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On the geometry of the state space in neoclassical thermodynamics

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THE PAPER contains certain topological and geometric properties of the space of states of an isolated thermodynamic system. The first part deals with the topological properties necessary for introduction in the state space of a structure of an infinitely dimensional differentiable manifold. In the second and third parts, we introduce the objects of heat and work and the flux of heat and power. This made it possible to prove the existence of a state function interpreted as the energy of the thermodynamic system satisfying the first law of thermodynamics.

W pracy przedstawiono pewne własności topologiczne i geometryczne przestrzeni stanów termodynamicznych układu izolowanego. Część pierwsza zawiera omówienie własności topologicznych, niezbędnych do wprowadzenia w przestrzeni stanów struktury nieskończenie-wymiarowej rozmaitości różniczkowalnej oraz dowodu istnienia na niej obiektów geometrycznych. W części drugiej i trzeciej wprowadzono obiekty ciepła i pracy, strumienia ciepła i mocy. Pozwoliło to w konsekwencji dowieść istnienia funkcji stanów, interpretowanej jako energia układu termodynamicznego, spełniającej pierwsze prawo termodynamiki.

В работе изложены некоторые топологические и геометрические свойства пространства состояний термодинамической изолированной системы. Первая часть содержит обсуждение топологических свойств, необходимых для введения в пространство состояний структуры бесконечно-мерного дифференциального многообразия и доказательства существования в нем некоторых геометрических объектов. Во второй и третьей частях введены объекты тепла и работы, теплового потока и мощности. В итоге это позволило доказать существование функции состояний, рассматриваемой как энергия термодинамической системы, которая удовлетворяет первому закону термодинамики.

Introduction

THE CONCEPT "neoclassical thermodynamics" as a name of a method does not exist long but its basic idea appeared in science many years ago and its originator was undoubtedly C. CARATHEODORY [2]. His theory was not the neoclassical thermodynamics in the exact sense of this word but it was exactly Caratheodory who first attempted to construct a phenomenological axiomatic thermodynamics in which the original concepts were the isolated system and its state. In later papers, the accessibility relation was introduced in the set of states as a relation of partial order and in this way there arose thermodynamics called by K. WILMAŃSKI [12] "neoclassical".

Before the papers [12, 13, 14] K. WILMAŃSKI, who presented the neoclassical thermodynamical procedure in a unified form, many authors attempted at such theories, for instance, R. GILES [6], J. B. COOPER [3], J. B. BOYLING [1]. However, the first theory is due to G. FALK and H. JUNG [5]. Their paper differs from the later papers first of all because they did not take as an original concept "the energetic isolation" but constructed an axiomatic thermodynamics for a non-isolated system. The introduction there of the concepts of state and accessibility relation in the state space is subject to serious doubts. In fact, as explained in details by R. GILES [6], the concept of states is correct only in the case

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of an isolated system. Moreover, in the set of "states" of a non-isolated system, the accessibility relation is not transitive and therefore is not a relation of partial order, see K. Wilmański [13].

In later papers, various authors confined their considerations to isolated systems. It is noteworthy, however, that there are two problems directly related to the mathematical structure of the proposed theories and indirectly to their range, namely the problem of topological properties of the state space and the properties of the accessibility relation. Both J. B. Cooper and J. B. Boyling assumed that the state space is a connected separable T_2-space. On the other hand, K. Wilmański [12] proved that every set of states with a relation of partial order is a Hausdorff space. It can easily be proved that there exist state spaces both disconnected and not separable, e.g. the discrete state space, space with isolated states, space of a system with a discontinuity.

In both above-mentioned papers, the authors introduced in the state space the accessibility relation not as a partial order but as a linear order. This is equivalent to the existence of a "process"([1]) containing all states of the system. This assumption would undoubtedly be useful but is too simplifying([2]).

In this paper, we assume that the state space is a set with a relation of partial order with a structure of an infinitely dimensional Banach differentiable manifold.

In the first part, we consider certain topological and geometric properties of the state space. The second part contains a method of determination of geometric objects on the infinitely dimensional manifold. In the third part, we introduce the objects of heat and work, the flux of the heat end power. The above makes it then possible to state the first law of neoclassical thermodynamics in a classical form. However, in the classical thermodynamics, this law was true for quasi-static processes only, while here, it appears as a necessary condition of existence in the state space of the accessibility relation and therefore is true for fast processes as well.

1. Topological and geometric properties of the state space

Consider a Boolean algebra Π with operations $\bigvee, \bigwedge, <$. The greatest element $\mathscr{B} \in \Pi$ is called the thermodynamic isolated system. Every element $\mathscr{P} \in \Pi \wedge \mathscr{P} < \mathscr{B}$ will be called a subsystem of system $\mathscr{B}$ (K. Wilmański [13]).

Axiom 1.1 *The set $\mathscr{S}$ is the space of states of an isolated thermodynamic system $\mathscr{B}$ if in $\mathscr{S}$ there is the accessibility relation $\underline{\rightarrow}$ with the following properties:*

1. $\bigwedge_{s \in \mathscr{S}} \quad s \underline{\rightarrow} s,$

2. $\bigwedge_{s^1, s^2, s^3 \in \mathscr{S}} \quad s^1 \underline{\rightarrow} s^2 \wedge s^2 \underline{\rightarrow} s^3 \Rightarrow s^1 \underline{\rightarrow} s^3,$

3. $\bigvee_{s^0 \in \mathscr{S}} \; \bigwedge_{s \in \mathscr{S}} \quad s \underline{\rightarrow} s^0.$

([1]) See Definition 2.3.

([2]) Example. Consider an isolated system divided by an isolating wall. Remove instantaneously the wall and immediately replace it. Is is readily observed that the processes of the system before and after the removal of the wall are not continuations of each other.

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DEFINITION 1.1. The thermodynamic simple process is the pair $(\pi_{(p)}; \underline{\rightarrow}) \overset{\text{df}}{=} p_{(p)}$, where $\pi_{(p)} \neq \{s\}$ is maximally linearly ordered subset of $\mathscr{S}$.

By $s^i \overset{\text{df}}{=} l(p_{(p)})$ and $s_f \overset{\text{df}}{=} r(p_{(p)})$ we denote the first and the last elements of the set $\pi_{(p)}$, respectively. These states will be called the initial (s^i) and the final (s^f) states, respectively, of the simple process $p_{(p)}$.

Hausdorff theorem implies immediately

LEMMA 1.1. $\bigwedge\limits_{s^1 s^2 \in \mathscr{S}} s^1 \underline{\rightarrow} s^2 \Rightarrow$ *exists* $p_{(p)} = (\pi_{(p)}; \underline{\rightarrow})$, *where* $s^1 = l(p_{(p)})$ *and* $s^2 = r(p_{(p)})$.

By $\mathbf{P}_{(p)}$ we denote hereafter the set of all simple processes of the system $\mathscr{B}$.

Following [12] we present two definitions concerning a thermodynamic process.

DEFINITION 1.2. The simple process $p_{(p)} = (\pi_{(p)}; \underline{\rightarrow})$ is called reversible when there exists an isomorphic inversely ordered simple process $p'_{(p)} = (\pi'_{(p)}; \underline{\rightarrow})$ such that $\pi_{(p)} = \pi'_{(p)}$.

DEFINITION 1.3. The thermodynamic process is every pair $(\pi; \underline{\rightarrow}) = p$, where

1. $\pi = \bigcup\limits_{i \in J} \pi^i_{(p)}$; $\quad p^i_{(p)} = (\pi^i_{(p)}; \underline{\rightarrow})$,
2. $\overline{\overline{J}} = \aleph_0$,
3. $\bigwedge\limits_{i \in J} l(p^{(i+1)}_{(p)}) = r(p^i_{(p)})$.

We then write $p = \bigoplus\limits_{i \in J} p^i_{(p)}$.

The process $p = (\pi; \underline{\rightarrow})$ is called reversible when there exists a family of reversible simple processes $\{p^i_{(p)}\}_{i \in J}$ such that $p = \bigoplus p^i_{(p)}$.

The definition of the thermodynamic process implies

LEMMA 1.2. $\bigwedge\limits_{p=(\pi;\underline{\rightarrow})} \bigwedge\limits_{s \in \pi} \bigvee\limits_{p_{(p)} \in \mathbf{P}_{(p)}}$ *such that* $s \in \pi_{(p)}$ *and* $p_{(p)}$ *is a simple subprocess of the process* p, *i.e.*, $\pi_{(p)} \subset \pi$.

Proof. Consider an arbitrary process $p = (\pi; \underline{\rightarrow})$. It follows from Definition 1.3 that there exists a family of simple processes $\{p^k_{(p)}\}_{k \in J}$ such that $p = \bigoplus p^k_{(p)}$. Since $\pi = \bigcup\limits_{k \in J} \pi^k_{(p)}$, then $\bigwedge\limits_{s \in \pi} \bigvee\limits_{p^j_{(p)}} s \in \pi^j_{(p)}$. Setting $p_{(p)} \equiv p^j_{(p)}$ we end the proof.

Let us now define the topology in the set of simple processes.

DEFINITION 1.4. The family

$$\mathscr{A}(p_{(p)}) \equiv \{p'_{(p)} \in \mathbf{P}_{(p)} : \pi'_{(p)} = \pi_{(p)} \cap L(r(p'_{(p)})) \vee \pi'_{(p)} = \pi_{(p)} \cap R(l(p'_{(p)}))\}$$

is a subbasis of the simple process $p_{(p)} = (\pi_{(p)}; \underline{\rightarrow})$, where

$$R(s) \equiv \{s^1 \in \mathscr{S}, s \underline{\rightarrow} s^1\}, \quad L(s) \equiv \{s^1 \in \mathscr{S}, s^1 \underline{\rightarrow} s\}.$$

It was proved in the paper [12] that for an arbitrary simple process $p_{(p)}$ the set $\pi_{(p)}$ with the above defined topology is a T_3-space.

Consequently, we have

THEOREM 1.1. $\mathscr{S}$ *is a* T_3-*space*.

In what follows we confine ourselves only to compact simple processes, i.e., processes such that if $p_{(p)} = (\pi_{(p)}; \rightleftarrows)$, then $\pi_{(p)}$ is a compact set. Obviously, this is restriction on the space $\mathscr{S}$ since it excludes processes with gaps. However, we shall soon find out that this restriction is both necessary and sufficient for our purposes.

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Consider the set of all simple processes $\mathbf{P}_{(p)}$ and an index set A; moreover,

$$\tilde{\pi}^{\alpha}_{(p)} \equiv \pi^{\alpha}_{(p)} \times \{\alpha\}.$$

Evidently, if $\alpha, \beta \in A$ and $\alpha \neq \beta$, then

$$\tilde{\pi}^{\alpha}_{(p)} \cap \tilde{\pi}^{\beta}_{(p)} = \phi.$$

Consequently, the set $\{\tilde{\pi}^{\alpha}_{(p)}\}_{\alpha\in A}$ is a family of compact, disjoint topological spaces, for $\bigwedge_{\alpha\in A} \lambda_\alpha : \tilde{\pi}^{\alpha}_{(p)} \to \pi^{\alpha}_{(p)};\ \lambda_\alpha(s, \alpha) = s$ is a homeomorphism.

It is known that every compact space is paracompact and a topological sum of paracompact spaces is paracompact. Moreover, the set $\{\pi^{\alpha}_{(p)}\}_{\alpha\in A}$ covers the whole $\mathscr{S}$. Let us therefore construct a topological sum $\bigoplus_{\alpha\in A} \tilde{\pi}^{\alpha}_{(p)}$ and define the map $\lambda\colon \bigoplus_{\alpha\in A} \tilde{\pi}^{\alpha}_{(p)} \overset{\text{on}}{\to} \mathscr{S}$, namely $\lambda|\tilde{\pi}^{\alpha}_{(p)} = \lambda_\alpha$, i.e. $\lambda(s, \alpha) = \lambda_\alpha(s, \alpha)$.

Let there be given in $\mathscr{S}$ a topology the subbasis of which is constituted by the sets $L(s)$, $R(s)$. It is readily observed that λ is then a closed map. On the other hand, it is known that if a closed map maps a paracompact space onto a T^1-space, then the latter is also paracompact (E. Michael [10]).

Thus we have proved

THEOREM 1.2. *$\mathscr{S}$ is a paracompact space.*(3)

It follows immediately that $\mathscr{S}$ is normal, for every paracompact space is normal (R. ENGELKING [5]).

Let us now introduce further axioms.

AXIOM 1.2. $\bigwedge_{\mathscr{P}\in\Pi} \ \bigvee_{f_{\mathscr{P}} : \mathscr{S} \overset{\text{on}}{\to} \mathscr{S}_{\mathscr{P}}}$ *such that*

1. $f_{\mathscr{S}} = id$,

2. $\mathscr{P}' < \mathscr{P} \wedge \bigwedge_{s_{\mathscr{P}}\in\mathscr{S}_{\mathscr{P}}} \ \bigwedge_{s_{\mathscr{P}'}\in\mathscr{S}_{\mathscr{P}'}} f^{-1}(s_{\mathscr{P}}) \cap f^{-1}(s'_{\mathscr{P}}) \neq \Theta \Rightarrow f^{-1}(s_{\mathscr{P}}) \subset f^{-1}(s'_{\mathscr{P}})$.

The elements $s_{\mathscr{P}} \in \mathscr{S}_{\mathscr{P}}$ will be called states of the subsystem $\mathscr{P}$. The properties of the function $f_{\mathscr{P}}$ imply imemdiately that $\bigwedge_{\mathscr{P}} \overline{\overline{\mathscr{S}}}_{\mathscr{P}} \leqslant \overline{\overline{\mathscr{S}}}$, [13].

AXIOM 1.3. *$\mathscr{S}$ is infinitely dimensional differentiable manifold of class $C^p (p \geqslant 2)$ modelled on the Banach space,* $\mathbf{B}$ *such that* $\bigwedge_{p=(\pi;\, \rightleftharpoons)}$ *there exists a piecewise single-valued C^p — morphism $\alpha\colon \pi \to \mathbf{I} \subset \mathbf{R}$.*

LEMMA 1.3. $\bigwedge_{p=(\pi;\, \rightleftharpoons)} \ \bigwedge_{s\in\pi} \ \bigwedge_{\alpha\in C^p(\pi)} \ \bigvee_{p_{(p)}\subset p} s \in \pi_{(p)}$ *and* $\alpha|_{\pi_{(p)}}$ is a C *— isomorphism.*

Proof. It follows from the properties of the function α that $\bigwedge_{s\in\pi}$ there exists a subprocess $p' = (\pi'; \rightleftharpoons)$ containing a state s such that $\alpha|_{\pi'}$ is an C^p — isomorphism. Since $p \in \mathbf{P}$, there exists a family of simple processes $\{p^i_{(p)}\}_{i\in J}$ such that $p = \bigoplus_{i\in J} p^i_{(p)}$. Hence

(3) λ is a closed map, $\bigoplus_{\alpha\in A} \pi^{\alpha}_{(p)}$ is a paracompact space, $\mathscr{S}$ is a T_1-space, for it is a T_3-space.

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$\bigwedge_{s \in \pi'} \bigvee_{p^k_{(p)} \in \mathbf{P}_{(p)}}$ such that $s \in \pi^k_{(p)}$. Let $\pi_{(p)} \equiv \pi' \cap \pi^k_{(p)} \subset \pi'$; then $s \in \pi_{(p)}$ and $\alpha|_{\pi_{(p)}}$ is a C^p — isomorphism.

The above lemma is equivalent to the assertion that every thermodynamic process is locally parametrisable.

In what follows, without affecting the generality of our considerations, we confine ourselves to a class of maps $\alpha|_{\pi_{(p)}}$, namely

DEFINITION 1.5. The parametrisation $\beta : \pi_{(p)} \to \mathbf{I}$ is called admissible when $\bigwedge_{s^1, s^2 \in \pi_{(p)}}$ $s^1 \xrightarrow{} s^2 \Rightarrow \beta(s^1) \leqslant \beta(s^2)$. Evidently, $\beta(s^1) = \beta(s^2)$, if and only if, $s^1 = s^2$.

2. Geometric objects on $\mathscr{S}$

We shall now present a method of defining geometric objects on the manifold $\mathscr{S}$. Our procedure is based on the theory of geometric objects on finitely dimensional manifolds [11] and generalised to the case of infinite dimension.

Observe first the following fact.

THEOREM 2.1. *In every open cover of $\mathscr{S}$ we can inscribe a partition of unity.*

Proof. Since $\mathscr{S}$ is a paracompact Hausdorff space, we can inscribe a partition of unity in every open cover.

Assume that the partition of unity on $\mathscr{S}$ is of class C^p.

Similarly to the case of a finite number of dimensions, we present a definition of the linear group [16].

DEFINITION 2.1. The linear Banach-Lie group $GL(\mathbf{B})$ is the set of all automorphisms of class C^{p-1} of the Banach space $\mathbf{B}$.

Evidently, $GL(\mathbf{B})$ is an open subset of the space $L(\mathbf{B})$, where $L(\mathbf{B})$ is the space of all linear and continuous maps of the space $\mathbf{B}$ into itself.

Let $\mathbf{U}(\mathscr{S}) \equiv \{(s; (U, \varphi))\}$, where $s \in U$ and (U, φ) is a map on $\mathscr{S}$. If $g \in GL(\mathbf{B})$, then $\langle (s; (U, \varphi)) | g \rangle = (s; (U, g \circ \varphi))$ and if $g' \in GL(\mathbf{B})$, then $\langle (s; (U, \varphi)) | g' \circ g \rangle = \langle (\langle (s; (U, \varphi)) | g \rangle) | g' \rangle$.

Let μ: $\mathbf{U}(\mathscr{S}) \to \mathscr{S}$ be given by the rule $\mu(s; (U, \varphi)) = s$. Hence $\mu(\langle \mathbf{u} | g \rangle) = \mu(\mathbf{u})$, where $\mathbf{u} \in U(\mathscr{S})$.

DEFINITION 2.2. We say that two elements $\mathbf{u}, \mathbf{u}' \in \mathbf{U}(\mathscr{S})$ are equivalent and we write $\mathbf{u} \sim \mathbf{u}'$ if, and only if,

1. $s = s'$,
2. $D_{\varphi(s)} (\varphi' \circ \varphi^{-1}) = id$,

$(D_{\varphi(s)} (\varphi' \circ \varphi^{-1})$ is the derivative in Frèchet sense of the function $\varphi' \circ \varphi^{-1}$ at the point $\varphi(s))$.

DEFINITION 2.3. The bundle of bases over the space $\mathscr{S}$ is the quotient space $\mathscr{F}^*(\mathscr{S}) = \mathbf{U}(\mathscr{S})/\mathscr{R}(\sim)$ with the canonical projection μ^*: $\mathscr{F}^*(\mathscr{S}) \to \mathscr{S}$, where $\mu^*([\mathbf{u}]) \equiv \mu(\mathbf{u})$ and $[\mathbf{u}] \in \mathscr{F}^*(\mathscr{S})$.

Thus we have the following lemma:

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LEMMA 2.1. *Let* $g \in GL(\mathbf{B})$ *and* $[\mathbf{u}]$, $[\mathbf{u}_1]$, $[\mathbf{u}_2] \in \mathcal{F}^*$; *then*

1. $\bigwedge_{[\mathbf{u}] \in \mathcal{F}^*(\mathcal{S})} \langle [\mathbf{u}] \| g \rangle = [\mathbf{u}] \Rightarrow g \equiv id,$

2. $\langle [\mathbf{u}_1]) g \rangle = [\mathbf{u}_2] \Rightarrow \mu^*([\mathbf{u}_1]) = \mu^*([\mathbf{u}_2]),$

3. $\mu^*([\mathbf{u}_1]) = \mu^*([\mathbf{u}_2]) \Rightarrow \bigvee_{g \in GL} \langle [\mathbf{u}_1] \| g \rangle = [\mathbf{u}_2],$

where the action of the group $GL(\mathbf{B})$ *in the space* $\mathcal{F}^*(\mathcal{S})$ *is defined in the canonical way.*

Let us now prove

THEOREM 2.2. $\mathcal{F}^*(\mathcal{S})$ *is a differentiable principal bundle over the basis space* $\mathcal{S}$ (4).

Proof. Let $\{(U_i, \varphi_i)\}$ be an atlas on $\mathcal{S}$ and consider the map $\Psi_i : \mu^{*-1}(U_i) \to L(\mathbf{B})$ such that

$$\Psi_i([\mathbf{u}]) = D_{\varphi_i(\mu^*([\mathbf{u}]))}(\varphi' \circ \varphi_i^{-1}) \quad \text{and} \quad [\mathbf{u}] \in \mu^{*-1}(U_i)$$

and, moreover,

$$(\mu^*([\mathbf{u}]); (U', \varphi')) \in \mu^{-1}(\mu^*([\mathbf{u}])).$$

Let $\Phi_i : \mu^{*-1}(U_i) \overset{\text{in}}{\to} U_i \times L(\mathbf{B})$, so that $\Phi_i([\mathbf{u}]) = (\mu^*([\mathbf{u}], \Psi_i([\mathbf{u}])$. Evidently, $\Phi_i(\mu^{*-1}(U_j)) = U_i \cup U_j \times GL(\mathbf{B})$ and therefore $\Phi_i \circ \Phi_j^{-1}(s, f) = (s, f \circ D_{\varphi_i(s)}(\varphi_j \circ \varphi_i^{-1}))$, since $\Psi_i([\mathbf{u}]) = \Psi_j([\mathbf{u}]) D_{\varphi_i(\mu^*([\mathbf{u}]))}(\varphi_j \circ \varphi_i^{-1})$, where $f \in GL(\mathbf{B})$. Since $D_{\varphi_i(s)}(\varphi_j \circ \varphi_i^{-1}) \in GL(\mathbf{B})$, $\Phi_i \circ \Phi_j^{-1}$ is a C^{p-1} — isomorphism.

Following S. LANG [8] we state the following theorem. "Let there be a space $\mathbf{E}$, a manifold $\mathcal{S}$ and the map $\mu: \mathbf{E} \to \mathcal{S}$. Assume that $\{U_i\}$ covers $\mathcal{S}$ and for each i $\mathbf{E}_i$ is a Banach space and a bijection τ_i: $\mu^{-1}(U_i) \to U_i \times \mathbf{E}_i$ such that $\bigwedge_{i,j} \bigwedge_{s \in U_i \cap U_j} (\tau_j \tau_i^{-1})_s$ is a linear topological isomorphism satisfying the condition 2b of the definition of the differentiable bundle. Then there exists on $\mathbf{E}$ a structure of a differentiable manifold such that μ is a morphism and $\mathbf{E}$ is a bundle."

Let $\Phi_i \equiv \tau_i$, $\Psi_i \equiv \tau_{is}$ and $\mathbf{E}_i \underset{i}{\equiv} L(\mathbf{B})$. Then all assumptions of the above theorem are satisfied and $\mathcal{F}^*(\mathcal{S})$ is a differentiable principal bundle over the basis $\mathcal{S}$.

Let $\mathbf{R}^k$ be a k-dimensional Euclidean space(5) with elements $q \in R^k$ and let σ: $\mathbf{R}^k \times GL(\mathbf{B}) \to \mathbf{R}^k$ be a differentiable action of the group $GL(\mathbf{B})$ in $\mathbf{R}^k$ satisfying the condition

$$\bigwedge_{g, h \in GL(\mathbf{B})} \sigma(\cdot, h) \circ \sigma(\cdot, g) = \sigma(\cdot, h \circ g).$$

DEFINITION 2.4. We say that two elements $([\mathbf{u}], q)$, $([\mathbf{u}'], q') \in \mathcal{F}^* \times \mathbf{R}^k$ are equivalent if, and only if, there exists an automorphism $g \in GL(\mathbf{B})$ such that $[\mathbf{u}'] = \langle [\mathbf{u}] \| g \rangle$ and $q' = \langle q | \sigma(\cdot, g) \rangle$.

(4) A differentiable principal bundle over the basis $\mathcal{S}$ is the space $\mathbf{F}$ with the morphism $\mu: \mathbf{F} \to \mathcal{S}$ such that

1. $\bigwedge_{s \in \mathcal{S}} \mu^{-1}(s) = \mathbf{F}_s$ has the structure of a Banach space,

2. $\bigwedge_i$ for each i the following map is given: $\tau_i : \mu^{-1}(U_i) \to U_i \times \mathbf{F}_i \subset \mathcal{S} \times \mathbf{F}_i$ ($\mathbf{F}_i$ is a Banach space),

satisfying the conditions

a) $\tau_{is} : \mu^{-1}(s) \to \mathbf{F}_i$ is a linear topological isomorphism,

b) the map $U_i \cap U_j \to L(\mathbf{F}_i, \mathbf{F}_j)$ associating with s the map $(\tau_j \tau_i^{-1})s$ is a morphism (S. LANG [8]),

(5) Instead of $\mathbf{R}^k$ we can take an arbitrary differentiable manifold.

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By $\mathbf{E}(\mathscr{S}, \mathbf{R}^k)$ we denote the quotient space $\mathscr{F}^* \times \mathbf{R} / GL(\mathbf{B})$.

Similarly to Theorem 2.2, we can prove

THEOREM 2.3. *$\mathbf{E}(\mathscr{S}, \mathbf{R}^k)$ is a differentiable vector bundle over $\mathscr{S}$ with a k-dimensional fiber and the canonical projection*

$$\mu_{\mathbf{R}^k} : \mathbf{E}(\mathscr{S}, \mathbf{R}^k) \to \mathscr{S}, \quad \mu_{\mathbf{R}^k}\big([\mathbf{u}], q\big) = \mu(\mathbf{u}).$$

DEFINITION 2.5. The manifold $\mathbf{E}(S, \mathbf{R}^k)$ is called a space of quantities of the type $\mathbf{R}^k$. The map $\lambda: \mathscr{S} \to \mathbf{E}$ such that $\mu_{\mathbf{R}^k} \circ \lambda = id$ is called the field of quantities of the type $\mathbf{R}^k$. If $\lambda \in C^1(\mathscr{S})$, then the field is called of class C^1.

The existence on $\mathscr{S}$ of a partition of unity of class C^p implies immediately the following corollary: $\mathscr{S}$ admits quantities of the type $\mathbf{R}^k$ of class C^p.

Consider now a class of geometric objects, namely objects with one component and a linear transformation law.

Let $k = 1$ and assume that $\sigma: \mathbf{R} \times GL(\mathbf{B}) \to \mathbf{R}$ is given by the formula $\sigma(q, g) = \Phi(g)q + \Psi(g)$, where $\Phi, \Psi: GL(\mathbf{B}) \to \mathbf{R}$ are functionals of class C^1 on $GL(\mathbf{B})$. Since $\sigma_g \circ \sigma_h = \sigma_{g \circ h}$ has to hold, the functionals Φ and Ψ must satisfy the system of equations

$$\bigwedge_{g, h \in GL(\mathbf{B})} \begin{aligned} \Phi(g \circ h) &= \Phi(g) \cdot \Phi(h), \\ \Psi(g \circ h) &= \Phi(h)\Psi(g) + \Psi(h). \end{aligned}$$

In the finite dimensional case it was proved [7] that the only objects of the type $(1, n, 1)$ are objects with the transformation law $\overline{\omega} = \omega + c\ln|\varphi(\mathrm{X})|$ or $\overline{\omega} = \varphi(\mathrm{X})\omega + c\big(\varphi(\mathrm{X}) - 1\big)$, where $\mathrm{X} = \det J_x$ and $c = \mathrm{const}$.

Consider therefore $\bigwedge_{g \in GL(\mathbf{B})} \varphi(g) = 1$. Then $\Psi(g \circ h) = \Psi(g) + \Psi(h)$ and $\sigma(q, g) = q + \Psi(g)$.

3. The first law of neoclassical thermodynamics

Consider an arbitrary subsystem $\mathscr{P} \in \Pi$ and define on $\mathscr{S}$ the following geometric objects:

$$\Gamma_{\mathscr{P}}: \mathscr{S} \to \mathbf{E}(\mathscr{S}, \mathbf{R}) \quad \sigma_g^{\Gamma}(q) = q - \frac{\mathrm{d}}{d\tau}\Psi_{\mathscr{P}}(g)$$

$$h_{\mathscr{P}}: \mathscr{S} \to \mathbf{E}(\mathscr{S}, \mathbf{R}) \quad \sigma_g^{h}(q) = q - \Psi_{\mathscr{P}}(g)$$

$$l_{\mathscr{P}}: \mathscr{S} \to \mathbf{E}(\mathscr{S}, \mathbf{R}) \quad \sigma_g^{l}(q) = q + \Psi_{\mathscr{P}}(g),$$

Here, $\Psi_{\mathscr{P}}: GL(\mathbf{B}) \to \mathbf{R}$ and $\Psi_{\mathscr{P}}(g \circ h) = \Psi_{\mathscr{P}}(g) + \Psi_{\mathscr{P}}(h)$.

The objects $h_{\mathscr{P}}$ and $l_{\mathscr{P}}$ will be called "the heat" and "the work" of the subsystem $\mathscr{P}$, respectively.

Let $\mathscr{F}$ be a linear space of vector functions and a subspace of the space $\mathbf{B}$; we shall call it the space of velocities of the system $\mathscr{B}$(6).

(6) For a continuum, this space was constructed in the paper by M. E. GURTIN, W. O. WILLIAMS, *On the first law of thermodynamics*, Arch. Rat. Mech. Anal. **42**, 2, 1971.

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Consider in $\mathbf{B}$ a family of linear subspaces $\{\mathbf{A}_k\}_{k\in K}$ and a family of linear continuous maps $\Phi_k: \mathbf{B} \to \mathbf{A}_k$ such that $\bigwedge_{b^1, b^2 \in \mathbf{B}} \Phi_k(b^1) \underset{k}{\equiv} \Phi_k(b^2) \Leftrightarrow b^1 = b^2$.

Assume that $\mathscr{F} \in \{\mathbf{A}_k\}_{k\in K}$ and for every $k \in K$ there exists an endomorphism $\varphi_k: \mathscr{S} \to \mathbf{A}_k$. The endomorphism $\varphi_{k^0}: \mathscr{S} \to \mathscr{F}$ will be denoted by V.

DEFINITION 3.1. We say that two states $s^1, s^2 \in \mathscr{S}$ are equivalent mod. V and we shall write $s^1 \underset{v}{\sim} s^2$, if and only if, for every subspace $\mathbf{A}_k \neq \mathscr{F}$ we have $\varphi_k(s^1) = \varphi_k(s^2)$.

It is readily observed that the relation is an equivalence relation. Let us therefore construct the quotient space $\mathscr{S}_v = \mathscr{S}/\mathscr{R}(V)$ which we call the non-mechanical state space; its elements are denoted by $s_v \in \mathscr{S}_v$.

AXIOM 3.1. *The objects $\Gamma_{\mathscr{P}}$, $h_{\mathscr{P}}$, $l_{\mathscr{P}}$ are defined in such a manner that there exists in $\mathscr{S}$ a configuration such that*

1. $\bigwedge_{s\in\mathscr{S}} \Gamma_{\mathscr{P}} = 0$
2. $\bigwedge_{s_v\in\mathscr{S}_v}$ *the map $s \to h_{\mathscr{P}}(s)$ is constant,*
3. $\bigwedge_{s_v\in\mathscr{S}_v}$ *the map $s \to l_{\mathscr{P}}(s)$ is linear.*

Here $s \in s_v$.

4. $\bigwedge_{s^0\in\mathscr{S}} \Gamma_{\mathscr{P}}(s^0) = \dfrac{d}{d\tau} h_{\mathscr{P}}(s)|_{s^0} = -\dfrac{d}{d\tau} l_{\mathscr{P}}(s)|_{s^0}$(7).

It follows immediately from Axiom 3.1 (1.4) that there exists on $\mathscr{S}$ a configuration such that $h_{\mathscr{P}}(s) \underset{s}{\equiv}$ const and $l_{\mathscr{P}}(s) \underset{s}{\equiv} 0$.

Definition 3.1. The heat flux to the subsystem $\mathscr{P}$ is the field $\mathscr{H}_{\mathscr{P}}: (\mathscr{S}, \mathbf{P}_{(p)}) \to \mathbf{E}(\mathscr{S}, \mathbf{R})$ such that

$$\bigwedge_{p_{(p)}\in\mathbf{P}_{(p)}} \bigwedge_{s\in\pi_{(p)}} \mathscr{H}_{\mathscr{P}}(s, p_{(p)}) = \frac{d}{d\tau} h_{\mathscr{P}}(\pi_{(p)})|_s - \Gamma_{\mathscr{P}}(s).$$

The power of the subsystem $\mathscr{S}$ is a similar field $\mathscr{L}_{\mathscr{P}}: (\mathscr{S}, \mathbf{P}_{\mathscr{P}}) \to \mathbf{E}(\mathscr{S}, \mathbf{R})$ such that

$$\bigwedge_{p_{(p)}\in\mathbf{P}_{(p)}} \bigwedge_{s\in\pi_{(p)}} \mathscr{L}_{\mathscr{P}}(s, p_{(p)}) = \frac{d}{d\tau} l_{\mathscr{P}}(\pi_{(p)})|_s + \Gamma_{\mathscr{P}}(s)\text{(8)}.$$

The above definition implies immediately that for every subsystem $\mathscr{P}$ we have $\sigma_g^{\mathscr{H}} = \sigma_g^{\mathscr{L}} = id$.

The following lemmas are implied by Axiom 3.1 and Definition 3.1.

LEMMA 3.1. $\bigwedge_{p\in\mathbf{P}} \bigwedge_{s\in\pi} \mathscr{H}_{\mathscr{P}}(s, p) = \mathscr{L}_{\mathscr{P}}(s, p) \equiv 0.$

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(7) Consider an arbitrary function $h: \mathscr{S} \to \mathbf{R}$ of class C^1 and an arbitrary simple process $p_{(p)} = (\pi_{(p)}; \to)$ and its admissible parametrisation $\beta: \pi_{(p)} \to I$. Then $\dfrac{d}{d\tau} h(s)/s^0 \equiv \lim\limits_{\beta(s)\to\beta(s^0)} \dfrac{\hat{h}(\beta(s^0)) - \hat{h}(\beta)(s))}{\beta(s^0) - \beta(s)}$, where $s, s^0 \in \pi_{(p)}$, $s \rightrightarrows s^0$ and $h(\beta(s)) \underset{s\in\pi_{(p)}}{\equiv} \hat{h}(\beta^{-1}(\beta(s)))$.

(8) Observe that the derivatives of the objects $h_{\mathscr{P}}$, $l_{\mathscr{P}}$ are with respect to admissible parametrisations.

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LEMMA 3.2. *Consider a simple process* $p_{(p)}$. *Let us select from the set* $\mathbf{P}_{(p)}$ *all simple processes* $p' = (\pi'; \underrightarrow{\ })$ *such that for every* $\pi'_{(p)}$ *there exists an order preserving isomorphism* $i: \pi_{(p)} \to \pi'_{(p)}$ *satisfying the condition* $\bigwedge_{s \in \pi_{(p)}} s_v = i(s)_v$. *Then, for every* $s \in \pi_{(p)}$

1) *the map* $i(\pi_{(p)}) \to \mathscr{H}_{\mathscr{P}}(i(s), i(\pi_{(p)}))$ *is constant,*

2) *the map* $i(\pi_{(p)}) \to \mathscr{L}_{\mathscr{P}}(i(s), i(\pi_{(p)}))$ *is linear.*

Thus the heat flux is independent of the velocity and the power is linear in it.

Observe that making use of the definitions of the objects $h_{\mathscr{P}}$ and $l_{\mathscr{P}}$ we can prove

COROLLARY 3.1. For every subsystem $\mathscr{P}$ there exists on $\mathscr{S}$ a real function $\varepsilon_{\mathscr{P}}: \mathscr{S} \to \mathbf{R}$ of class C^p such that $\varepsilon_{\mathscr{B}} = \text{const}$.

Proof. Let $\varepsilon_{\mathscr{P}}(s) \underset{s}{\equiv} h_{\mathscr{P}}(s) + l_{\mathscr{P}}(s)$. It follows from the properties of the objects $h_{\mathscr{P}}$ and $l_{\mathscr{P}}$ (p. 771) that $\varepsilon_{\mathscr{P}}$ is a function of state, such that $\varepsilon_{\mathscr{B}} = h_{\mathscr{B}} + l_{\mathscr{B}} = \text{const}$.

The function $\varepsilon_{\mathscr{P}}: \mathscr{S} \to \mathbf{R}$ will be called the energy of the subsystem $\mathscr{P}$.

THEOREM 3.1. *For every subsystem* $\mathscr{P} \in \Pi$, *the first law of thermodynamics is satisfied, i.e., "a change of the energy of the subsystem is equal to the sum of the heat flux and power of the subsystem". In other words,*

$$\bigwedge_{p_{(p)} \in \mathbf{P}_{(p)}} \bigwedge_{s \in \pi_{(p)}} \frac{d}{d\tau} \varepsilon_{\mathscr{P}}(\pi_{(p)})|_s = \mathscr{H}_{\mathscr{P}}(s, p_{(p)}) + \mathscr{L}_{\mathscr{P}}(s, p_{(p)}) = \mathscr{E}_{\mathscr{P}}(s, p_{(p)}).$$

The quantity $\mathscr{E}_{\mathscr{P}}(s, p_{(p)})$ *will be called the energy flux to the subsystem* $\mathscr{P}$.

Let us finally note one simple fact.

LEMMA 3.3. *Let* $p_{(p)} \in \mathbf{P}^r_{(p)}$ *(the set of reversible processes) and let* $p'_{(p)}$ *be the inverse process. Then*

$$\bigwedge_{\mathscr{P} \in \Pi} \bigwedge_{s \in \pi_{(p)}} \begin{aligned} \mathscr{H}_{\mathscr{P}}(s, p_{(p)}) &= -\mathscr{H}_{\mathscr{P}}(s, p'_{(p)}) \\ \mathscr{L}_{\mathscr{P}}(s, p_{(p)}) &= -\mathscr{L}_{\mathscr{P}}(s, p'_{(p)}) \end{aligned}$$

Proof. Let $\beta: \pi_{(p)} \to I$ be an admissible parametrisation of the process $p_{(p)}$. Since $p'_{(p)}$ is inverse to $p_{(p)}$, there exists between $\pi_{(p)}$ and $\pi'_{(p)}$ an isomorphism inverting the order of these sets. Thus the natural parametrisation for the process $p'_{(p)}$ is the map $\beta': \pi'_{(p)} \to I$ of the form $\beta'(s) = 1 - \beta(s)$.

Denote $f(\beta(s)) = 1 - \beta(s)$. Obviously, we have $\frac{d}{d\beta(s)} f(\beta(s)) = -1$.

It follows from Axiom 3.1 and Definition 3.1 that there exists in $\mathscr{S}$ a configuration in which $\mathscr{H}_{\mathscr{P}}(s, p_{(p)}) = \frac{d}{d\tau} h_{\mathscr{P}}(\cdot)|_s$. Making use of the definition of the derivative of the objects $h_{\mathscr{P}}$, we have

$$\frac{d}{d\tau} h_{\mathscr{P}}(\pi'_{(p)})|_s = \frac{d}{d\tau} h_{\mathscr{P}}(\pi_{(p)})|_s \cdot \frac{d}{d\tau} f(\cdot)|_{\beta(s)} = -\frac{d}{d\tau} h_{\mathscr{P}}(\pi_{(p)})|_s.$$

Similarly, we can prove that $\frac{d}{d\tau} l_{\mathscr{P}}(\pi'_{(p)})|_s = -\frac{d}{d\tau} l_{\mathscr{B}}(\pi_{(p)})|_s$ Hence $\mathscr{H}_{\mathscr{P}}(s, p_{(p)}) = -\mathscr{H}_{\mathscr{P}}(s, p'_{(p)})$ and $\mathscr{L}_{\mathscr{P}}(s, p_{(p)}) = -\mathscr{L}_{\mathscr{P}}(s, p'_{(p)})$. These relations hold only in the fixed configuration but since $\sigma^{\mathscr{H}}_g = \sigma^{\mathscr{L}}_g = id$ they are also true in every configuration.

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