29.

NOTE ON CONTINUANTS.

[*Messenger of Mathematics*, VIII. (1879), pp. 187—189.]

To find the number of terms in the cumulant or continuant $(a_1, a_2, \ldots a_n)$, we may proceed as follows:

(1) There is the term $a_1 a_2 a_3 \ldots a_n$.

(2) The number of terms of the first order of degradation, that is, obtained by leaving out any pair of consecutive elements, is $n-1$, say $u_{n,1}$.

(3) The number of terms of the second order of degradation obtained by leaving out any two pairs of such, that is, by leaving out the first and second and some other pair of those that follow the second, the second and third and a pair of those that follow the third, the third and fourth and a pair of those that follow the fourth and so on, is

$$u_{n-2,1} + u_{n-3,1} + u_{n-4,1} + \ldots,$$

and, consequently,

$$(n-3) + (n-4) + (n-5) + \ldots$$

$$= \frac{(n-2)(n-3)}{2} =, \text{ say, } u_{n,2}.$$

(4) The number of the third order of degradation is in like manner

$$u_{n-2,2} + u_{n-3,2} + u_{n-4,2} + \ldots,$$

that is

$$= \frac{(n-4)(n-5)}{1.2} + \frac{(n-5)(n-6)}{1.2} + \ldots$$

$$= \frac{(n-3)(n-4)(n-5)}{1.2.3},$$

and so in general

$$u_{n,\,r} = u_{n-2,\,r-1} + u_{n-3,\,r-1} + u_{n-4,\,r-1} + \dots$$
$$= \frac{(n-r)(n-r-1)\dots(n-2r-1)}{1.2\dots r}.$$

Hence, the total number is

$$1 + (n-1) + \frac{(n-2)(n-3)}{1.2} + \frac{(n-3)(n-4)(n-5)}{1.2.3} + \dots.$$

Verification. In general

$$(i \sin\theta + \cos\theta)^n - (i\sin\theta - \cos\theta)^n$$
$$= 2\cos\theta\,\{(2i\sin\theta)^{n-1} + (n-2)(2i\sin\theta)^{n-3} + \frac{(n-3)(n-4)}{2}(2i\sin\theta)^{n-5} + \dots\},$$

for we know that

$$\cos\theta\,\{(2\sin\theta)^{n-1} - (n-2)(2\sin\theta)^{n-3} + \frac{(n-3)(n-4)}{2}(2\sin\theta)^{n-5} - \dots\}$$

$= (-1)^{\frac{1}{2}(n-1)} \cos n\theta$, or $(-1)^{\frac{1}{2}(n-2)} \sin n\theta$, according as n is odd or even.

Hence, putting

$$i\sin\theta + \cos\theta = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5},$$

so that

$$i\sin\theta - \cos\theta = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5},$$
$$2i\sin\theta = 1,$$

and

$$2\cos\theta = \sqrt{5},$$

$$1 + (n-1) + \frac{(n-2)(n-3)}{1.2} + \frac{(n-3)(n-4)(n-5)}{1.2.3} + \dots$$
$$= \frac{(\frac{1}{2} + \frac{1}{2}\sqrt{5})^{n+1} - (\frac{1}{2} - \frac{1}{2}\sqrt{5})^{n+1}}{\sqrt{5}}.$$

But because

$$(a_1, a_2, \dots a_n) = a_n(a_1, a_2, \dots a_{n-1}) + (a_1, a_2, \dots a_{n-2}),$$

if u_n is the number of terms in $(a_1, a_2, \dots a_n)$,

$$u_n = u_{n-1} + u_{n-2},$$

with the initial conditions

$$u_0 = 1, \quad u_1 = 1.$$

Solving this difference-equation, we shall obtain

$$u_n = \frac{1}{\sqrt{5}}\{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5})^{n+1} - (\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5})^{n+1}\},$$

agreeing with the preceding result.

Corollary 1. The value of the continuant of the nth order $(x, x, \dots x)$, is

$$x^n + (n-1)\, x^{n-2} + \frac{(n-2)(n-3)}{1\,.\,2}\, x^{n-4} + \dots,$$

which admits also of the clumsy representation

$$[\{\tfrac{1}{2}x + \tfrac{1}{2}\sqrt{(x^2+4)}\}^{n+1} - \{\tfrac{1}{2}x - \tfrac{1}{2}\sqrt{(x^2+4)}\}^{n+1}] \div \sqrt{(x^2+4)}.$$

Corollary 2. The value of the pro-continuant of the nth order

$$(2\cos\theta,\ 2\cos\theta, \dots 2\cos\theta),$$

is

$$\frac{\sin(n+1)\,\theta}{\sin\theta}.$$

By the pro-continuant is to be understood what a continuant becomes, when in its representative determinant, the oblique lines of negative units are all changed into positive units so that the matrix has two precisely similar bands of units one above and one below the diagonal line and in opposition with it.

Corollary 3. The integral of the partial-difference equation

$$u_{x+1,\,y} - u_{x,\,y} - u_{x-1,\,y-1} = 0,$$

limited by the conditions

$$u_{x,\,0} = 1,\ \ u_{x+1,\,x+1} = 0,$$

is

$$u_{x,\,y} = \frac{\Pi\,(x-y)}{\Pi\,(x-2y)\,\Pi y}.$$