ &=\alpha''H-\alpha'G, \\ &\text{\&c.}\,; \end{aligned}$$

and the equation becomes

$$\begin{vmatrix} \alpha''H-\alpha'G, & \beta''H-\beta'G, & \gamma''H-\gamma'G \\ \alpha\,F-\alpha''H, & \beta\,F-\beta''H, & \gamma\,F-\gamma''H \\ \alpha'G-\alpha\,F, & \beta'G-\beta\,F, & \gamma'G-\gamma\,F \end{vmatrix}=0.$$

Now the minor $(\beta F-\beta''H)(\gamma'G-\gamma F)-(\gamma F-\gamma''H)(\beta'G-\beta F)$ is equal to

$$GH\,(\beta'\gamma''-\beta''\gamma')+HF\,(\beta''\gamma-\beta\gamma'')+FG\,(\beta\gamma'-\beta'\gamma),$$

i.e. to

$$K\,(GHA+HFA'+FGA'')\,;$$

and expressing the other minors in a similar form, the equation to be proved is

$$\left.\begin{array}{l}(GHA + HFA' + FGA'')(B\gamma - C\beta) \\ + (GHB + HFB' + FGB'')(C\alpha - A\gamma) \\ + (GHC + HFC' + FGC'')(A\beta - B\alpha)\end{array}\right\} = 0,$$

i.e.

$$HF\begin{vmatrix} A', & B', & C' \\ A\,, & B\,, & C \\ \alpha\,, & \beta\,, & \gamma \end{vmatrix} + FG\begin{vmatrix} A'', & B'', & C'' \\ A\,, & B\,, & C \\ \alpha\,, & \beta\,, & \gamma \end{vmatrix} = 0.$$

The first determinant is

$$-\{\alpha(BC' - B'C) + \beta(CA' - C'A) + \gamma(AB' - A'B)\} = -\frac{1}{K}(\alpha\alpha'' + \beta\beta'' + \gamma\gamma'') = -\frac{1}{K}G,$$

and the second determinant is

$$\{\alpha(B''C - BC'') + \beta(C''A - CA'') + \gamma(A''B - AB'')\} = \frac{1}{K}(\alpha\alpha' + \beta\beta' + \gamma\gamma') = \frac{1}{K}H,$$

and we have therefore identically

$$HF(-G) + FG(H) = 0.$$

The corresponding theorem in geometry of three dimensions is that a tetrahedron and its reciprocal have to each other a certain relation, viz. the four lines joining the corresponding angles are generating lines of a hyperboloid, or, what is the same thing, the four lines of intersection of corresponding faces are generating lines of a hyperboloid. The demonstration would show how the theorem in determinants is to be generalised.

2, *Stone Buildings, Lincoln's Inn, February*, 1855.