76.

A WORD ON NONIONS.

[*Johns Hopkins University Circulars*, I. (1882), pp. 241, 242; II. (1883), p. 46.]

In my lectures on Multiple Algebra I showed that if u, v are two matrices of the second order, and if the determinant of the matrix $(z + yv + xu)$ be written as

$$z^2 + 2bxz + 2cyz + dx^2 + 2exy + fy^2$$

then the necessary and sufficient conditions for the equation $vu + uv = 0$ are the following, namely,

$$b = 0, \quad c = 0, \quad e = 0.$$

If to these conditions we superadd $d = 1, f = 1$, and write $uv = w$, then

$$u^2 = -1, \quad v^2 = -1, \quad w^2 = -1, \quad uv = -vu = w, \quad vw = -wv = u, \quad wu = -uw = v;$$

and 1, u, v, w form a quaternion system. The conditions above stated will be satisfied if

$$\text{Det.}\,(z + yv + xu) = z^2 + y^2 + x^2,$$

which will obviously be the case if

$$v = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \quad u = \begin{vmatrix} 0 & \theta \\ \theta & 0 \end{vmatrix},$$

where $\theta = \sqrt{(-1)}$. For then

$$z + yv + xu = \begin{vmatrix} z & y + x\theta \\ -y + x\theta & z \end{vmatrix}.$$

Hence the matrices

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & \theta \\ \theta & 0 \end{vmatrix} \quad \begin{vmatrix} -\theta & 0 \\ 0 & \theta \end{vmatrix}$$

construed as complex quantities are a linear transformation of the ordinary quaternion system $1, i, j, k$; that is to say, if we form the multiplication table

	λ	μ	ν	τ
λ	λ	μ	0	0
μ	0	0	λ	μ
ν	ν	τ	0	0
τ	0	0	ν	τ

$$\lambda + \tau = 1 \qquad -\mu + \nu = i$$

$$-\theta\lambda + \theta\tau = k \qquad \theta\mu + \theta\nu = j.$$

Since u, v contain between them 8 letters subject to the satisfaction of 5 conditions, the most general values of λ, μ, ν, τ ought to contain 3 arbitrary constants; but it is well-known that any particular (i, j, k) system may be superseded by a $\lambda(i', j', k')$ system, where i', j', k' are orthogonally related linear functions of i, j, k; and as this substitution introduces just 3 arbitrary constants, we may, by aid of it, pass from the system of matrices above given, to the most general form. The general expression for the matrices containing 3 arbitrary constants may also be found directly by the method given in my lectures, which will be reproduced in the memoir on Multiple Algebra in the *Mathematical Journal.* What goes before is by way of introduction to the *word* on Nonions which follows.

Just as the necessary and sufficient condition that u, v, two matrices of the second order, may satisfy the equations $vu = -uv$, $u^2 = 1$, $v^2 = 1$, is that the determinant to $z + yv + xu$ may be $z^2 + y^2 + x^2$, so I have proved that the necessary and sufficient condition, in order that we may have $vu = \rho uv$, $u^3 = 1$, $v^3 = 1$ (u, v being matrices of the third order, and ρ an imaginary cube root of unity) is that the determinant to $z + yu + xv$ may be $z^3 + y^3 + x^3$; but if we make

$$u = \begin{vmatrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0 \end{vmatrix}, \qquad v = \begin{vmatrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix},$$

then
$$z + yu + xv = \begin{vmatrix} z & 0 & y + x \\ \rho y + \rho^2 x & z & 0 \\ 0 & \rho^2 y + \rho x & z \end{vmatrix}$$

of which the determinant is

$$z^3 + (y + x)(\rho y + \rho^2 x)(\rho^2 y + \rho x) = z^3 + y^3 + x^3.$$

Hence there will be a system of Nonions (precisely analogous to the known system of quaternions) represented by the 9 matrices

$$\begin{array}{ccccc} & & 1 & & \\ & u & & v & \\ u^2 & & uv & & v^2 \\ & u^2v & & uv^2 & \\ & & u^2v^2 & & \end{array}$$

and just as in the preceding case the 8 terms ± 1, $\pm u$, $\pm v$, $\pm uv$ form a closed group, so here the 27 terms obtained by multiplying each of the above 9 by 1, ρ, ρ^2 will form a closed group. The values of the 9 matrices will easily be found to be

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0 \end{vmatrix} \qquad \begin{vmatrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & \rho^2 & 0 \\ 0 & 0 & \rho \\ 1 & 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho \\ \rho & 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho^2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & \rho \\ \rho & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix}$$

These forms can be derived from an algebra given by Mr Charles S. Peirce (*Logic of Relatives*, 1870).

I will only stay to observe that as the condition of the Determinant to $z + uy + vx$ (which for general values of u, v is a general cubic with the coefficient of z^3 unity) assuming the form $z^3 + y^3 + x^3$, implies the satisfaction of 9 conditions, and as u, v between them contain 18 constants, the most general form of a system of Nonions must contain $18 - 9$, or 9 arbitrary constants; but how these can be obtained from the particular form of the system above given, remains open for further examination.

[*Note.* For the remark made above] "These forms can be derived from an algebra given by Mr Charles S. Peirce (*Logic of Relatives*, 1870)," read "Mr C. S. Peirce informs me that these forms can be derived from his *Logic of Relatives*, 1870." I know nothing whatever of the fact of my own personal knowledge*. I have not read the paper referred to, and am not

* I have also a great repugnance to being made to speak of Algebras in the plural; I would as lief acknowledge a plurality of Gods as of Algebras.

acquainted with its contents. The mistake originated in my having left instructions for Mr Peirce to be invited to supply in my final copy for the press, such reference as he might think called for. He will be doing a service to Algebra by showing in these columns how he derives my forms from his logic*. The application of Algebra to Logic is now an old tale—the application of Logic to Algebra marks a far more advanced stadium in the evolution of the human intellect; the same may be said as regards the application by Descartes of Analysis to Geometry, and the reverse application by Eisenstein, Dirichlet, Cauchy, Riemann, and others, of Geometry to Analysis—so that if Mr Peirce accomplishes the task proposed to him (his ability to do which I do not call into question), he will have raised himself as far above the level of the ordinary Algebraic logicians as Riemann's mathematical stand-point tops that of Descartes.

It is but justice to Boole's memory to recall the fact that, in one of his papers in the *Philosophical Transactions*, he has made a reverse use of logic to establish a certain theorem concerning inequalities, which is very far from obvious, and which I think he states it took him ten years to deduce from purely algebraical considerations, having previously seen it through logical spectacles—I mean, by the aids to vision afforded him by his logical calculus: this theorem I believe (or at least did so when it was present to my mind) must of necessity admit of a much more comprehensive form of statement.

* I had understood Mr Peirce to say that these forms were actually contained in his memoir.