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ON AN APPLICATION OF THE NEW ATOMIC THEORY TO THE GRAPHICAL REPRESENTATION OF THE INVARIANTS AND COVARIANTS OF BINARY QUANTICS,—WITH THREE APPENDICES.

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[The figures are given on p. 163.]

BY the *new* Atomic Theory I mean that sublime invention of Kekulé which stands to the *old* in a somewhat similar relation as the Astronomy of Kepler to Ptolemy's, or the System of Nature of Darwin to that of Linnæus;—like the latter it lies outside of the immediate sphere of energetics, basing its laws on pure relations of form, and like the former as perfected by Newton, these laws admit of exact arithmetical definitions.

Casting about, as I lay awake in bed one night, to discover some means of conveying an intelligible conception of the objects of modern algebra to a mixed society, mainly composed of physicists, chemists and biologists, interspersed only with a few mathematicians, to which I stood engaged to give some account of my recent researches in this subject of my predilection, and impressed as I had long been with a feeling of affinity if not identity of object between the inquiry into compound radicals and the search for "Grundformen" or irreducible invariants, I was agreeably surprised to find, of a sudden, distinctly pictured on my mental retina a chemico-graphical image serving to embody and illustrate the relations of these derived algebraical forms to their primitives and to each other which would perfectly accomplish the object I had in view, as I will now proceed to explain.

To those unacquainted with the laws of atomicity I recommend Dr Frankland's *Lecture Notes for Chemical Students*, vols. 1 and 2, London (Van Voorst), a perfect storehouse of information on the subject arranged in the most handy order and put together and explained with true scientific accuracy and precision. On the algebraical side of the subject my readers may consult Salmon's *Lessons on Higher Algebra*, Clebsch's *Binären Formen* or Faà de Bruno's treatise more elementary than the former, *Sur les formes binaires* (Turin, 1876). I propose also to run a course of articles on the Invariantive Theory, beginning from the beginning, through the pages of this Journal, from my own particular point of view, which will be found, I hope, considerably to simplify the subject.

Any binary quantic may be denoted by a single letter with a number attached corresponding to its degree, and may therefore be adumbrated by a chemical symbol with corresponding *valence*. Thus hydrogen, chlorine, bromine, or potassium will serve to denote so many distinct binary linear forms; oxygen, zinc, magnesium, &c., binary quadrics; boron, gold, thallium, cubics; carbon, lead, silicon, tin, quartics; nitrogen, phosphorus, arsenic, antimony, &c., quintics; sulphur, iron, cobalt, nickel, &c., sextics. The sixth appears to be the highest degree of valency at present recognizable in natural substances.

The factors of any algebraical form may be regarded as in some sense the analogues of the rays of atomicity in the equivalent chemical atom—these rays being what Dr Frankland, according to his nomenclature, would have to designate as free bonds; such rays between two consecutive atoms in a molecule are conceived as blending in some manner so as to represent some unknown kind of special relation existing between them; they may then with propriety be called bonds or lines of connexion.

An invariant of a form or system of algebraical forms must thus represent a saturated system of atoms in which the rays of all the atoms are connected into bonds. Thus, for example, O_2 (oxygen combined with itself) will represent a quadratic invariant of a quadric. Its graph is seen in Fig. 1 (a). Potash, a combination of potassium, oxygen and hydrogen, having for its graph that of Fig. 2, will represent the invariant to a system of one quadratic and two linear forms which is linear in each set of coefficients. This is in fact the *Connective* between the given quadratic and another obtained by taking the product of the two linear forms. Phosphorus and arsenic are quinquivalent, but form "tetratomic molecules." An isolated element of phosphorus may possibly, therefore, be represented by the graph of Fig. 3, which will correspond, if the figure is indecomposable (which requires examination to determine), to the quart-invariant of a quintic, and the same for arsenic. So too the graph to nitric anhydride (Fig. 4) may possibly serve to express the resultant of a binary quadric and quintic, or this blended with any other invariant of the system included under the same type [10: 5, 2; 2, 5]*. And in general, the Jacobian to any two quantics will be completely expressed by their two corresponding atoms connected by a pair of bonds. Nitric acid has for its graph that of Fig. 5. This will

* 10 is the weight; 5, 2 the degree and order in the coefficients of the quintic; 2, 5 the degree and order in the coefficients of the quadric. See p. [151].

correspond to an invariant of a quintic, quadric and linear form of the first order in the coefficients of each extreme and of the third order in those of the middle form. Such an invariant as is well known (by virtue of a general principle about to be stated), is, in substance, the same thing as a lineo-cubic linear covariant of a quintic and quadric. The general arithmetical rule (also hereafter to be set forth) for determining the number of asyzygetic derivatives of a given type, enables us to see that such a covariant exists and is monadelphic. It may readily be obtained by making the given quintic (after substituting $\frac{d}{dy}$ and $-\frac{d}{dx}$ for x and y respectively) operate on the cube of the given quadratic.

The general principle above referred to, which is extremely easily proved from the partial differential equation (but which I believe I was the first to enunciate), is that every covariant of one quantic or several simultaneous quantics may be transformed into an invariant of the same quantic or set of quantics enlarged by the addition thereto of one additional linear form; the degree in the variables becoming replaced by the order in the new set of coefficients, and the orders in the original sets of coefficients remaining unchanged.

Thus, covariants might altogether be dispensed with and invariants alone made the object of study. But algebraists have found and will continue to find it more convenient to dispense with the additional linear form and to retain in use covariants as well as invariants. With me, covariants are to be regarded as simple emanations, so to say, from differentiants which are functions of the coefficients alone, and of which invariants are merely a particular species satisfying a certain condition of maximum; this is why the properties of invariants can with difficulty be made out so long as they are studied alone; it was only by contemplating the whole group of differentiants simultaneously, that I was enabled, after a suspense of more than a quarter of a century, to set on an irrefragable basis Professor Cayley's fundamental arithmetical theorem for calculating the number of asyzygetic invariants and covariants to a given quantic, and also the more general theorem which I have shown applies to a system of quantics*.

I will here give this rule, as it may be useful to us in the sequel. First, for a single quantic.—Let i be its degree, j the order of any covariant, w its weight (that is, the weight of its root-differentiant). Then we may call its type $[w: i, j]$. Now let us, in general, employ $(m: i, j)$ to signify *the number of ways* in which m can be made up with j parts of which each is either 0, 1, 2, 3, &c. up to i, and let us use the symbol $\Delta\,(m: i, j)$ to denote $(m: i, j) - \{(m-1): i, j\}$; then $\Delta\,(w: i, j)$ is the number of arbitrary

* The demonstration is given in a paper inserted in the *Philosophical Magazine* for March of this year [p. 117, above].

numerical parameters in the most general covariant or invariant answering to the type $[w: i, j]$. It is a known theorem in partitions of numbers that $(m: i, j) = (m: j, i)$, from which it follows that the number of arbitrary parameters remains unaltered when the degree of the primitive and the order of the derivative are interchanged. It is sometimes more convenient to use the degree of the derivative in lieu of the weight to express its type; let then ϵ be the degree, so that $\epsilon = ij - 2w$; then I shall employ, when desirable, $[i, j : \epsilon]$ to signify the same thing as $[w : i, j]$. If there be several quantics, the type may be expressed in like manner by $[w : i, j ; i', j' ; \text{\&c.}]$, or by $[i, j ; i', j' ; \text{\&c.} : \epsilon]$. The rule for finding the number of independent parameters, or the most general covariant or invariant corresponding to either of these types, then becomes as follows. Let $(m: i, j ; i', j' ; \text{\&c.})$ denote the number of ways in which m can be made up of j elements each comprised between 0 and i, combined with j' elements each comprised between 0 and i', and so on, and let $\Delta\,(m: i, j; i', j'; \text{\&c.})$ denote $(m: i, j; i', j'; \text{\&c.}) - (m-1 : i, j; i', j'; \text{\&c.})$. The number of parameters in question is $\Delta\,(w : i, j ; i', j' ; \text{\&c.})$ and I may observe that the value of Δ remains unaltered when *any one* i is interchanged with the corresponding j, and consequently when any number of i's are interchanged, each respectively with its corresponding j. This theorem of reciprocity for a single quantic is due to M. Hermite. The above statement, applicable to a quantic system, constitutes a notable and important generalization of it. In Note D to Appendix 2, it will be shown that this theorem still further generalized by employing the method of Emanation (virtually the same thing as Regnault's law of substitution) admits of the following simple chemico-algebraical statement. *In an algebraical compound (in an algebraical sense) m n-valent atoms may be replaced by n m-valent ones.* But it should be observed that this replacement involves an entire reconstruction of the representative graph and conveys the notion of respondence or contraposition rather than similarity of type. (See Appendix 2.)

It may be well here (as it will be useful in the sequel) to say a few words more on these differentiants in their relation to covariants. Every covariant may be regarded as arising from either of two differentiants, as from a root. One, the coefficient of the highest power of x, is called a differentiant in x; the other, the coefficient of the highest power of y, a differentiant in y. It is not, for ordinary purposes such as present themselves in this study, requisite to consider more than one of these at a time, and for greater brevity it will be understood that, unless I give notice to the contrary, a *differentiant* will always be understood to mean one in x. I shall also suppose, when dealing with a single binary quantic, that the successive coefficients beginning with the highest power of x, are $a, b, c, \dots h, k, l$ multiplied successively by the binomial coefficients proper to the degree of the form.

A differentiant, D, may then be defined as a rational integer function of the coefficients of equal weight in all its terms in respect to either variable subject to satisfy the equation

$$\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}+\ldots\right)D=0.$$

An invariant again may be regarded as a rational integer isobaric function of the coefficients which is a differentiant both in regard to x and y, but it may be best defined as a differentiant (meaning in one of the variables as x) to a given form or form-system whose weight (in respect of the selected variable) is the greatest possible that its order in the coefficients admits of. [The doubleness of the character and the symmetry, direct or skew, of a differentiant satisfying this condition of maximum then become matter of deduction from the definition.] To each covariant corresponds but one differentiant (in a given variable), and *vice versa*, to each differentiant will correspond only one covariant. In fact, D being the differentiant in x, the covariant taking its rise in D is

$$Dx^{\epsilon}+\Omega\,.\,Dx^{\epsilon-1}y+\frac{1}{1\,.\,2}(\Omega.)^2Dx^{\epsilon-2}y^2+\ldots,$$

where $\Omega.$ represents the operator,

$$\left(l\frac{d}{dk}+2k\frac{d}{dh}+3h\frac{d}{dg}+\ldots\right)$$

if D belongs to a simple quantic, and

$$\Sigma\left(l\frac{d}{dk}+2k\frac{d}{dh}+\ldots\right)$$

if it belongs to a quantic system, and where ϵ is $ij-2w$ for a single quantic, and $\Sigma ij-2w$ for a quantic system, i representing the degree of any one form in the variables, j the order of the differentiant in the corresponding set of coefficients, and w the weight of the differentiant. As ϵ can never become negative, we see that the maximum value of w, when each i and its corresponding j is given, will be $\frac{1}{2}ij$ for one form, and $\frac{1}{2}\Sigma ij$ for a form system. By the weight of any covariant I shall understand the weight of the differentiant in which it may be regarded as originating. Precisely as algebraists find their advantage in using covariants when invariants alone might be made to suffice, chemists find theirs in the use of organic or inorganic compound radicals, as unsaturated forms capable of becoming saturated by the addition of the right number of monad elements to the unsatisfied atoms, that is, those through which a sufficient number of *bonds* do not pass to exhaust their valency. Thus, for example, Hydroxyl H – O – is the linear covariant of the quadratic form oxygen, and the linear form hydrogen; this, combined with the linear form potassium, expresses the invariant potash denoted by H – O – K.

As the free valence of a single atom corresponds to the degree of a single quantic, so the free valence of a molecule formed by an aggregate of atoms will express the degree of the corresponding covariant. Let us understand by the *toti-valence* of a molecule the sum of the absolute valences of the separate atoms of which it is composed. This toti-valence will obviously correspond to the sum, Σij, above mentioned. Since every bond or connecting line in the graph passes through two atoms, this toti-valence must be equal to the free valence of the molecules increased by twice the number of bonds; but Σij is the toti-valence, and ϵ (the degree of the covariant) is the number of unsatisfied bonds, and we have already stated in effect that ϵ increased by twice the weight of the root differentiant (which for brevity we call the weight of the covariant) is equal to Σij; hence the weight of a covariant (meaning that of its root differentiant), represented by any chemicograph, is the number of bonds or connecting lines between the atoms.

Let us consider an invariant or a covariant belonging to a type containing only one numerical parameter, which I shall call a monadelphic form*. Then this is either decomposable into factors or not; in the former case it may be termed composite, in the latter case prime. When prime its graph will also be prime, when composite its graph will be composite in a sense which will be made more clear by one or two examples. Let us take as a first example a graph composed of four triadic atoms of the same name, as in Fig. 6, where each atom, for instance, represents boron and in ordinary chemical symbolism would be denoted by the same letter B, but where for facility of reference I use four different letters to mark the positions of the several atoms. This corresponds to a covariant of a cubic for which the complete type, if we use the weight or number of bonds, is [4 : 3, 4], or, if we use the free valency, is [3, 4 : 4]. Now for a cubic the fundamental types, expressed in terms of the order and degree alone, omitting the constant number 3, which refers to the given degree, are

$$1 \,.\, 3$$
$$4 \,.\, 0$$
$$2 \,.\, 2$$
$$3 \,.\, 3.$$

Consequently, there is but one covariant corresponding to the given graph, and that is the product of the primitive by the covariant whose order and degree are each 3, the well-known skew covariant of $(a, b, c, d \between x, y)^3$ whose root or base is the differentiant $ad^2 - 3abc + 2b^3$.

* The type itself may also be termed a monadelphic type: so I shall speak when necessary of diadelphic, triadelphic, &c. types and designate any forms contained under such types as diadelphic, triadelphic, &c. forms. A family comprising many brothers, or any member of such a family, may each without doing violence to the laws or usage of language be termed *polyadelphic*.

It must be well understood that the bonds are not rigid, but capable of being curved or bent into any desired form. In this case the mode of decomposition is self-evident; for the skew covariant is represented by the triangle of Fig. 7, and we have only to draw out the elastic bond AC into the position ADC and place the atom D anywhere upon it to obtain the given graph. On the contrary the skew covariant itself is indecomposable and its graph ABC is obviously so too. Now let us consider the graph of Fig. 8. If the atoms at the angles are all triadic, there is no free valency, and the figure represents the invariant to a cubic form corresponding to 4 . 0 in the above table. It will be found, on trial, impossible to decompose it. But now suppose the atoms to be tetradic, the graph will represent a covariant of the fourth order and of the fourth degree to a quartic, each atom having one degree of valency unsatisfied. The fundamental derivatives of a quartic, of which all others are algebraical combinations, are represented in the following table of order and degree

1 . 4

2 . 0

3 . 0

2 . 4

3 . 3.

The complete covariant answering to the graph will therefore be $\lambda U + \mu V$, where, λ, μ being arbitrary numbers, U is the product of the primitive (1 . 4) by the cubinvariant 3 . 0, and V the product of the Hessian 2 . 4 by the quadrinvariant 2 . 0. Since, on making either $\lambda = 0$ or $\mu = 0$, the covariant breaks up and in two different ways into factors, we ought to expect that the graph should be capable of two corresponding modes of decomposition, and such we shall easily see is the case. For 1°, the invariant 3 . 0 may be represented by the graph of Fig. 9. Now imagine the three points E, F, G to come together and blend at D, and at D place a fourth atom. The given graph is thus recovered. Observe that this could not be done for the case of triads (corresponding to a cubic form) because, in the figure last referred to, the valence at each atom A, B, C is quadrivalent. Next, for the decomposition corresponding to the case of $\lambda = 0$ where the covariant breaks up into 2 . 0 multiplied by 2 . 4, the decomposition will be more easily followed by considering the graph to be pulled out into the form seen in Fig. 10. We may conceive this as the superposition of two carbon graphs, one in which the carbon atoms are at A and B connected by the *four* bonds AB, ACB, BDA, $ACDB$ denoting the quadrinvariant, and another in which the carbon atoms C, D are connected by the *two* bonds CAD, CBD, leaving two degrees of valence free at each atom and thus representing the quadro-quart-invariant or Hessian of the primitive.

I will now pass to the very interesting case which corresponds to one of the proposed graphs for benzole (or rather for the compound radical obtained by striking off its hydrogen atoms), a sextivalent hexad molecule of carbon—not the one proposed by Kekulé and which I believe still commands the general assent of chemists, but that suggested by Ladenburg* and put by him under the form of a wedge or prism. As, however, the question is one purely of colligation or linkage in the abstract, it is sufficiently described as a hexagon in which the three pairs of opposite angles are joined, or, if we please, as two triangles in which each angle of one is connected with a corresponding angle of the other. In regard of the atomicity theory, all these modes of colligation are identical, and the supposition that there is any real difference between them, or that figures in space are distinguishable from figures in a plane (as I heard suggested might be the case by a high authority at a meeting of the British Association for the Advancement of Science, where I happened to be present), is a departure from the cautious philosophical views embodied in the theory as it came from the hands of its illustrious authors and continued to be maintained by their sober-minded successors and coadjutors, and affords an instructive instance of the tendency of the human mind to the worship, as if of self-subsistent realities, of the symbols of its own creation.

The order (or number of atoms) being 6 and the unexhausted valences (one at each atom) also 6, we must turn to our table of fundamental derivatives to the quartic and shall find that the combination 6.6 is not amongst them, but that it can be obtained, and in only one way, by composition of the combinations therein contained. It is, in fact, the product of the cubic invariant 3.0 by the skew covariant 3.6, which has the very same *root* $a^2d - 3abc + 2b^3$ as the skew covariant to the cubic and accordingly has the same graph, namely a simple triangle. (It may be well to remark here incidentally, that it follows as an immediate consequence from the conditioning partial differential equation, that a root-differentiant to any quantic or system of quantics of given degree or degrees remains such to every other system in which one or more of those degrees is augmented.) On the other hand the cubic invariant has for its graph a triangle in which each line is doubled or looped. I shall show that Ladenburg's graph for the radical to benzole may be obtained by the superposition of these two forms. Let $ABC\gamma\beta\alpha$ represent a sextivalent tetradic hexad (Fig. 11); ABC, with the three loops $A\alpha\gamma C$, $C\gamma\beta B$, $B\beta\alpha A$, will represent a saturated triple atom of carbon, or the cubinvariant of a binary quartic. Again, $\alpha\gamma\beta$ taken alone will represent a sextivalent compound atom, or the fundamental skew covariant of the quartic, and the superposition of the two figures obviously gives the graph as it stands.

Another form of the product of the same two graphs would be a triangle inscribed in another, as in Fig. 12. Here $\alpha\beta\gamma$, as before, is the sextivalent

* *Berichte der deutschen chemischen Gesellschaft*, 1869, 141. I am indebted for this reference to my able colleague, Professor Ira Remsen.

molecule and ABC with the additional bonds $A\beta C$, $B\gamma A$, $C\alpha B$, the saturated one.

A simple hexagon of triadic atoms (Fig. 13) being sextivalent will serve to represent a derivative from a cubic of the sixth order and sixth degree. Such a covariant, in its most general form, will contain two parameters and be represented by $\lambda U^3 + \mu V^2$ where U is the Hessian $2 . 2$ and V the skew cube covariant $3 . 3$, and it is easy to see that this figure may be decomposed either into 3 bivalent, or 2 trivalent graphs. Thus AB, CD, EF, with the additional bonds $BCDEFA$, $DEFABC$, $FABCDE$, will represent the former; two atom groups such as A, C, E (with the bonds ABC, $AFEDC$, CDE, $CBAFE$, EFA, $EDCBA$) and B, D, F (with the bonds BCD, $BAFED$, DEF, $DCBAF$, FAB, $FEDCB$) the other. The first method of regarding the hexagon as a combination of three dyads may perhaps be admitted to throw some light on what Dr Frankland styles the two distinct molecular weights of sulphur. When two atoms of sulphur, regarded as bivalent, are combined by two loops, we have a representation of an isolated element of it as "a diatomic molecule." When three of these letters, regarded now as submolecules, are combined, or multiplied together into the hexagon, we have a representation of the isolated element as "a hexatomic molecule." More generally, let μ be the number of solutions of the equation in positive integers $2x + 3y = m$, then μ arbitrary parameters will enter into the most general representation of a covariant to a cubic of the order m in the coefficients and the degree m in the variables. Its graph will be a simple polygon of m sides and this will be capable of being decomposed, in μ essentially distinct ways, into elementary graphs consisting either, of binary groups or, ternary groups exclusively or, the two sorts of groups intermixed.

It may be easily shown (see Appendix 3) that every covariant of a binary form multiplied by a suitable power of its primitive, is capable of being represented by a rational integer function of covariants consisting, in addition to the primitive, of covariants exclusively of the second and third orders in the coefficients. I have already given an example of the mode in which a graph may be augmented by an additional atom corresponding to the multiplication of a covariant by the primitive.

The important proposition above referred to (given in Clebsch's *Binären Formen*) amounts then to affirming that any homogeneous graph augmented by a suitable number of atoms of the same, may be decomposed, in one or more ways, into bilooped dyads and single-sided triangles. Such a proposition ought to admit of graphical proof. The theorem has considerable graphical importance because it enables us, in some cases at least, to discriminate the true from the spurious graphs, or as we might say, pseudographs, representing a given type. Thus, it serves to show that Fig. 14 and not Fig. 15 is the graph to the discriminant of a cubic; for, in accordance with Clebsch's theorem, this discriminant, namely

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd,$$

multiplied by a^2 becomes equal to the square of $a^2d - 3abc + 2b^3$, together with four times the cube of $ac - b^2$, and consequently its graph, after combination with two additional points, should be decomposable, at will, into 3 double-looped lines, or into 2 single-lined triangles, which is the case with Fig. 14, inasmuch as its combination with two points gives rise to a simple hexagon, but not with Fig. 15.

If we call the apices of the two figures, 14, 15, a, b, c, d, the true graph (on substituting negative signs for bonds and prefixing a sign of summation) reads as

$$\Sigma\,(a-b)^2(c-d)^2(a-c)(b-d),$$

which is the cubinvariant of the quartic whose roots are a, b, c, d, so that a graph to an invariant of the type [3, 4 : 0] gives the algebraical expression in terms of the roots of an invariant of the reciprocal type [4, 3 : 0]. On the other hand, the pseudograph treated in the same way reads as

$$\Sigma\,(a-b)(b-c)(c-d)(d-a)(a-c)(b-d),$$

the value of which is zero; a similar remark may probably be found to be true of reciprocal graphs of invariants in general. This is abundantly confirmed by subsequent investigation; see remarks at end of Appendix 1.

So again, if we take the graph of Fig. 42, which represents an invariant to the type [3, 2; 1, 2 : 0], it reads off into

$$\Sigma\,(B_1 - B_2)^2(B_1 - H_1)(B_2 - H_2),$$

belonging to the reciprocal type [2, 3; 2, 1 : 0], and the Σ is in fact the discriminant of one binary quadratic multiplied by the connective between it and another.

So if we take the graph represented in (a), Fig. 45,

$$\Sigma\,(O_1 - O_2)(O_1 - H)(O_2 - K)$$

will represent an invariant to the type [2, 2; 1, 1; 1, 1 : 0]. If, however, we were to substitute H_1, H_2 in lieu of H and K, so as to form the hydroxyl graph of Fig. 45 (b), it would not be true that $\Sigma\,(O_1 - O_2)(O_1 - H_1)(O_2 - H_2)$ would represent an invariant to the type [2, 2; 2, 1 : 0]; on the contrary it would be zero. But hydroxyl is *not an invariant*, for to the combination of a quadratic and a linear form there appertains no invariant of the second degree in the coefficients of each of them. This may be easily proved by the rule I have given at the commencement of this paper. I have gone through this calculation for the benefit of those new to the subject and to show how the arithmetical "rule of multiplicity" is to be applied. Had I been writing solely for algebraists it would have been unnecessary to prove so familiar a fact. We have here

$$i=2,\quad j=2,\quad i'=1,\quad j'=2,\quad w=\frac{ij+i'j'}{2}=3.$$

To find $(w: i,\ j;\ i',\ j')$ we have to count the combinations

$$\begin{array}{c|c} 2.1 & 0.0 \\ 2.0 & 0.1 \\ 1.1 & 0.1 \\ 1.0 & 1.1; \end{array}$$

the number of these is 4. Again to find $(w-1: i,\ j;\ i';\ j')$ we have to count the combinations

$$\begin{array}{c|c} 2.0 & 0.0 \\ 1.1 & 0.0 \\ 1.0 & 0.1 \\ 0.0 & 1.1, \end{array}$$

of which the number is also 4. Hence

$$\Delta(3:\ 2,\ 2;\ 1,\ 2)=4-4=0.$$

So that hydroxyl, being of the type [3: 2, 2; 1, 2], cannot be an invariant.

So far then the supposed law is safe; but I think I see other difficulties in the way of its application to heteronymous types, so that if it shall be capable of being made universally applicable, other parts of the graphical theory, as it has been laid down, will possibly require reconsideration. What I advance is to be regarded not as dogmatic but as tentative and open to correction.

It is obvious that not every chemico-graph, potential or even actual, corresponds to an invariantive derivative. Of this I have already given examples. Were the case otherwise we should have surprised the secret of nature, for, as we know how to obtain all possible fundamental forms to binary quantics, we should know *à priori* all possible compound radicals. As a matter of fact the cases of algebraical invariance in nature seem to be rare and rather the exception than the rule. Thus while muriatic acid (H – Cl), is an invariant, self-saturating hydrogen (H – H), is a non-invariant, there being a linear invariant to two linear forms but not to a single one. In like manner ozone (Fig. 16) is also non-invariantive, there being no cubic invariant to a quadratic form. But there is an essential difference to be observed between the two cases. A graph consisting of a single or an odd number of bonds between two atoms of the same kind can *never*, for any species of such atoms, be invariantive, because no covariant of the second order in the coefficients can have an *odd* weight. If that were possible, then, by the theorem of reciprocity, a quadratic function could have an invariant or covariant of an odd weight, which is, of course, not true. Whereas a triangle of n-ads, although it does not picture an invariant when $n=2$, does do so when $n=3$ or any higher number. When an homonymous graph is given in weight (the number of bonds) and in order (the number of atoms) two of the elements of its type (w: i, j) say w, j are known and the third i is left indeterminate. For all values of i which make $\Delta(w: i, j)$ greater than zero, there will be one or a plurality of such graphs according to the value of Δ. If no value of i makes Δ greater than zero, there will be no such graph possible, but it is not necessary, to ascertain this, to make an indefinite number of trials, for it is obvious that for all values of i equal to or greater than w, Δ has the same value, namely $\Delta(w: \infty, j)$, since the condition that a number w shall not be made up of numbers greater than i, when i is equal to w, becomes nugatory.

It will be instructive to consider the case of $w=5, j=3$, and consequently the free valence $\epsilon = 3i - 10$; this implies that i must be at least equal to 4. But if we take $i=4$, $\epsilon=2$, as there is no covariant to a binary quartic whose order is 3 and degree 2, we may be sure that $\Delta(5: 4, 2)=0$. Hence we have only to consider the case of $i=w=5$, $\epsilon=5$. $\Delta(5: 5, 3)$ is the number of covariants of the fifth order and fifth degree to a cubic of which there is but *one*, formed by the multiplication together of the Hessian and skew-covariant. If now we proceed to form the graph corresponding to the type [5: 5, 3], we have the choice of two figures, 17, 18. In the former figure there are three degrees of vacancy from saturation at A and one at each of the points B, C. In the latter, one at A and two at each of the points B and C. The graph, we must recollect, is to correspond to a cubic covariant of the fifth degree to a fifthic which is unique and indecomposable. This enables us to fix upon the true representation. It cannot be the graph of Fig. 17, for that may be considered as generated by the combination of one isolated nitrogen atom with two atoms of nitrogen, B, C, connected by five bonds; two of these being subsequently welded together and bent out into the angle having A at its vertex. [The hypothetical nitrogen pair exists in chemistry but not as an algebraical invariant.] Hence the true figure can but be that given in Fig. 18, where the free valence is separated into the parcels 2, 1, 2, and not as in Fig. 17 into the parcels 1, 3, 1. And it should be observed that, for all higher values of i beyond 5, this will continue to be the one and only true graph to the corresponding covariant. It thus appears that every given homogeneous graph has an intrinsic character of capability or incapability of respondence to algebraical in- or co-variance, irrespective of the particular valence assigned to its atoms, and it is natural to suppose that there must be some immediate intrinsic criterion for determining this character, so as to dispense with the necessity of any algebraical considerations to establish it; but if such criterion exists, I have not yet been able to make out what it is*. In common with this view we may consider the theory of reciprocity of algebraical derived forms. It has already been stated that to every m-ad of n-ad atoms having a given number of bonds corresponds an n-ad of m-ad atoms with the same number of bonds. As for example, to a quasi carbon-ad (so to say) of sulphur will correspond a quasi sulphur-ad of carbon, the number of bonds and consequently the amount of free atomicity remaining the same in the two molecules. This suggests the possibility of there being some mode of passing from a graph to its reciprocal (this reciprocity being seemingly of quite a different kind from that which connects correlated girders or frameworks in graphical statics). I offer the subjoined instance of such transformation tentatively and with a view to stimulate inquiry, rather than as possessing any assurance of the validity of the process employed.

Suppose the case of $i = 4$, $j = 2$, $w = 4$; the one and only corresponding graph will be a system of 4 bonds connecting two atoms A, B. If now we take a pair of these bonds, stretch them out, weld them together and form a knot between them at C, and in like manner convert the other pair of bonds into a pair knotted at D, we shall have a graph consisting of a simple quadrilateral which will correspond to the case of $i = 2$, $j = 4$.

Again, suppose $i = 6$, $j = 4$, $w = 12$. We may consider either of the graphs quasi in Figures 19, 20. In the first of these figures we may take four bonds connecting respectively AC, CB, AD, DB, stretch and weld them together and form a knot between them at a new point E which will then be attached by four bonds to the atom $ABCD$. I mean that we may stretch out AC, CB, to meet in E (Fig. 21) and have EC common, and in like manner stretch out AD, DB to E and have ED common and then knot together the four bonds of the strings at E. In like manner we may form another knot F with bonds through AB, BC, AD, DC, and shall thus obtain the reciprocal graph of Fig. 21, where now $i = 4$, $j = 6$, $w = 12$. So again it will be found that we may distort Fig. 20 (if I can trust to my recollection of the result of previous work) in two different ways into a reciprocal graph.

At the risk of provoking the ire or ridicule of my chemical friends and the chemical public, I will venture to throw out a few remarks on the substructure, so to say, of the accepted theory of atomicity and to offer a suggestion as to a possible mode of getting rid of some imperfections under which it appears at present to labour. First there is the inconsistency of admitting the isolated existence of single atoms of mercury, cadmium and zinc, as monads with their bonds or tails absorbed or suppressed or else swinging loose and unsatisfied in direct opposition (as it seems to me) to the fundamental postulate of the theory. Next, one cannot get over a somewhat uncomfortable feeling at the representation of isolated oxygen in the state

* The law of reciprocity, however, exemplified above can obviously be made to supply the criterion in question.

of ozone by a triangular graph, which, although conceivable, is supported by no analogous case unless that of baric peroxide, or any similar graph, be regarded as such. Thirdly, there is the vague and unsatisfactory (not to say unthinkable) explanation of the variability of the valence of a given atom by what Dr Frankland calls "the very simple and obvious assumption that one or more pairs of bonds belonging to the atom of an element can unite and having saturated each other become, as it were, latent."

Now these stumbling-blocks to the acceptance of the theory may be removed by one simple, clear and unifying hypothesis, which will in no wise interfere with any actually existing chemical constructions. It is this: leaving undisturbed the univalent atoms, let every other n-valent atom be regarded as constituted of an n-ad of *trivalent* atomicules arranged along the apices of a polygon of n sides. Thus, sextivalent, quinquivalent and quadrivalent atoms in their state of maximum valence will be represented by Figures 22, 23, 24, where the letters denote *trivalent atomicules*. When the valence is reduced by two we need only conceive any one of the side loops doubled or a new loop as formed by the coalescence of a pair of free bonds or tails, and when in the Figures 22 and 23 the valence is reduced by 4, we may in like manner either suppose existing loops doubled, or fresh ones inserted, or both changes to go on simultaneously, by the coalescence of two pairs of tails. We have thus a conceivable and conformable-to-analogy method of accounting for the variability in question. So likewise, a trivalent atom with maximum state of valence will be represented by Fig. 25, and when univalent by Fig. 26. Again, an isolated zinc element will have for its graph Fig. 1 (b), the two letters Z signifying the zinc atomicules, and so in like manner isolated cadmium and mercury may be represented. On the other hand O_2, isolated oxygen in its ordinary state, will be represented by the graph of Fig. 27, whilst ozone will have for its representative graph the well known Kekuléan hexad (which, in its importance to chemistry, would seem to vie with Pascal's mystic hexagons to geometry) represented in Fig. 28, where as in Fig. 27, each letter O represents an atomicule of oxygen. So an isolated element of carbon would be represented by the graph of Fig. 29.

This hypothesis of atomicules, if unobjectionable on other grounds, would not be open to the charge of having any tendency to disturb or complicate the existing graphology; for we should still be at perfect liberty to substitute for the graphs (a) of Figures 30, 31, 32 the abridged notation (b), and should naturally do so when considering the relations of atoms to each other. The beautiful theory of atomicity has its home in the attractive but somewhat misty border land lying between fancy and reality and cannot, I think, suffer from any not absolutely irrational guess which may assist the chemical enquirer to rise to a higher level of contemplation of the possibilities of his subject. I have therefore ventured to make the above suggestion.

Chemical graphs, at all events, for the present are to be regarded as mere translations into geometrical forms of trains of priorities and sequences having their proper *habitat* in the sphere of order and existing quite outside the world of space. Were it otherwise, we might indulge in some speculations as to the directions of the lines of emission or influence or radiation or whatever else the bonds might then be supposed to represent as dependent on the manner of the atoms entering into combination to form chemical substances. Such not being the case, what follows is to be considered as having relation to mere *algebraical* atoms, or atomicules (quantics) and their bonds which may be regarded as represented by the linear factors of such quantics.

Let us consider a symmetrical trivalent atomicule whose three bonds or rays make angles of 120° with each other. Calling τ, τ', τ'', the tangents of the angles which the axis of y makes with its rays, we have

$$\tau' = \frac{\tau + \sqrt{(3)}}{1 - \sqrt{(3)}\,\tau}, \qquad \tau'' = \frac{\tau - \sqrt{(3)}}{1 + \sqrt{(3)}\,\tau},$$

so that its equation will be easily found to be

$$(1 - 3\tau^2)\,x^3 + (9\tau - 3\tau^3)\,x^2y + (9\tau^2 - 3)\,xy^2 + (\tau^3 - 3\tau)\,y^3 = 0,$$

which may be identified with the standard form

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$$

by writing $\qquad a = 1 - 3\tau^2 = -c, \qquad b = 3\tau - \tau^3 = -d.$

Suppose the three atomicules to become condensed into a single atom after the manner of the graph of Fig. 25. The combination will be represented by the cubic covariant (see Tables des Invariants et Covariants, Table V, annexed to Faà de Bruno's *Théorie des Formes Binaires*)

$$(a^2d - 3abc + 2b^3)\,x^3 + (3abd - bac^2 - 3b^2c)\,x^2y$$
$$+ (3bc^3 + 6b^2d - 3acd)\,xy^2 + (3bcd - ad^2 + 2c^3)\,y^3,$$

which, for the present case, becomes

$$2\,(1 + \tau^2)^3\,[(3\tau - \tau^3)\,x^3 + (9\tau^2 - 3)\,x^2y + (3\tau^3 - 9\tau)\,xy^2 + (1 - 3\tau^2)\,y^3].$$

Hence the new ray-directions will have for their equation

$$-dx^3 + 3cx^2y - 3bxy^2 + ay^3 = 0,$$

or the pencil of the atom will be identical with that of each of the separate atomicules, but accompanied with a rotation (whatever that may mean) of the whole pencil of rays through a right angle in its own plane. Again, suppose that only two atomicules are brought into connexion as in (a) of Fig. 30. The quadricovariant which expresses the atom (Faà de Bruno *ante*) is

$$(ac - b^2)\,x^2 + (ad - bc)\,xy + (bd - c^2)$$

which here becomes $\qquad -(1 + \tau^2)^3\,(x^2 + y^2).$

Hence the ray-directions will be given by the equation

$$y^2 + x^2 = 0, \qquad y = \pm x\sqrt{(-1)},$$

which we may, if we please, according to the usual convention concerning the

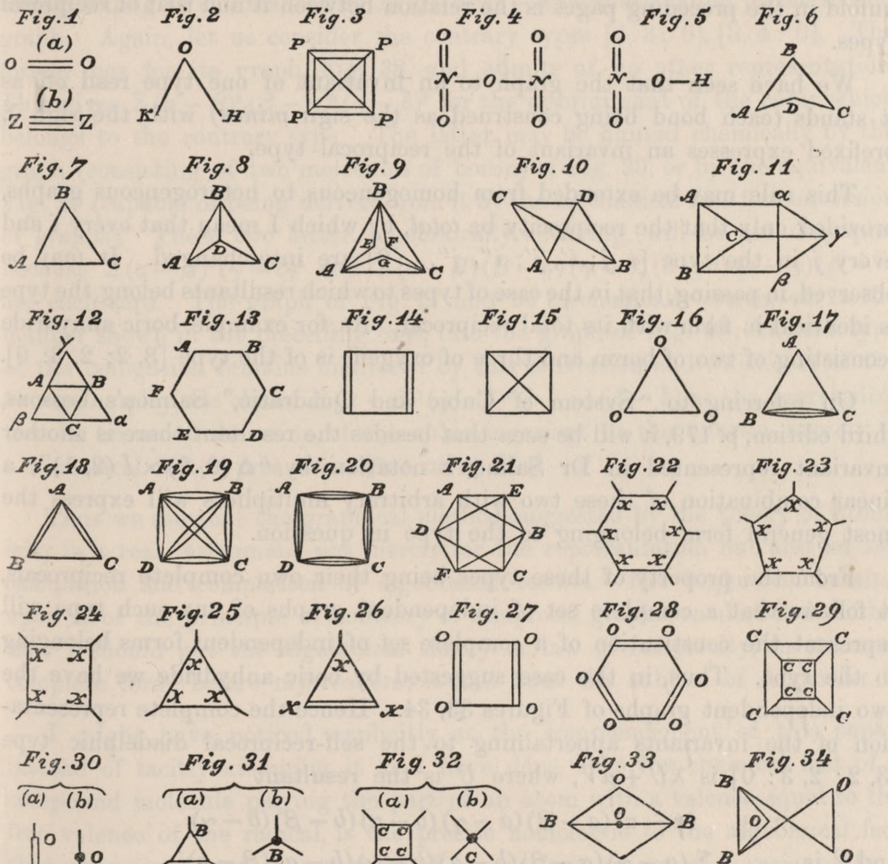

Fig. 1 (a) (b); Fig. 2; Fig. 3; Fig. 4; Fig. 5; Fig. 6; Fig. 7; Fig. 8; Fig. 9; Fig. 10; Fig. 11; Fig. 12; Fig. 13; Fig. 14; Fig. 15; Fig. 16; Fig. 17; Fig. 18; Fig. 19; Fig. 20; Fig. 21; Fig. 22; Fig. 23; Fig. 24; Fig. 25; Fig. 26; Fig. 27; Fig. 28; Fig. 29; Fig. 30 (a) (b); Fig. 31 (a) (b); Fig. 32 (a) (b); Fig. 33; Fig. 34; Fig. 35; Fig. 36; Fig. 37; Fig. 38; Fig. 39; Fig. 40; Fig. 41; Fig. 42; Fig. 43; Fig. 44; Fig. 45 (a) (b)

square root of minus unity, explain by supposing that the original rays are situated in planes perpendicular to the joining line XX, and that these are replaced by two rays lying in opposite directions along the line XX, where the atomicules are condensed into one atom. But it would be idle to pursue this speculation further.

The most remarkable point in the theory which I have endeavoured to unfold in the preceding pages is the relation between it and that of reciprocal types.

We have seen that the graph to an invariant of one type read off as it stands (each bond being construed as the sign *minus*) with the sign Σ prefixed expresses an invariant of the reciprocal type.

This rule may be extended from homogeneous to heterogeneous graphs, provided only that the reciprocity be *total*, by which I mean that every i and every j in the type $[i, j; i', j'; i'', j''\ldots : 0]$ are interchanged. It may be observed, in passing, that in the case of types to which resultants belong, the type is identical in form with its total reciprocal. As, for example, boric anhydride (consisting of two of boron and three of oxygen) is of the type $[3, 2;\ 2, 3:\ 0]$.

On referring to "System of Cubic and Quadratic," Salmon's Lessons, third edition, p. 179, it will be seen that besides the resultant there is another invariant represented in Dr Salmon's notation by "$\Delta(0, 2) \times I(2, 1)$"; a linear combination of these two with arbitrary multipliers will express the most general form belonging to the type in question.

From the property of these types being their own complete reciprocals, it follows that a complete set of independent graphs of any such type will represent the constitution of a complete set of independent forms belonging to the type. Thus, in the case suggested by boric anhydride we have the two independent graphs of Figures 33, 34. Hence the complete representation of the invariants appertaining to the self-reciprocal diadelphic type $[3, 2;\ 2, 3:\ 0]$ is $\lambda U + \mu V$, where U is the resultant

$$(a-\alpha)(a-\beta)(a-\gamma)(b-\alpha)(b-\beta)(b-\gamma)$$

and V is
$$\Sigma(a-\gamma)(a-\beta)(b-\alpha)(b-\gamma)(b-a)(\beta-\alpha).$$

U is derived from the graph of Fig. 33 by replacing the several O's by α, β, γ, and the B's by a, b, and V in like manner from the graph of Fig. 34. This latter graph is replaceable by the disjoined graph of Fig. 35, to which, by the rule for combination of graphs, it is easily seen to be equivalent.

Hence, instead of $\lambda U + \mu V$ we may write $\lambda V + \mu V'$ where

$$V' = \Sigma(\alpha-\beta)^2(a-b)^2(a-\gamma)(b-\gamma);$$

a, b of course will be understood to be the roots of a general quadric and α, β, γ of a general cubic. A very good similar instance of this kind of equivalence is afforded by the quadrinvariant of a quartic whose type is $[4, 2:\ 0]$. The reciprocal of this, namely $[2, 4:\ 0]$, may be represented, either by the connected graph of Fig. 36, or by the disjoined one of Fig. 37, and accordingly the noted quadrinvariant $ae - 4bd + 3c^2$ may be expressed (to a numerical factor près) either by the symmetrical function

$$\Sigma (a-c)(a-d)(b-c)(b-d)$$

corresponding to the first, or by $\Sigma (a-b)^2 (c-d)^2$ corresponding to the second graph. Again, let us consider the contrary types [4, 3 : 0], [3, 4 : 0]. The former has for its graph Fig. 38, and admits of no other representation. This gives $\Sigma (\alpha-\beta)^2 (\beta-\gamma)^2 (\gamma-\delta)^2$ for the discriminant of the cubic which belongs to the contrary type. The latter may be figured chemically by the graph (consisting of two molecules of boron) of Fig. 39, or by the equivalent Fig. 27 (capable of being derived from it by the mechanical rule for conversion of graphs). These two latter, algebraically speaking, will be pseudographs, because $\Sigma (\alpha-\beta)^3 (\gamma-\delta)^3$ and $\Sigma (\alpha-\beta)(\beta-\gamma)(\gamma-\delta)(\delta-\alpha)(\alpha-\gamma)(\beta-\delta)$ are each zero. The graph of Fig. 27 may be mechanically converted, in the manner shown in the preceding case, into the graph of Fig. 40; but the type of the colligation remains unaltered by this conversion and whichever of the two we employ, we obtain $\Sigma (\alpha-\beta)^2 (\gamma-\delta)^2 (\alpha-\gamma)(\beta-\delta)$ as the representation in terms of the roots, of the cubic invariant to the quartic, namely to a numerical factor près $ace - b^2e - ad^2 + 2bcd - c^3$.

Thus we see that the graphical method suggested by the theory of atomicity is a real instrument not merely for the representation but also for the calculation and comparison of algebraical results. The important bearing upon it of the principle of contrary or reciprocal graphs, renders it desirable that I should put the algebraical theory or law of reciprocity, in its most complete form, before my readers; it will form the subject of Appendix 2.

I might have noticed explicitly at the commencement of this paper, instead of tacitly assuming it as I have done, that the chemical fact of a compound molecule playing the part of an atom with a valence equal to the free valence of the radical, is the precise homologue to the algebraical fact that every invariant or covariant of a covariant, or set of covariants, to a quantic, or system of quantics, is itself an invariant or covariant to such quantic, or system of quantics; and again that Regnault's chemical principle of substitution and the algebraical one of emanation* are identical; and again, the modern notion of two semi-molecules, simple or compound, combining or uniting to form a chemical substance is tantamount to the construction of an invariant, the connective (or in Professor Gordan's language, the final "Ueberschiebung") of a quantic, or of the derivee of a quantic or a set of quantics,

* By which I mean in this place the operation upon an invariant or covariant of the symbol $(a'\delta_a + b'\delta_b + \ldots)$ performed any number of times in succession; a, b, for instance, may refer to Hydrogen $(ax+by)$ and a', b' to Chlorine $(a'x+b'y)$, and then the emanantive operator, according to a notation used, if I mistake not, by Professor Clerk Maxwell in his theory of poles, might be denoted by $Cl\delta_H$.

with itself. So again, it will hereafter be seen* that Hermite's law of reciprocity applied to quantic systems and stated in its widest terms, amounts to affirming in chemical language that in any compound an arbitrarily selected group of m n-adic atoms may be replaced by a group of n m-adic atoms, but how far this law of replacement has objective validity in the chemical sphere, I am not able to say.

Attention might also have been called to the fact that every chemicograph may, for anything that has been shown to the contrary, and probably in all cases does admit of algebraical interpretation, provided that each given atom however often repeated in a graph counts as a distinct quantic with its own distinct set of coefficients. I do not know whether chemists are of opinion that every chemico-graph exists or is capable of existence in nature; if this is not the case, the condition of the possibility of such existence (should it be discovered) must admit of being stated in mathematical terms. The condition for its existence in algebra may be gathered from what precedes, to be certainly for monadelphic types and probably in all cases, as follows, namely: *if the difference between every two letters of an algebraically existent graph be raised to the power whose index is the number of bonds connecting them, the permutation sum of the product of those powers must not vanish.* Finally, an irreducible covariant is the homologue of a compound radical. Thus we see that chemistry is the counterpart of a province of algebra as probably the whole universe of fact is, or must be, of the universe of thought.

APPENDIX 1.

Remarks on Differentiants Expressed in Terms of the Differences of the Roots of their Parent Quantics.

Since the preceding matter was written, in dwelling upon the law of reciprocal graphs, I came to what appeared to be a formidable difficulty in the way of its reception, a very lion in my path, so formidable that, for a time, I thought that it would be necessary, either to abandon this law, or else to admit the unwelcome conclusion that not every type of invariant was susceptible of graphical representation.

But further consideration has shown me that this apprehension was

* In Note D to Appendix 2. The proposition stated in the text results from the joint effect of the law of substitution or emanation combined with Hermite's law extended to quantic systems.

entirely groundless owing to an algebraical fact on which I had not previously reflected, but which this difficulty forced upon my notice. The difficulty in question arose out of the expressions given by M. Hermite and le père Joubert respectively for the skew invariants of the binary quintic and sextic. I shall first address myself to the consideration of the former. Following Dr Salmon's notation (Lessons, Third Edition, p. 230), let α, β, γ, δ, ϵ be the roots of a quintic, and let

$$F = (\alpha-\beta)(\alpha-\epsilon)(\delta-\gamma) + (\alpha-\gamma)(\alpha-\delta)(\beta-\epsilon)$$

$$G = (\alpha-\beta)(\alpha-\gamma)(\epsilon-\delta) + (\alpha-\delta)(\alpha-\epsilon)(\beta-\gamma)$$

$$H = (\alpha-\beta)(\alpha-\delta)(\epsilon-\gamma) + (\alpha-\gamma)(\alpha-\epsilon)(\delta-\beta).$$

Then it will be found as will presently be shown that the product $F.G.H$ is a symmetrical function of the four roots β, γ, δ, ϵ, consequently, on forming four other similar products symmetrical in respect to α, γ, δ, ϵ: α, β, δ, ϵ: α, β, γ, ϵ: α, β, γ, δ respectively, the product of these five products will be symmetrical in respect to α, β, γ, δ, ϵ and being a function of the differences of the roots of order 18 and of weight 45, that is of the type [45 : 5, 18], must be (paying no attention to a mere numerical factor) I, the skew invariant to the quintic.

Now consider the type reciprocal to this [45 : 18, 5] (monadelphic like the preceding), and expressing the invariant of the fifth order to an octodecadic. Suppose this has a graph. It will follow from the law of reciprocal graphs that I may be expressed under the form

$$\Sigma(\alpha-\beta)^a(\alpha-\gamma)^b(\alpha-\delta)^c(\alpha-\epsilon)^d(\beta-\gamma)^e(\beta-\delta)^f(\beta-\epsilon)^g(\gamma-\delta)^h(\gamma-\epsilon)^k(\delta-\epsilon)^l,$$

where $a+b+c+\ldots = 45$ and each letter α, β, γ, δ, ϵ is conditioned to appear the same number of times, which at first might seem contradictory to what has just been established, but in reality is in perfect accordance with it. For imagine the product of the 15 quantities

$$FGHF'G'H'F''G''H''F'''G'''H'''F^{IV}G^{IV}H^{IV}$$

to be actually written out giving rise to 2^{15}, or 32768 terms, and to each of these terms prefix the sign Σ indicating that the sum is to be taken of the 120 values which it assumes on permuting the five letters α, β, γ, δ, ϵ. The sum of all these partial sums is $120I$; hence some, at least, of them cannot vanish. Let ΣT be any one that does not vanish. Then ΣT is a function of the differences of the roots of the same weight and order as the entire expression; it is therefore to a numerical factor près identical with I, just as every fragment of a mirror is itself a mirror, or as every particle of diamond dust, a diamond.

Thus, as many distinct non-vanishing forms as there may be of ΣT, so many different graphs to the quint-invariant of a binary octodecadic shall we be able to construct agreeing respectively with the different representations of I of the form

$$\Sigma(\alpha-\beta)^a(\alpha-\gamma)^b(\alpha-\delta)^c\ldots$$

and it is probable that the virtual equivalence of all these several graphs may admit of being made out by inspection, as we saw was the case with the two graphs (one dissociated, the other connected) corresponding to the two algebraical representatives of the quadrinvariant of a quartic. Thus, what seemed, at first sight, to be fatal to the admissibility of the algebraico-graphical theory only serves to set in a clearer light its value as an instrument of research.

If we analyse M. Hermite's form of the skew invariant* to the quintic we shall see that it depends upon this simple but not obvious fact, that writing

$$F=(c,\ d)(a-b)+(a,\ b)(c-d)$$

$$G=(b,\ d)(a-c)+(a,\ c)(d-b)$$

$$H=(b,\ c)(a-d)+(a,\ d)(b-c)$$

and interpreting any such quantity as (a, b) to mean either 1 or $(a+b)$ or ab the product FGH is a symmetrical function of a, b, c, d, because on interchanging any two letters (say for example c, d) that one of the three quantities F, G, H (in this example H) in which those two letters are affected with the same sign, will remain unaltered in value whilst the other two (here G and F) change, each into the negative of the other.

Consequently we may interpret (a, b) to mean $(e-a)(e-b)$ and then the product of the five products corresponding to FGH is a function of the coefficients which expressed in terms of the differences of the roots will be of the weight 15 and of the order $1.6+4.3$ or 18 because in one of the five products each letter will enter in six dimensions and in each of the other four products in three dimensions; thus in FGH, e^6 will appear, but in each of the other four products e^3 will be the highest power of e. Hence the quindenary product is the invariant in question. No further step is necessary, the proof is complete as stated.

This remark will enable us to illustrate the process of transformation, which I have compared with grinding a diamond into dust, by an example

* I am wont to compare in my mind this symmetrical and translucent form to the Pitt Diamond and Père Joubert's to the Koh-i-Noor. In Note D to Appendix 2 a method is given whereby these forms may be transmuted into one another subject, however, to the bare possibility that the one, put into the algebraical alembic at a certain stage of the process, instead of passing into the other may, so to say, evaporate and be reduced to nothing. In the theory of forms, all-embracing Zero is the source and reconciler of contradictions, because, algebraically speaking, *everything is contained in nothing*, and so in a morphological sense "nought is everything" though not "everything is nought."

that can be completely pursued to the end. For let us now regard a, b, c, d as the roots of a binary quartic; then

$$\{(a-b)+(c-d)\}\,\{(a-c)+(d-b)\}\,\{(a-d)+(b-c)\}$$

will be a differentiant thereto of weight three and order three; it will, in fact, represent the root-differentiant of the skew sextic covariant.

Imagine this multiplied out without disturbing the marks of coupling so as to give eight terms or fragments analogous to the 32768 fragments spoken of in the preceding case. These terms will be of only four different patterns, one of the pattern $(a-b)(a-c)(a-d)$, three of the pattern $(a-b)(a-c)(b-c)$, three of the pattern $(a-b)(b-c)(d-b)$ and one of the pattern $(c-d)(d-b)(b-c)$. Prefixing Σ to each of these pattern terms to signify the sum resulting from the 24 permutations of a, b, c, d, we know *à priori* that not all of these can be zero since a linear function of them will be 24 times the differentiant in question, and on examination we find that the second and fourth Σ will vanish, but that the first and third will not. Accordingly, we shall have two new expressions

$$\Sigma\,(a-b)(a-c)(b-c), \qquad \Sigma\,(a-b)(b-c)(b-d),$$

each of which represents a differentiant of the same type as the original one, and this type being monadelphic or henparametric, the original product and these two sums will only be different representations of the same differentiant. Thus we see that each independent form belonging to a given type is susceptible (when expressed as a function of the differences of the roots) of a number of distinct phases, or, as we may express it, an algebraical form, in this theory, is in general polyphasic and accordingly its Icon or linkage exponent will be in general polygraphic, and each phase will have its own appropriate graph. It is a work of some difficulty, in general, to recognize the substantial identity of the different phases of the same algebraical form, and in like manner it may not, in all cases, be easy to recognize the substantial identity of the different graphs of its Icon, but sufficient has been shown to indicate the possibility and method of establishing such identity. The more I study Dr Frankland's wonderfully beautiful little treatise the more deeply I become impressed with the harmony or homology (I might call it, rather than analogy) which exists between the chemical and algebraical theories. In travelling my eye up and down the illustrated pages of "the Notes," I feel as Aladdin might have done in walking in the garden where every tree was laden with precious stones, or as Caspar Hauser when first brought out of his dark cellar to contemplate the glittering heavens on a starry night. There is an untold treasure of hoarded algebraical wealth potentially contained in the results achieved by the patient and long continued labour of our unconscious and unsuspected chemical fellow-workers.

We have seen that M. Hermite's beautiful expression for the skew invariant of the quintic proves its own character. A similar analysis may be applied to père Joubert's equally beautiful and even more remarkable expression for that of the sextic. M. de Bruno's statement of this, Table IV[10], contains two very perplexing typographical errors, namely, 4th line from foot of page, in V_0, $x_1x_2(x_\infty + x_0 - x_3 - x_2)$ should read $x_1x_2(x_\infty + x_0 - x_3 - x_4)$, and 3rd line from foot of page, in W_0, $x_2x_4(x_2 + x_3 - x_\infty - x_0)$ should be $x_2x_4(x_1 + x_3 - x_\infty - x_0)$. Moreover, the form in which the expression is presented in M. de Bruno's pages tends to mask its true nature and to suggest an analogy, which has no existence in fact, between it and M. Hermite's form; the latter is intrinsically a quinary group of triadic products, but such representation in the case of M. Joubert's form is purely conventional and confusing, it really being a single indecomposable quindenary product. Call a, b, c, d, e, f the six roots of a sextic, and let ab; cd; ef be any one of the 15 *duadic synthemes** which can be formed with them, and

$$F = \pm \left\{\begin{array}{l} ab\,.(c+d-e-f) \\ +cd\,.(e+f-a-b) \\ +ef\,.(a+b-c-d) \end{array}\right\}$$

The external sign is arbitrary, but must be considered as *determined* once for all for each of the 15 values of F. The product of these 15 values is a symmetrical function of the roots. For suppose any two letters, as a, b, to be interchanged; then three of the factors F in which a and b are coupled will undergo no change, but the remaining twelve will evidently be resoluble into six pairs reciprocally related, so that each F of a pair is transformed either into the other or into its negative and on either supposition the product of the pair remains unaltered in value. Also this product is a differentiant, for $\Sigma\delta_a$ operating on any one factor evidently reduces it to zero. It is also of the weight 45 and of the order 15. Hence the product of the fifteen values of F is the skew invariant to the sextic.

It seems desirable to make the *differentiantive* character of the form self-apparent. This may be done by virtue of the remark that $\pm F$ may be replaced by the form

$$\left\{\begin{array}{l} (a-d)(b-f)(c-e) + (a-f)(b-d)(c-e) \\ +(a-c)(b-e)(d-f) + (a-e)(b-c)(d-f) \\ +(a-c)(b-f)(d-e) + (a-f)(b-c)(d-e) \\ +(a-d)(b-e)(c-f) + (a-e)(d-b)(c-f) \end{array}\right\}$$

* A duadic syntheme of $2n$ letters is a combination of n duads containing between them all the letters. In it the order of the duads and of the letters in each duad is disregarded. Hence the number of such is $\frac{\Pi 2n}{2^n \Pi n}$ or $1.3.5\ldots(2n-1)$. For an odd number of letters simple synthemes do not exist but in lieu of them we may construct diplo-synthemes containing every letter taken twice over.

This sum contains 64 terms, of which 48 are the terms in F taken 4 times over, and the other 16 are the 8 quantities $ace, bdf, acf, bde, bce, adf, bcf, ade$, each appearing twice with opposite signs. If we expand the product of the 15 values of F, we shall obtain 35,184392,568832, or upwards of 35 billions of terms distributable among a certain number of patterns; on prefixing Σ to one of each pattern a certain number of such sums will be zero, but the remaining ones of which there must be some (and there will probably be a very large number) will all be (except as to a numerical multiplier) identical with each other and with père Joubert's formula. We see by these examples that there is a sort of polymorphism or pheno-polymorphism, as it may be termed, which is of a much more superficial character than and ought to be carefully distinguished from true polymorphism, eteo-polymorphism as we may call it, and this distinction as it has a marked bearing upon the theory of algebraical linkages, it is reasonable to expect may not be without importance in the study and construction of chemical graphs. Although I have been dealing, in what precedes, with particular cases, the reasoning is general in its nature and leads to conclusions which I will proceed to express in exact terms.

Let us understand by a permutation-sum of a function of letters belonging to one or more sets ($n, n', n'', \ldots$ being the number of letters in the respective sets) the sum of the $\Pi n \Pi n' \Pi n'' \ldots$ values which the function assumes when the letters in each several set are permuted *inter se*; and let us understand by a monomial differentiant one which (with the usual convention as to $a = 1$) may be expressed as a permutation-sum of a single product of differences of roots of the parent quantic, or quantic system; then in the first place it has virtually been proved, in what precedes, and is undoubtedly true that every monadelphic differentiant is monomial, and it may easily be proved in like manner that a differentiant of multiplicity k may be represented by the sum of k monomial differentiants.

For greater simplicity let us confine ourselves to the case of monadelphic invariants and let us consider any two such belonging to reciprocal types; then the algebraical value of either one, in terms of the roots of its parent quantic or quantic system, will be represented by the permutation-sum of the product of the differences of every two letters in the other taken as many times as there are connecting bonds between them, such letters being for this purpose regarded as the roots in question. Hence also we may derive the rule previously given for determining whether or not any given graph, in which the number of bonds is equal to half the toti-valence, represents or not an algebraical invariant—the condition of its doing so being that the permutation-sum of the product of the differences between the connected letters (each bond giving one such difference) shall be other than zero. This rule will stand good whether the type of the graph be monadelphic or not.

A very simple instance occurs to me of the monomial law for monadelphic types. Let α, β, γ be the roots of a cubic. It will easily be found that the type (4 : 3, 4) to which

$$\{(\alpha-\beta)^2+(\alpha-\gamma)^2+(\beta-\gamma)^2\}^2$$

belongs is monadelphic; prefix to it the sign of summation, which is merely equivalent to multiplying it by 6. It will not be a monomial permutation-sum as it stands, but it may be replaced by $2\Sigma(\alpha-\beta)^2(\alpha-\gamma)^2$ or $\Sigma(\alpha-\beta)^4$ each of which monomial sums is a half of

$$\{(\alpha-\beta)^2+(\alpha-\gamma)^2+(\beta-\gamma)^2\}^2.$$

POSTSCRIPT. Subsequently to the printing of the foregoing sheets I have seen in an editorial notice in the English Journal *Nature* (Feb. 14, 1878) a statement of the claims of Dr Frankland to be the discoverer and first promulgator of the law of atomicity, and I appear unconsciously to have done injustice to this great English chemist by attributing the discovery to Kekulé. I derived my impression on the subject from the popular belief and from the account of it given by Wurz in his *Histoire des doctrines chémiques.* If the facts of the case are as set forth in *Nature* and admit of no qualifying statements, I am unable to understand how such a discovery as that of valence or atomicity, which furnishes the master-key to our knowledge of the transformations of matter and raises chemistry to the rank of a mathematical and predictive science (it was previously only arithmetical), can have escaped receiving the award of a Copley Medal from the society in whose Transactions it appeared. I can hardly imagine that, if the first announcement and proof of universal gravitation or the circulation of the blood had been communicated to the world in a paper inserted in the *Philosophical Transactions* in these days, its author would have failed to receive for it the highest mark of recognition in the power of the Royal Society of London to bestow, and in my humble judgment the law of atomicity in its far-reaching importance and the labour, and mental acumen required for its discovery, stands fully on a level with either of these great landmarks in the history of natural science. It seems also from the same article in *Nature* that my distinguished friend, Professor Crum Brown, to whose personal teaching at Edinburgh I owe the very slight acquaintance with the subject I can lay claim to, was the first to use the admirable method of chemico-graphs.

The conception of hydro-carbon graphs as "trees with nodes, branches and terminals" and the indispensable notion of constructing them by starting from "an intrinsic central node or pair of nodes, so as to get rid of the otherwise unsurmountable difficulty of having to recognize equivalent forms appearing several times over in the same construction," are exclusively my own and were used by me in my communications with Professor Crum Brown on the subject and stated by me in a letter to Professor Cayley, who has adopted them as the basis of his own isomerical researches. In the account of this method given in German chemical journals I am informed that all reference (or at least all adequate reference) to my name as the author of it "fine by degrees and beautifully less," has at length entirely evaporated. M. Camille Jordan was led by quite a different order of considerations and with quite a different object in view to a discovery of the same centres before me, but I was not acquainted with this fact when I rediscovered them and made the application above mentioned. The idea of this application stands in the same relation to Professor Cayley's perfected use of it, as his idea of the use to be made of the equation $\Delta(w:\ i, j) =$ the number of linearly independent covariants of the type $[i, j:\ ij - 2w]$ stands to my completed method founded thereon, for obtaining the scale and connecting syzygies of the irreducible covariants to a quantic, laying me thereby under an obligation which I should take it in very ill part if any translator of my papers on the subject failed to acknowledge in unmistakable terms.

The hydro-carbon graphs, it may be noticed, belong to the limiting case of chemico-graphs; where no cyclical system of bonds connects any groups of atoms in a graph, it becomes an arborescence.

I have found it a profitable exercise of the imagination, from a philosophical point of view, to build up the conception of an *infinite* arborescence and to dwell on the relations of time and causality which such a concept embodies. An example of the good to be gained by these limitless mental constructions (new tracts and highways, so to say, opened out in the all-embracing "grand continuum" which we call space) is afforded by the valuable applications to the theory of local probability and the integral calculus in general made by Professor Crofton (my successor at Woolwich) of his new idea of an infinite reticulation (warp and woof), every finite portion of which contains an infinite number of meshes, being formed by the crossings of two sets of parallel lines all infinitely extended in both directions and those of the same set equidistant and infinitely near to each other. So the largest idea of an arborescence is that of an infinite number of nodes with an infinite number of branches proceeding from each of them.

APPENDIX 2.

Note on M. Hermite's Law of Reciprocity.

I take for granted that the treatise of M. Faà de Bruno represents this theory as it at present stands, in which case it seems to have made no advance since it was first promulgated by M. Hermite in his well known paper in the *Cambridge and Dublin Mathematical Journal*, 1854. It will be seen, however, I think from what follows, that it admits of being presented in a somewhat simpler and more general form. It rests essentially on the proposition of reciprocity in the theory of partitions that $(w: i, j) = (w: j, i)$, from which it follows as an immediate consequence that the number of arbitrary constants in the general covariant (or invariant) whose type is $[w: i, j]$, is the same as that whose type is $[w: j, i]$ since that number will be $\Delta(w: i, j) = \Delta(w: j, i)$ for each. Let now $\phi(a, b, c, \ldots l)$ be any differentiant of the order j in the coefficients, and of the weight w to a binary quantic $F(x, y)$ of the degree i in the variables; then ϕ is the root of a single covariant whose order is j and degree in the variables $ij - 2w$. Let ϕ be expressed (as from the definition of a differentiant must necessarily be possible) as a function of the differences of the roots $\alpha_1, \alpha_2, \ldots \alpha_i$ of F when y is made unity. For any difference $\alpha_p - \alpha_q$ substitute $\frac{d}{dx_p} \cdot \frac{d}{dy_q} - \frac{d}{dx_q} \cdot \frac{d}{dy_p}$, and let ϕ be converted into $\dot{\phi}$ by this substitution. Now operate with $\dot{\phi}$ upon the product of the i forms $G(x_1, y_1)$, $G(x_2, y_2), \ldots G(x_i, y_i)$, $G(x, y)$ signifying the general form of the degree j in the variables, and after the operation has been performed turning each subscript x into x and each subscript y into y, after the manner of Professor Cayley's original method of generating invariants or covariants as "Hyperdeterminants;" we shall thus obtain an in- or co-variant to a form of the degree j which will be of the order i in the coefficients and of the degree $ij - 2w$ in the variables, for there are w factors in $\dot{\phi}$ and each factor is of the second dimension in two of the x's and the corresponding two y's. Thus we shall have passed from a form of the type $[i, j: ij - 2w]$ to another of the type $[j, i: ij - 2w]$, or which is the same thing, from one of the type $[w: i, j]$ to another of the type $[w: j, i]$.

This latter may be called the *image* of the first. For facility of reference, let the number of arbitrary parameters in the one and the other type be called the multiplicity. If we repeat upon this image the process by which it was deduced from its primitive, we shall obviously get back the original type, but it by no means follows that if the multiplicity exceed unity, we shall get back the primitive form itself. It may be possible to revert to the same type without reverting to the same individual specimen of it *; and such, we shall presently see, is what in general happens.

Before proceeding further I shall give a very simple methodical rule for finding the image to any given invariantive form. Since, for any given value of i, the form and its image are each given when their root-differentiants are respectively given, it will be sufficient to assign the law for passing from the differentiant of the primitive to that of its image.

*** Just as, if I rightly understand the explanation given of fluorescence, a ray of light may give birth to some other form of motion and that again to another ray of light but of a different colour from the first. The theory of reciprocity treated of in the text is, in fact, a theory of alternate generation.**

For this purpose, let the given in- or co-variant be expressed in terms of symmetrical functions of the roots of the quantic when the leading coefficient (a), is made equal to unity. Then it will consist of terms, any one of which, apart from its numerical coefficient, will be of the form

$$\Sigma\,(\alpha_1\alpha_2\ldots\alpha_\lambda)^0\,(\beta_1\beta_2\ldots\beta_\mu)^1\,(\gamma_1\gamma_2\ldots\gamma_\nu)^2\,(\delta_1\delta_2\ldots\delta_\pi)^3\ldots$$

$\alpha_1\alpha_2\ldots\alpha_\lambda$, $\beta_1\beta_2\ldots\beta_\mu$, $\gamma_1\gamma_2\ldots\gamma_\nu$, &c. being all distinct and comprising between them *all* the i roots and of course $\mu+2\nu+3\pi+$ &c. will be equal to the weight; to pass from a differentiant expressed in terms of roots of a given quantic to the expression in terms of coefficients of the allied quantic of its image it will be found that the only thing necessary is to change any such factor as α^λ (where α is any root of the given quantic) into C_λ, the coefficient of the term containing y^λ in the allied one. This rule is a consequence (obtainable by ordinary algebraical processes) from the method above explained, where it is to be borne in mind that in order to obtain the image from the given form we have only to substitute for each root α_κ which occurs in ϕ, the fraction $\dfrac{dx_\kappa}{dy_\kappa}$ and to multiply the result by such a power of $\dfrac{d}{dy_1}\cdot\dfrac{d}{dy_2}\cdots\dfrac{d}{dy_\kappa}$, as will just serve to make it integral. A much simpler demonstration of this rule will be given in the sequel, and it will be shown that it not only holds good for deriving the leading term of the reciprocal (in the case of a covariant) from that of the primitive (that is, the root-differentiant of the one from the root-differentiant of the other) but that it is applicable to deriving the whole of one expression from the whole of the other.

As an example, take the differentiant whose type is [3: 3, 3], the root or base of the skew covariant to a cubic $(a, b, c, d \between x, y)^3$. Its value is $a^2d-3abc+2b^3$; expressed in terms of the roots α, β, γ, making $a=1$, this becomes

$$\alpha\beta\gamma-\frac{3\,(\alpha+\beta+\gamma)\,(\alpha\beta+\alpha\gamma+\beta\gamma)}{9}+2\,\frac{(\alpha+\beta+\gamma)^3}{27},$$

$$\text{or } \frac{1}{27}\left\{27\alpha\beta\gamma-9\,(\alpha+\beta+\gamma)\,(\alpha\beta+\alpha\gamma+\beta\gamma)+2\,(\alpha+\beta+\gamma)^3\right\},$$

$$\text{or } \frac{1}{27}\left\{2\Sigma\alpha^3-3\Sigma\alpha^1\beta^2+12\alpha\beta\gamma\right\},\text{ that is, } \frac{1}{27}\left\{2\Sigma\alpha^0\beta^0\gamma^3-3\Sigma\alpha^0\beta^1\gamma^2+12\alpha^1\beta^1\gamma^1\right\}.$$

Applying the rule, this becomes converted into

$$\frac{1}{27}\left\{6C_0^2C_3-18C_0C_1C_2+12C_1^3\right\},$$

or, reverting to the letters a, b, c, d, the image becomes the primitive affected with the factor $\dfrac{6}{27}$ and may be seen to be its own conjugate. Or again, let the primitive be the discriminant of a cubic, that is,

$$\frac{1}{27}\,(\alpha-\beta)^2\,(\alpha-\gamma)^2\,(\beta-\gamma)^2 \text{ or } (\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha-\alpha\beta^2-\beta\gamma^2-\gamma\alpha^2)^2;$$

this is equal to

$$\Sigma\,(\alpha^2\beta^4 + 2\Sigma\alpha\beta^2\gamma^3 - 2\Sigma\alpha^3\beta^3 - 6\alpha^2\beta^2\gamma^2 - 2\Sigma\alpha\beta\gamma^4).$$

Hence, by our rule, the image will be

$$\frac{1}{27}\,(6c_0c_2c_4 + 12c_1c_2c_3 - 6c_0c_3{}^2 - 6c_2{}^3 - 6c_1{}^2c_4),$$

or, using a, b, c, d, e in lieu of c_0, c_1, c_2, c_3, c_4, we obtain the form

$$-\frac{6}{27}\,(ace + 2bcd - ad^2 - c^3 - b^2e),$$

that is, $-\dfrac{2D}{9}$, where D is the well known quadrinvariant to a quartic

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

Treating this quadrinvariant as a function of the roots of a biquadratic form and proceeding as before to form its image, we shall obtain a second image which will be a numerical multiple of the original invariant.

But now let us consider the case of polyadelphic forms belonging to reciprocal types and for greater brevity, as the calculations are necessarily long, take a quantic of the self-contrary type $[w : i, i]$, as, for example $[6 : 4, 4]$ which belongs to the covariant of the fourth order and fourth degree to a quartic. This will be diadelphic; its general form is a linear combination of two products, one of the quartic itself by its cubinvariant, the other of the Hessian by the quadrinvariant. It will therefore have for its leading coefficient the differentiant

$$\lambda a\,(ace + 2bcd - ad^2 - c^3 - b^2e) + \mu\,(ac - b^2)\,(ae - 4bd + 3c^2),$$

say $\lambda U + \mu V$. Let us first find the image of U. Expressed in terms of the roots $\alpha, \beta, \gamma, \delta$, it is

$$\frac{1}{6}\,(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)\,(\alpha\beta\gamma\delta)$$

$$+ \frac{1}{48}\,(\alpha + \beta + \gamma + \delta)\,(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)\,(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$$

$$- \frac{1}{16}\,(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)^2 - \frac{1}{216}\,(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)^3$$

$$- \frac{1}{16}\,(\alpha + \beta + \gamma + \delta)^2\,\alpha\beta\gamma\delta,$$

which is

$$\frac{[6]\,(\alpha\beta\gamma^2\delta^2)}{6} + \frac{[24]\,(\alpha\beta^2\gamma^3) + [48]\,(\alpha\beta\gamma^2\delta^2) + [12]\,(\alpha^2\beta^2\gamma^2) + [12]\,(\alpha\beta\gamma\delta^3)}{48}$$

$$- \frac{[4]\,(\alpha^2\beta^2\gamma^2) + [12]\,(\alpha\beta\gamma^2\delta^2)}{16}$$

$$-\frac{[6](\alpha^3\beta^3)+[90](\alpha\beta\gamma^2\delta^2)+[72](\alpha\beta^2\gamma^3)+[24](\alpha^2\beta^2\gamma^2)+[24](\alpha\beta\gamma\delta^3)}{216}$$

$$-\frac{[4](\alpha\beta\gamma\delta^3)+[12](\alpha\beta\gamma^2\delta^2)}{16},$$

where any term, as for example $[48](\alpha\beta\gamma^2\delta^2)$, means the sum of the quantities of the type $\alpha\beta\gamma^2\delta^2$ each taken a sufficient number of times to make up 48 combinations, so that it is identical in meaning with $8\Sigma(\alpha\beta\gamma^2\delta^2)$ in the common notation. This convention is useful in saving the unnecessary labour of performing divisions in this first part of the process which have to be exactly reversed by multiplications in the transformation process which follows. The value of the above sum is, for purposes of transformation, equivalent to

$$\frac{1}{36}\left\{3\alpha\beta\gamma^2\delta^2+6\alpha\beta^2\gamma^3-4\alpha^2\beta^2\gamma^2-4\alpha\beta\gamma\delta^3-\alpha^3\beta^3\right\},$$

which gives for the image of U

$$\frac{1}{36}(3b^2c^2+6abcd-4ac^3-4db^3-a^2d^2)$$

or $\frac{1}{36}(U-V)$, where it will be observed that $(V-U)$ is identical with the discriminant to $(a, b, c, d \between x, y)^3$. Let us now proceed to find the image of $(U-V)$. Using σ to denote the sum of the combinations of $\alpha, \beta, \gamma, \delta$ taken i and i together, where $\alpha, \beta, \gamma, \delta$ are the roots of the general quartic, we have

$$\begin{aligned}U-V&=\frac{\sigma_1{}^2\sigma_2{}^2}{192}+\frac{\sigma_1\sigma_2\sigma_3}{16}-\frac{\sigma_2{}^3}{54}-\frac{\sigma_3\sigma_1{}^3}{64}-\frac{\sigma_3{}^2}{16}\\&=\frac{1}{1728}(9\sigma_1{}^2\sigma_2{}^2+108\sigma_1\sigma_2\sigma_3-32\sigma_2{}^3-27\sigma_3\sigma_1{}^3-108\sigma_3{}^2).\end{aligned}$$

Expanding and transforming, it will be found that the image of $(U-V)$ is $\left(\frac{21}{432}U-\frac{1}{432}V\right)$ and the second image of U which is $\frac{I(U-V)}{36}$ does not revert to the form U.

As a simpler example we may take the covariant to a quartic, still of the fourth order in the coefficients as before, but of the eighth degree in the variables. This will have for its root-differentiant

$$\lambda a^2(ae-4bd+3c^2)+\mu(ac-b^2)^2, \text{ say } \lambda U+\mu V.$$

Here
$$U=\sigma_4-\frac{\sigma_1\sigma_3}{4}+\frac{\sigma_2{}^2}{12}=\frac{1}{12}(12\sigma_4-3\sigma_1\sigma_3+\sigma_2{}^2),$$

and for the purpose of transformation is equivalent to

$$\begin{aligned}&\frac{1}{12}\left\{12\alpha\beta\gamma\delta-3(4\alpha\beta\gamma\delta+12\alpha^2\beta\gamma)+6\alpha\beta\gamma\delta+6\alpha^2\beta^2-12\alpha^2\beta\gamma\right\}\\&\qquad=\frac{1}{12}\left\{6\alpha\beta\gamma\delta+6\alpha^2\beta^2-12\alpha^2\beta\gamma\right\}.\end{aligned}$$

Hence, using I to denote "image of,"

$$IU = \frac{1}{2}\left\{b^4 + a^2c^2 - 2acb^2\right\} = \frac{1}{2}V.$$

Again
$$V = \left(\frac{\sigma_2}{6} - \frac{\sigma_1^2}{16}\right)^2$$

$$= \frac{1}{48^2}\left\{3\alpha^2 + 3\beta^2 + 3\gamma^2 + 3\delta^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)\right\}^2,$$

which, for purposes of transformation, will be found equivalent to

$$\frac{1}{48^2}\left\{36\alpha^4 + 132\alpha^2\beta^2 - 48\alpha^2\beta\gamma - 144\alpha^2\beta + 24\alpha\beta\gamma\delta\right\}.$$

Consequently

$$IV = \frac{1}{192}\left\{3a^3e + 11a^2c^2 - 4ab^2c - 12a^2bd + 2b^4\right\}$$

$$= \frac{1}{192}\left\{3a^2(ae - 4bd + 3c^2) + 2(b^2 - ac)^2\right\}$$

$$= \frac{1}{192}(3U + 2V).$$

Let now $\lambda : \mu$ be so chosen that

$$I(\lambda U + \mu V) = \rho(\lambda U + \mu V).$$

This gives
$$\frac{\mu U}{64} + \left(\frac{\lambda}{2} + \frac{\mu}{96}\right)V = \rho(\lambda U + \mu V),$$

or
$$\frac{\mu^2}{64} - \frac{\lambda\mu}{96} - \frac{\lambda^2}{2} = 0,$$

that is,
$$3\mu^2 - 2\lambda\mu - 96\lambda^2 = 0.$$

The two values of $\frac{\mu}{\lambda}$ derived from this equation are 6 and $-\frac{16}{3}$. The corresponding values of ρ will be 6 and $-\frac{1}{12}$. There are thus two definite systems of $\lambda : \mu$, and no more, which will make $\lambda U + \mu V$ self-conjugate and it is obvious that there will be no other values of $\lambda : \mu$ which will make

$$I^2(\lambda U + \mu V) = \rho(\lambda U + \mu V),$$

for, $I^2 U$ and $I^2 V$ being determinate linear functions of U, V, we shall have a quadratic equation for determining $\lambda : \mu$, but the two values of $\lambda : \mu$ which make $\lambda U + \mu V$ self-conjugate must satisfy this equation, and hence there can be no others. Reverting to the preceding example of the type [6 : 4, 4], we have found

$$IU = \frac{1}{36}U - \frac{1}{36}V$$

$$I(U - V) = \frac{21}{432}U - \frac{1}{432}V.$$

Hence $$IV = -\frac{9}{432}U - \frac{11}{432}V,$$

and making $$I(\lambda U + \mu V) = \rho(\lambda U + \mu V),$$

the equation for finding ρ will be

$$\begin{vmatrix} \frac{12}{432} - \rho\ , & -\frac{12}{432} \\ -\frac{9}{432} & , \ -\frac{11}{432} - \rho \end{vmatrix} = 0,$$

whence $$\rho_1 = -\frac{1}{27},\quad \rho_2 = \frac{5}{144};$$

also, since $$\left(\frac{12}{432}\lambda - \frac{9}{432}\mu\right) = \rho\lambda,$$

we shall have $$\frac{\lambda_1}{\mu_1} = -\frac{9}{28},\quad \frac{\lambda_2}{\mu_2} = 3.$$

What intrinsic peculiar properties are possessed by the principal forms* is a question as to which we are at present quite in the dark, as are we also with regard to the general character of the equation in ρ. It were much to be wished that some one would work out the case of a triadelphic type, as for example the type of covariants of the 6th order in the coefficients and the 6th degree in the variables, to a sextic. It might be supposed from the two preceding examples that the values of ρ are necessarily rational, but it will be shown hereafter that such is not the case.

It is easy to see that the relation between any form belonging to a given type of multiplicity 2 or 3 and its second image may be geometrically represented by means of a quadric curve or surface. Thus suppose the multiplicity is three, and that the three values of ρ are A, B, C. Construct an ellipsoid or hyperboloid whose semiaxes are $\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}, \frac{1}{\sqrt{C}}$. Draw r any radius vector making angles α, β, γ with the principal axes, p a perpendicular from the centre upon the tangent plane at the point where r meets the quadric, making angles λ, μ, ν with these axes. Then if

$$K(\cos\alpha U + \cos\beta V + \cos\gamma W)$$

be any given form of the system for which U, V, W are the principal forms,

$$\frac{K}{pr}(\cos\lambda U + \cos\mu V + \cos\nu W)$$

will be its second image. And we may say that, if a form lies in the

* By a principal form (in general), as hereafter stated in the text, I mean one which is the reciprocal of its first image in the sense that it bears a numerical ratio to its second image. The numerical quantity by which it must be multiplied to give the second image, I call a principal multiplier.

direction of the axis of instantaneous rotation, its second image will lie in the perpendicular upon the invariable plane: or more simply if by the direction of a form $\lambda U + \mu V + \nu W$ we understand that of a straight line whose direction cosines are as $\lambda : \mu : \nu$ and by its modulus $\sqrt{(\lambda^2 + \mu^2 + \nu^2)}$, we may say that if a radius vector of the ellipsoid (or other quadric) represent the direction and modulus of an in- or co-variant the corresponding radius vector of the polar reciprocal to the quadric will represent the direction and modulus of its second image.

The true nature of the reciprocity theorem, in the general case where i, j have any values whatever, is now obvious. Let $U_1, U_2, \ldots U_q$ be independent forms belonging to the type $[w: i, j]$, whose multiplicity is q, and $V_1, V_2, \ldots V_q$ as many forms belonging to the reciprocal type $[w: j, i]$. We may, by virtue of the transformation process, express each IU in terms of linear functions of the forms V and *vice versa*, so that each I^2U will be a known linear function of all the U's. For clearness sake suppose $q = 3$ and let

$$I^2U_1 = aU_1 + bU_2 + cU_3$$
$$I^2U_2 = a'U_1 + b'U_2 + c'U_3$$
$$I^2U_3 = a''U_1 + b''U_2 + c''U_3.$$

Now make

$$I^2(\lambda U_1 + \mu U_2 + \nu U_3) = \rho(\lambda U_1 + \mu U_2 + \nu U_3).$$

We shall have for finding ρ the equation

$$\begin{vmatrix} (a-\rho), & b &, & c \\ a' &, & (b'-\rho), & c' \\ a'' &, & b'' &, & (c''-\rho) \end{vmatrix} = 0,$$

and then the three systems of values of $\lambda : \mu : \nu$, which make the second image of $\lambda U_1 + \mu U_2 + \nu U_3$ coincide to a numerical factor près, with itself, will be rational functions of the respective roots. So, in general, when the multiplicity of the type $[w: i, j]$ is q, there will be in general q special forms, and no more, which have reciprocal forms belonging to the type $[w: j, i]$, and if the interchangeable elements, i, j are equal, then these q forms will all be self-conjugate. It is conceivable that in certain cases the equation in ρ may have equal roots; in that event each such equality would introduce a corresponding indeterminateness in the forms admitting of conjugates. For example, if the multiplicity were 2 and the two roots of ρ equal, that would signify that *every* form belonging to the type would have a conjugate—a fact analogous to an ellipse becoming a circle, or an ellipsoid a spheroid—and so in general.

A form having a conjugate, that is, whose second image is a numerical multiplier of itself, may be called a principal form. If the multiplicity of the type is q, there will be q such. All but these will give rise to an endless succession of images such that any $q+1$ of an even order (the form itself included among these) will be connected by a linear equation. That the succession is endless is clear from the consideration that if an image, say of the $(2p)$th rank, is identical (to a numerical factor près) with the form, we have an equation of the qth degree for finding the values of the systems of multipliers λ, μ, ν of U, V, W; therefore there are only q such systems, but the systems which satisfy $I^2F=\rho F$ must also satisfy $I^{2p}F=\rho' F$, and consequently there are no others.

To illustrate this, suppose

$$I^2U=aU+bV$$
$$I^2V=cU+dV;$$

then

$$I^4U=(a^2+bc)\,U+(ab+bd)\,V$$
$$I^4V=(ca+ad)\,U+(cb+d^2)\,V.$$

If now we put

$$\begin{vmatrix} a-\rho, & b \\ c\quad , & d-\rho \end{vmatrix}=0,$$

to find the values of $\lambda:\mu$ which make $I^2(\lambda U+\mu V)=\rho\,(\lambda U+\mu V)$ we have

$$(a-\rho)\,\lambda+c\mu=0.$$

In like manner, if we make

$$\begin{vmatrix} a^2+bc-\rho, & ab+bd \\ ca+ad\quad , & cb+d^2-\rho \end{vmatrix}=0,$$

to find the values of Λ and M which make $I^4(\Lambda U+\mathrm{M}V)=R\,(\Lambda U+\mathrm{M}V)$, we have

$$(a^2+bc-R)\,\Lambda+(ca+ad)\,\mathrm{M}=0,$$

and it will be found that

$$a-\rho=\frac{a-d}{2}\pm\frac{1}{2}\sqrt{\{(a-d)^2+4bc\}}$$

$$a^2+bc-R=\frac{a^2-d^2}{2}\pm\frac{a+d}{2}\sqrt{\{(a-d)^2+4bc\}},$$

so that the values of $\lambda:\mu$ and $\Lambda:\mathrm{M}$ are the same, and such we know *à priori* must be the case.

It ought to be noticed that the method explained in the preceding pages furnishes a complete solution of the problem following. Given any in- or co-variant, say of the jth order in the coefficients to a form Q of the ith degree, to find the process of differentiation which performed upon the product

$$Q\,(x_1, y_1)\,.\,Q\,(x_2, y_2)\,.\,\ldots\,Q\,(x_j, y_j)$$

shall produce the j-partite-emanant of the in- or co-variant so given, and it proves incidentally that every binary in- or co-variant may be represented as a hyperdeterminant. To make this clear, let us call the above product, or rather that product divided by $(\Pi i)^j$, the j-ary norm of Q and denote it by NQ. Again, let G be any given differentiant to the type $[w: j, i]$, say $G(\rho_1, \rho_2, \ldots \rho_j)$ which is necessarily identical with

$$G\{0; (\rho_2 - \rho_1); (\rho_3 - \rho_1); \ldots (\rho_j - \rho_1)\}.$$

For $\rho_\kappa - \rho_1$ write $\frac{d}{dx_\kappa} \cdot \frac{d}{dy_1} - \frac{d}{dy_\kappa} \cdot \frac{d}{dx_1}$ and let the quantity so formed be called the hyperdeterminant to G and be denoted by HG. Then if E be any principal form to the type $[w: i, j]$, of the multiplicity q and belonging to a quantic Q, and G be its first image, we shall have

$$(HG)(N_jQ) = \rho F,$$

where ρ is one of the roots of a known equation of the qth degree in ρ. Consequently, since any form belonging to the given type is a linear function of its q principal forms, every such form may be expressed by means of the hyperdeterminant

$$\Sigma_{\lambda=q}^{\lambda=1} \frac{c_\lambda}{\rho_\lambda} (HG_\lambda)\, NQ,$$

the given form being supposed to be expressible by $\Sigma_{\lambda=q}^{\lambda=1} c_\lambda F_\lambda$, where F is any one of the q principal forms.

It follows from what has been shown above that in general from any one particular given form belonging to a type of multiplicity q may be deduced the $(q-1)$ others (by taking the successive second images) and thus the general form obtained; the exception is when the given form happens to be a linear function of less than q of the principal forms. A further consequence is that any in- or co-variant given in terms of the roots of its quantic may be converted by explicit processes into a function of the coefficients. Thus, for example, suppose that the multiplicity of the type is 3; call the given form R_0 and the successive second images R_1, R_2, R_3, R_4. These latter will be all known by the rule of transformation and we shall have R_4 a known linear function of the three preceding forms, say equal to

$$\alpha R_1 + \beta R_2 + \gamma R_3.$$

Hence if we put $\qquad R_0 = \lambda R_1 + \mu R_2 + \nu R_3,$

we must have $\qquad R_1 = \lambda R_2 + \mu R_3 + \nu(\alpha R_1 + \beta R_2 + \gamma R_3);$

hence $\qquad \nu = \frac{1}{\alpha}, \quad \mu = -\frac{\gamma}{\alpha}, \quad \lambda = -\frac{\beta}{\alpha}$

and thus R_0, given in terms of the roots, becomes known in terms of the coefficients of its quantic. And so in general, q being the multiplicity, $(q+1)$ forms deduced from the given function of the roots will serve to determine its value as a function of the coefficients. In fact by regarding R_0 as a linear function of the principal forms, it is easy to see it and all its successive secondaries (that is, second images) form a recurring series, the scale of relations being

$$R_0 - \Sigma \frac{1}{\rho} R_1 + \Sigma \frac{1}{\rho^2} R_2 - \Sigma \frac{1}{\rho^3} R_3 + \ldots = 0,$$

where $1 : \rho$ is the ratio of any principal form to its immediate secondary. Thus E_0 being given in terms of the roots and consequently $E_1, E_2, \ldots E_q$, in terms of the coefficients, E_0 becomes known in terms of the coefficients and of the quantities $\Sigma \frac{1}{\rho}, \Sigma \frac{1}{\rho^2}, \ldots$; these latter are identical with the quantities previously mentioned and furnish the simplest means of forming the equation in ρ, which (if we agree to call $\rho_1, \rho_2, \ldots \rho_q$ the moduli of the several principal forms $F_1, F_2, \ldots F_q$, that is, the ratios of their respective second images to themselves) may be termed the modular equation for any given type*.

It might have been useful, had I thought of it in time, and may be useful when the subject comes again under consideration, to treat a form and its second image, in which the type is restored as *antecedent* and *consequent*, and to describe the first image as the *alternate* form to the primitive, inasmuch as we pass, by what biologists term alternate generation, from one type to the other. It has been shown, in what precedes, that the transformation by images at each second step leads back to the original type, but, contrary to what might have been supposed, does not in general imply the resuscitation of the individual form.

The theorem of reciprocity has been seen to be, in its essence, a theorem of differentiants, and ought therefore to admit of being proved by means of the necessary and sufficient partial differential equation to which differentiants are subject. This may be done as follows. If we call $\epsilon_0, \epsilon_1, \epsilon_2, \ldots \epsilon_r$ the successive elements to a binary quantic expressed in its customary form, so that ϵ_r is the coefficient of the term containing y^r divested of its numerical binomial coefficient, and if we write

$$U = \frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma} + \ldots,$$

where $\alpha, \beta, \gamma, \ldots$ are the roots of the quantic, it is very easily proved that

$$U\epsilon_r = -r\epsilon_{r-1}\dagger.$$

Let $C\Sigma\alpha^r\beta^s\gamma^t \ldots$ be any term in a given differentiant F, the indices $r, s, t, \ldots$ being any whatever with no condition as to their being distinct from each

* But it will be better to adhere to the previous convention and to designate the ρ's as the principal multipliers and the equation in ρ as the principal equation.

† In fact it may easily be proved by the ordinary rule for the change of one system of independent variables into another that, if $a_1, a_2, \ldots a_i$ be the roots of $(\epsilon_0, \epsilon_1, \epsilon_2, \ldots \epsilon_i \between x, y)^i$,

$$\Sigma \frac{d}{da} = -\sum_{q=0}^{q=i} q\epsilon_{q-1} \frac{d}{d\epsilon_q}.$$

other, and let $N(r, s, t, \ldots)$ signify the number of combinations comprised in Σ; also let $CN(r, s, t, \ldots) \,.\, \epsilon_r \epsilon_s \epsilon_t \ldots$ be called the image of the term above written and G the image of F, that is, the sum of the images of the several terms in F; where it must be observed that the ϵ quantities do not necessarily refer to roots the same in number or name as the roots $\alpha, \beta, \gamma, \ldots$. Now suppose that we have any term, such as $Q\Sigma\alpha^l\beta^m\gamma^n \ldots$ in UF, where U refers to the given roots $\alpha, \beta, \gamma, \ldots$ and means $\frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma} + \ldots$. This term must arise from terms of the several forms

$$\left.\begin{array}{l} A\,\Sigma\alpha^{l+1}\,\beta^m\,\gamma^n \ldots \\ B\,\Sigma\alpha^l\,\beta^{m+1}\,\gamma^n \ldots \\ C\,\Sigma\alpha^l\,\beta^m\,\gamma^{n+1} \ldots \end{array}\right\} \text{in } F;$$
$$\&c. \;\&c. \ldots$$

corresponding to these there will be the images

$$\left.\begin{array}{l} AN(l+1, m, n, \ldots)\, \epsilon_{l+1} \,.\, \epsilon_m \,.\, \epsilon_n \ldots \\ BN(l, m+1, n, \ldots)\, \epsilon_l \,.\, \epsilon_{m+1} \,.\, \epsilon_n \ldots \\ CN(l, m, n+1, \ldots)\, \epsilon_l \,.\, \epsilon_m \,.\, \epsilon_{n+1} \ldots \end{array}\right\} \text{in } G,$$
$$\&c. \;\&c. \ldots$$

where G belongs to a quantic whose type is reciprocal to that of F, and it is clear that the effect of operating upon F with U will be to give

$$Q = A\rho N(l+1, m, n, \ldots)(l+1) + B\rho N(l, m+1, n, \ldots)(m+1)$$
$$+ C\rho N(l, m, n+1, \ldots)(n+1) + \&c. \ldots$$

ρ being a number easily determinable, but which there is no occasion to express. Again if $R\epsilon_l \,.\, \epsilon_m \,.\, \epsilon_n \ldots$ be the correlative term in G, we have by virtue of the formula $U\epsilon_r = -r\epsilon_{r-1}$, where the operator U refers to the roots of the quantic of reciprocal type,

$$(-)^w R = AN(l+1, m, n, \ldots)(l+1) + BN(l, m+1, n, \ldots)(m+1)$$
$$+ CN(l, m, n+1, \ldots)(n+1) + \&c. \ldots$$

Consequently, since on account of the identity $F=0$, we must have $Q=0$ for every term $Q\Sigma\alpha^l \,.\, \beta^m \,.\, \gamma^n \ldots$, we must also have $R = \rho^{-1} Q = 0$ and therefore, this being true for all the arguments $\epsilon_l \,.\, \epsilon_m \,.\, \epsilon_n \ldots$, we must have $UG = 0$. Hence, when any quantity F is a differentiant of a given quantic, its image (as defined in the text) is also a differentiant to a quantic of reciprocal type to the given one. This is the simplest method of establishing the theorem, but still the method originally employed in the note is valuable as serving to establish the important proposition that every in- or co-variant of a binary quantic is a hyperdeterminant.

I will proceed to show that for a system of two or more quantics of degrees $i, i', i'', \ldots$, we may pass from a covariant of the type $[w\colon i, j;\, i', j';\, i'', j'';\, \ldots]$

to one of the type $[w:j,i;\ i',j';\ i'',j'';\ldots]$ by taking its image in respect to the quantic whose indices, i, j, are to be interchanged precisely according to the same rule as if there were no other quantic present. As regards the law of reciprocity, a combination of quantics is analogous to a mixture of gases, according to Dalton's view, each playing the part, as it were, of a vacuum in respect to the other.

Let $[w: i,j;\ i',j';\ldots]$ be the type, $[w: j,i;\ i',j';\ldots]$ one of the antitypes, $(\epsilon_0, \epsilon_1, \epsilon_2, \ldots \epsilon_j \between x, y)^j$ the general form of the jth degree, $\alpha, \beta, \gamma, \ldots$ its roots when $\epsilon_0 = 1$. Let $\eta_r = (-)^r \epsilon_r$; then, since

$$\Sigma \frac{d}{d\alpha}\epsilon_r = -r\epsilon_{r-1}$$

$$\Sigma \frac{d}{d\alpha}\eta_r = r\eta_{r-1}.$$

Let D be any differentiant of the given type, $a, b, c, \ldots$ the roots of the quantic of degree i, $a', b', c', \ldots$ the roots of the quantic of degree i', with the usual convention as to the leading coefficients becoming unities. Let $\Sigma a^l b^m \ldots$, $\Sigma a'^{l'} b'^{m'} \ldots \Sigma \ldots$ be the arguments of any term in

$$\left(\Sigma \frac{d}{d\alpha} + \Sigma \frac{d}{d\alpha'} + \ldots\right) D,$$

say UD, then the coefficient of the term last written will arise from operating with U upon

$$\left.\begin{array}{l} A \,.\, \Sigma a^{l+1} \,.\, b^m \;\ldots\; \Sigma a'^{l'} \;b'^{m'} \,.\, \&c. \ldots \\ +\,B \,.\, \Sigma a^l \,.\, b^{m+1} \ldots\; \Sigma a'^{l'} \;b'^{m'} \,.\, \&c. \ldots \\ +\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ +\,A'\; \Sigma a^l \,.\, b^m \;\ldots\; \Sigma a'^{l'+1} b'^{m'} \,.\, \&c. \ldots \\ +\,B'\; \Sigma a^l \,.\, b^m \;\ldots\; \Sigma a'^{l'} \;b'^{m'+1} \,.\, \&c. \ldots \\ \qquad\qquad \&c. \ldots \qquad\qquad \&c. \ldots \end{array}\right\}$$

and the value of the coefficient will be

$$\left.\begin{array}{l} A\ (l \;+1)\ N(l+1, m, \ldots)\ N(l', m', \ldots\ldots\ldots) \ldots \\ +\,B\ (m\; +1)\ N(l, m+1, \ldots)\ N(l', m', \ldots\ldots\ldots) \ldots \\ +\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ +\,A'\ (l' \;+1)\ N(l, m, \ldots\ldots)\ N(l'+1, m', \ldots) \ldots \\ +\,B'\ (m'+1)\ N(l, m, \ldots\ldots)\ N(l', m'+1, \ldots) \ldots \\ +\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$

$$\div N(l, m, \ldots)\ N(l', m', \ldots) \ldots$$

To these feeders or contributory terms will correspond, in the image,

$$\begin{array}{l} AN\ (l+1, m, \ldots)\ \eta_{l+1} \,.\, \eta_m \;\ldots\; \Sigma a'^{l'} \;.\; b'^{m'} \;\ldots \\ +\,BN\ (l, m+1, \ldots)\ \eta_l \,.\, \eta_{m+1} \ldots\; \Sigma a'^{l'} \;.\; b'^{m'} \;\ldots \\ +\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ +\,A'N\,(l, m, \ldots\ldots)\ \eta_l \,.\, \eta_m \;\ldots\; \Sigma a'^{l'+1} \,.\, b'^{m'} \;\ldots \\ +\,B'N\,(l, m, \ldots\ldots)\ \eta_l \,.\, \eta_m \;\ldots\; \Sigma a'^{l'} \,.\, b'^{m'+1} \ldots \\ +\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

and it is obvious that by operating upon this with the U corresponding to its roots we shall obtain the argument $\eta_l . \eta_m \ldots \Sigma a'^{l'} . b'^{m'}$. &c.... affected with the very same coefficient as that above written, except that in its denominator the factor, $N(l, m, \ldots)$, will not appear. Hence, when D is a differentiant of the given type, its image (obtained by expressing the i set of coefficients in terms of roots and then replacing every power, ρ^q, of any such root, ρ, by η_q, leaving all the other coefficients unchanged) will also be a differentiant of the type transformed by interchanging i with its conjugate j*.

When there is but one quantic the effect of substituting ϵ_q instead of η_q will evidently only be to introduce a common factor $(-)^w$ into each term, which is immaterial and we may accordingly in that case reflect ρ^q into ϵ_q. Of course, in the general case, if all the letters i are simultaneously interchanged with the letters j, a similar conclusion follows.

As an example, let us take the two quadratics,

$$ax^2 + 2bxy + cy^2,$$

$$\alpha x^2 + 2\beta xy + \gamma y^2,$$

their resultant $(a\gamma - c\alpha)^2 + 4(a\beta - b\alpha)(c\beta - b\alpha)$, belongs to the type $[4 : 2, 2 ; 2, 2]$ which is its own reciprocal whichever of the interchangeable elements we permute. This resultant, treating a as unity, will be equal to

$$(\alpha\rho_1^2 + 2\beta\rho_1 + \gamma)(\alpha\rho_2^2 + 2\beta\rho_2 + \gamma)$$

$$= \alpha^2\rho_1^2\rho_2^2 + 2\beta\alpha(\rho_1^2\rho_2 + \rho_1\rho_2^2) + 4\beta^2\rho_1\rho_2 + \alpha\gamma(\rho_1^2 + \rho_2^2) + 2\beta\gamma(\rho_1 + \rho_2) + \gamma^2$$

the image of which will be

$$\alpha^2\epsilon_2^2 - 4\alpha\beta\epsilon_1\epsilon_2 + 4\beta^2\epsilon_1^2 + 2\alpha\gamma\epsilon_0\epsilon_2 - 4\beta\gamma\epsilon_0\epsilon_1 + \gamma^2\epsilon_0^2,$$

or as we may write it,

$$\alpha^2c^2 - 4\alpha\beta bc + 4\beta^2b^2 + 2\alpha\gamma ac - 4\beta\gamma ab + a^2\gamma^2,$$

which is $(c\alpha - 2b\beta + a\gamma)^2$, the square of the well known connective. Again, if we combine $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ with $\alpha x + \beta y$, we have the invariant

$$a\beta^3 - 3b\alpha\beta^2 + 3c\alpha^2\beta - d\alpha^3, \text{ say } I,$$

* Thus the rule of images for passing from a differentiant of a given type belonging to a single quantic to one of the opposite type is extended to the case of passing from a differentiant of a given type belonging to a system of quantics to any associated type, that is, to any type in which one or more of the numbers i chosen at discretion is or are interchanged with the corresponding numbers j, and it will presently be seen that this implies the extension of the rule without any alteration from differentiants or invariants to covariants of a quantic or system of quantics. In Note A it will further be shown that for any inversions whatever (or, to speak more accurately, for any cycle of inversions leading back to the original type), although the principal multipliers change their values as the cycle of inversion changes, the *principal forms themselves remain the same*,—a surprising conclusion but very easily proved. In other words, however many quantics there may be in the parent system, there is never more than one single set of principal forms of derivatives to it of a given type. A cycle of arbitrarily intercalated pairs of reversals (here of successive i's and j's), by which a type returns to itself, comes under the category, "Verschlingung," or "Knotting" of Gauss, Listing and Tait.

belonging to the type [3: 3, 1; 1, 3]. Write $\alpha = 1, \beta = -\rho$; this becomes

$$-a\rho^3 - 3b\rho^2 - 3c\rho - d,$$

of which an image, say J, belonging to the type [3: 3, 1; 3, 1],

$$a\epsilon_3 - 3b\epsilon_2 + 3c\epsilon_1 - d\epsilon_0$$

is the connective of

$$\left\{ \begin{array}{c} ax^3 + 3bx^2y + 3cxy^2 + dy^3 \\ \epsilon_0 x^3 + 3\epsilon_1 x^2 y + 3\epsilon_2 xy^2 + \epsilon_3 y^3 \end{array} \right\}.$$

Similarly $\quad (a^2d - 3abc + 2b^3)\beta^3 \ldots + \ldots + (d^2a - 3dbc + 2c^3)\alpha^3,$

say I, belonging to the type [6: 3, 3; 1, 3], will have for a reciprocal

$$(a^2d - 3abc + 2b^3)\epsilon_3 + \ldots (d^2a - 3dbc + 2c^3)\epsilon_0,$$

say J, belonging to the type [6: 3, 3; 3, 1]. The graph of I will be that of Fig. 41 and the graph of J, that of Fig. 42, where I use B and G (the initials of boron and gold, instead of Au for the latter) and H (the initial of hydrogen) to represent the algebraical atoms (that is quantics) of valencies (that is degrees) 3, 3, and 1 respectively. Prefixing Σ to the I graph and substituting G_1, G_2, G_3, the three roots of G, for H, H', H'' and B_1, B_2, B_3 for B, B', B'' we obtain

$$\Sigma (B_1 - B_2)(B_1 - B_3)(B_2 - B_3)(B_1 - G_1)(B_2 - G_2)(B_3 - G_3),$$

which by inspection is the root representative of J, and prefixing Σ to the J graph and substituting H for G, we obtain in like manner

$$\Sigma (B_1 - B_2)^2 (B_2 - B_3)(H - B_1)(H - B_3)^2,$$

as the root representative of I.

It may be observed that Fig. 43 is, algebraically speaking, a pseudograph of J, for its reading would give zero for the value of I.

It follows as an immediate consequence from the preceding extension of the law of images to quantic-systems, that the rule for deducing the first term of the reciprocal to a covariant from that of the covariant itself by writing η_r for α^r holds good as a rule for deducing each term of the one from the corresponding term of the other. To see this we have only to recall that every covariant to a quantic or quantic system may be regarded as an invariant of a new system containing the given quantic or system augmented by a linear quantic whose coefficients are y and $-x$.

Note A to Appendix 2.

Completion of the Theory of Principal Forms.

In the case of a derivative from a system of k parent quantics, it at first sight would seem that since reversion (the act of forming the second image, or process, as we may term it, of double reflexion) may be effected in regard to each system of coefficients separately, the method in the text ought to furnish in general k distinct systems of principal forms, but this is a mere mirage of the understanding which disappears as soon as the question is submitted to close examination. There is always an unique set of μ forms (μ being the multiplicity of the type) which revert unchanged (barring a numerical multiplier) whichever system of coefficients undergoes double reflexion. But a caution is necessary for the right interpretation of this statement. $U, V, W\ldots$ may be the principal forms in regard to one set of coefficients, $\lambda U+\mu V, W\ldots$, or $\lambda U+\mu V+\nu W\ldots$, where λ, μ, ν are indeterminate, in regard to some other. In any such case we may still say that $U, V, W\ldots$ is the principal system in regard to both sets and so in general. We have an example of this if we take any covariant to a single quantic Q and translate it into an invariant of Q and a linear form L. If $U, V, W\ldots$ are principal forms in respect to Q, $\lambda U+\mu V+\nu W+\ldots$ (that is the absolutely general form of the type) may be easily shown to undergo reversion in respect to L unaltered. $U, V, W\ldots$ may consequently still be seen to be a principal form system in respect to Q and L, as each of these quantities is unaltered by reversion in respect either to Q or to L.

Suppose now a diadelphic system of which U, V are the principal forms quâ one set of coefficients. Let R denote a reversion quâ this set, R' quâ some other set. Let $RU=aU$, $RV=bV$ and suppose $R'U=\alpha U+\beta V$. Then

$$R'RU=a\alpha U+b\beta V \text{ and } RR'U=a\alpha U+b\beta V.$$

But by the nature of the process of reversion $RR'=R'R$; hence $a\beta=b\beta$. If $a=b$, every linear combination of U, V is a principal form quâ R. Hence the principal form quâ the R' set, is such for both sets. But if a is not equal to b, we must have $\beta=0$. Hence U will be a principal form quâ R' as well as R, and the same will be true of V. For if

$$R'V=\gamma U+\delta V$$
$$RR'V=a\gamma U+b\delta V$$
$$R'RV=R'bV=b\gamma U+b\delta V.$$

Therefore $a\gamma=b\gamma$ and $\gamma=0$. Thus U, V will each of them be common as principal forms to each set. I have gone through the same somewhat tedious process of proof for triadelphic forms and with the same result. The very beautiful conclusion follows that whatever the multiplicity of a type may be and whatever number of sets of coefficients it involves, there is always *a single system of principal forms* common to all the sets*.

* Suppose there are k quantics in the parent system and that a derivative type μ is given. Each simple inversion of a pair of permutable indices (i, j) will give rise to a distinct principal equation; there will therefore be k such equations. Let ρ be a root of one of these, σ a root of any other. Then a principal form may be expressed as a linear function of any μ independent special forms connected by coefficients which are rational integer functions of ρ. Hence σ may be found as a rational function of ρ; but in like manner ρ may be found as a rational function of σ. Hence ρ, σ must be related by an equation of the form

$$A\rho\sigma+B\rho+C\sigma+D=0,$$

Note B to Appendix 2.

Additional Illustrations of the Law of Reciprocity.

Acetic aldehyde contains two atoms of carbon, one of oxygen and four of hydrogen*. It thus corresponds to the quartic covariant of a quadratic and quartic, linear and quadratic in respect to the coefficients of the first and second respectively; such a form exists algebraically (*Higher Algebra*, third edition, p. 200) and may easily be proved to be monadelphic. Let us treat it as an invariant: if we were to take for its graph a triangle of which C, C, O were the apices and attach two atoms of hydrogen to each C, the permutation-sum of the product of the differences of the connected letters is zero; this then is a pseudograph. A true graph of it is given by the figure

$$\begin{array}{c} H\cdot C\cdot O\cdot H \\ : \\ H\cdot C\cdot H \end{array},$$

where each single dot between two letters means a single bond and the two dots between the upper and lower C's stand for a pair of bonds between them. This belongs to the invariantive type [4, 2; 2, 1; 1, 4: 0], the complete reciprocal to which is [2, 4; 1, 2; 4, 1: 0]. The constitution of the latter in terms of the roots is found from the above graph by writing O for C, C for H and H for O and is accordingly

$$\Sigma\,(O-O')^2\,(O-C)\,(O-O')\,(O'-C'')\,(O'-H)\,(H-C),$$

where the factor $(O-O')^2$ may be put outside the sign of summation. We may therefore take for its graph a detached molecule of oxygen + a molecule of formic acid, which latter contains two of oxygen, one of carbon and two of hydrogen

$$\begin{array}{c} H\cdot C\cdot O\cdot H \\ : \\ O \end{array}$$

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and thus we see that all the k principal equations are homographically related, that is, that each may be obtained from any other by a substitution of the form

$$\rho=\frac{C\sigma+D}{A\sigma+B}.$$

In a word, the multiplicity μ (whatever the *diversity* k) determines the number of principal forms; and the k sets of principal multipliers are given by k algebraical equations of the μth degree, homographically transformable into one another.

* I originally took chloral as the subject of this investigation, being interested in examining its algebraical constitution in consequence of having had personal experience of its use as an escharotic. But for greater simplicity I have substituted acetic-aldehyde of which chloral is a third emanant, three hydrogen atoms of the former being replaced by three of chlorine in the latter.

will be a graph of it, from which, turning O into C, H into O and C into H we obtain

$$\Sigma (C-O')^2 (C''-H)(C'''-O')(C'''-H)(H-O)$$

as the value, in terms of its roots, of the algebraical equivalent to acetic aldehyde. The graph for formic acid, it may be noticed, exists algebraically (*Higher Algebra,* p. 300).

Instead of the dissociated molecules of oxygen and formic acid, we may exhibit them combined in the graph

$$\begin{array}{c} C\cdot O\cdot O\cdot O\cdot H \\ \vdots \\ O \end{array}$$

which will give another form to the value of the reciprocal in question, namely

$$\Sigma (C-H)^2 (H-O)(H-C')(C'-C'')(C''-C''')(C'''-O)$$

which, not being zero and the type being monadelphic*, must be in a pure numerical ratio to the sum above written.

Chemistry has the same quickening and suggestive influence upon the algebraist as a visit to the Royal Academy, or the old masters may be supposed to have on a Browning or a Tennyson. Indeed it seems to me that an exact homology exists between painting and poetry on the one hand and modern chemistry and modern algebra on the other. In poetry and algebra we have the pure idea elaborated and expressed through the vehicle of language, in painting and chemistry the idea enveloped in matter, depending in part on manual processes and the resources of art for its due manifestation.

A peculiar case might possibly arise in applying the theory of principal forms to a self-reciprocal type $[w: i, i]$ which it is proper to mention. For greater simplicity suppose the type to be diadelphic and let M, N be forms of the type which satisfy the equations

$$IM=\rho M, \qquad IN=\rho' N;$$

* As an exercise the reader may satisfy himself that this type is monadelphic by the direct application of the rule for finding the multiplicity. It corresponds to a quadratic covariant of the type [2, 4 ; 4, 1 : 2], which is the same (introducing the weight $\frac{2\,.\,4+4\,.\,1-2}{2}$ in lieu of the degree) as the type [5 : 2, 4 ; 4, 1] and has the same multiplicity μ by the law of reciprocity as the type [5 : 4, 2 ; 4, 1], namely, the difference between the number of modes of composing 5 and of composing 4 with two of the numbers 0, 1, 2, 3, 4 and with one of a *distinct* set of the same numbers. The arrangements for the weight 5 will be

4. 1 : 0, 4. 0 : 1, 3. 2 : 0, 3. 1 : 1, 3. 0 : 2, 2. 2 : 1, 2. 1 : 2, 2. 0 : 3, 1. 1 : 3, 1. 0 : 4,

and for the weight 4,

4. 0 : 0, 3. 1 : 0, 3. 0 : 1, 2. 2 : 0, 2. 1 : 1, 2. 0 : 2, 1. 1 : 2, 1. 0 : 3, 0. 0 : 4.

The numbers of the combinations in the two sets of arrangements are respectively 10 and 9. Hence $\mu=10-9=1$, or the type is monadelphic. The same result of course follows from the known fundamental scale for a quadro-biquadratic system.

the M and N have tacitly been defined to be the principal forms for such a type. Now in general this definition merges into and is coincident with the definition of principal forms for the general case, namely, that I^2M and I^2N must be multiples of M and N and the latter condition might be substituted for the former. But this is not always true, for if $\rho + \rho' = 0$, we shall have

$$I^2M = \rho^2 M, \qquad I^2N = \rho^2 N,$$

and consequently, $$I^2(M + \lambda N) = \rho^2(M + \lambda N),$$

so that if we were to follow the general definition the principal forms might become indeterminate, whereas by following the definition special to the self-reciprocal case they are determinate. Thus for example, suppose that P, Q, two particular forms of the type, satisfy the equations

$$IP = \rho Q, \qquad IQ = \sigma P;$$

the principal forms will then be

$$\surd(\sigma)\, P + \surd(\rho)\, Q \text{ and } \surd(\sigma)\, P - \surd(\rho)\, Q,$$

and the two principal multipliers become $\surd(\rho\sigma)$ and $-\surd(\rho\sigma)$, so that the principal forms according to the general definition would be indeterminate, but according to the definition proper to self-reciprocal forms strictly determinate.

Let us, as a final example of self-reciprocal type, consider the type [10: 5, 5] which is the same as [5, 5: 5] and corresponds to the covariant of the fifth order in the coefficients and of the fifth degree in the variables to a quintic. This is diadelphic, as may be found by consulting the table of irreducible forms for the quintic, which will show that it can arise only from the multiplication of the parent quantic itself by its quartinvariant or from that of the quadratic quadricovariant by the cubic cubo-covariant or from a linear combination of the two products. But without this, the same conclusion may be arrived at by direct calculation of the value of (10: 5, 5) – (9: 5, 5) and the multiplicity will be found to be 18 – 16, or 2 as premised. Let us take as our special forms,

$$P = (ae - 4bd + 3c^2)(ace + 2bcd - ad^2 - c^3 - b^2e),$$

$$Q = a(a^2f^2 - 10abef + 4acdf + 16ace^2 - 12ad^2e + 16b^2df + 9b^2e^2 - 12bc^2f$$
$$- 76bcde + 48bd^3 + 48c^3e - 32c^2d^2),$$

where $\dfrac{Q}{a}$ is the quartinvariant J given by Salmon, p. 207 (third edition), being in fact the discriminant of the quadricovariant whose root-differentiant is $ae - 4bd + 3c^2$. Call α, β, γ, δ, ϵ the five roots of the quintic and make $a = 1$. Q contains the term f^2 which is the image of $\alpha^5\beta^5$ which can only arise from combinations of the coefficients into which d, e, f none of them enter. But all the terms of Q contain d, e, or f, moreover P has no term containing f^2, therefore IQ does not contain Q but is simply a multiple of P. Again ce^2, which enters into P, is the image of combinations of the form

$\alpha^2\beta^4\gamma^4$, and the only term in Q which can give rise to such combinations is $-32c^2d^2$, or

$$-\frac{32}{10^4}(\Sigma\alpha\beta)^2(\Sigma\alpha\beta\gamma)^2,$$

and each such combination will have unity for its coefficient and their number is 30. Hence

$$IQ=-\frac{30.32}{10000}P=-\frac{12}{125}P.$$

Again, Q contains $-10bef$, and bef is the image of such root-combinations as $\alpha^5\beta^4\gamma$ (60 in number) the only terms in P capable of producing which are $10bc^3d$ and $-3c^5$ or $\frac{1}{5000}\Sigma\alpha(\Sigma\alpha\beta)^3\Sigma\alpha\beta\gamma-\frac{3}{100000}(\Sigma\alpha\beta)^5$. And bef does not appear in P, hence one part of IP will be

$$\left(\frac{60}{-50000}+\frac{3.5.60}{1000000}\right)Q,\quad\text{or } -\frac{3}{10000}Q.$$

Again, ce^2 is the image of such combinations as $\alpha^4\beta^4\gamma^2$ (30 in number) and the only terms in P giving rise to such are $-3c^5-8b^2cd^2+10bc^3d-3c^2d^2$; $-3c^5$ is $-\frac{3}{100000}(\Sigma\alpha\beta)^5$ and will give rise to $-\frac{3.20.30}{100000}ce^2$ in IP; $-8b^2cd^2$ is $-\frac{8}{25000}(\Sigma\alpha)^2(\Sigma\alpha\beta)(\Sigma\alpha\beta\gamma)^2$ and will give rise to $-\frac{2.8.30}{25000}ce^2$ in IP; $10bc^3d$ is $\frac{10}{50000}\Sigma\alpha(\Sigma\alpha\beta)^3\Sigma\alpha\beta\gamma$ and will give rise to $\frac{7.10.30}{50000}ce^2$ in IP; $-3c^2d^2$ is $-\frac{3}{10000}(\Sigma\alpha\beta)^2(\Sigma\alpha\beta\gamma)^2$ and will give rise to $-\frac{3.30}{10000}ce^2$ in IP. Hence the total coefficient of ce^2 in IP is

$$-\frac{9}{500}-\frac{12}{625}+\frac{21}{500}-\frac{9}{1000}=\frac{-90-96+210-45}{5000}=-\frac{21}{5000},$$

and consequently, since P contains the term ce^2 and Q the term $16ce^2$, if $IP=\theta P-\frac{3}{10000}Q$,

$$\theta-\frac{3.16}{10000}=-\frac{21}{5000},\text{ so that }\theta=\frac{3}{5000},$$

and therefore $$IP=\frac{3}{5000}P-\frac{3}{10000}Q,$$

and thus the equation for finding the principal multipliers ρ is

$$\begin{vmatrix}\frac{3}{5000}-\rho, & -\frac{3}{10000}\\ -\frac{12}{125}, & -\rho\end{vmatrix}=0,$$

or, if $$\rho=\frac{3\sigma}{10000},\quad\begin{vmatrix}2-\sigma, & -1\\ -320, & -\sigma\end{vmatrix}=0.$$

Thus $\sigma^2 - 2\sigma - 320 = 0$, the roots of which are irrational. I have thought it advisable to set out the work in this example with some explicitness in order to remove an impression that might otherwise arise from the examples which precede, that the principal multipliers and consequently the principal forms, for self-reciprocal types, necessarily contain only rational numbers.

The work is very much longer for the case of non-self-reciprocal types The simplest example of such that presents itself to my mind is that of the sextinvariant of a quartic and the quartinvariant of a sextic, for either of which the type is diadelphic. The discussion of this case forms the subject of the annexed Note, for all the calculations of which I am indebted to the labour and skill of Mr F. Franklin, Fellow of Johns Hopkins University. For the sake of brevity the steps of the work have been suppressed and only the final results set out.

Note C to Appendix 2.

On the Principal Forms of the General Sextinvariant to a Quartic and Quartinvariant to a Sextic.

Let

$$L = (ae - 4bd + 3c^2)^3 = \left[\frac{1}{2^3 \cdot 3} \Sigma (\alpha - \beta)^2 (\gamma - \delta)^2\right]^3,$$

$$M = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}^2 = (ace + 2bcd - ad^2 - b^2e - c^3)^2 = \left[\frac{1}{2^4 \cdot 3^3} \Sigma (\alpha - \beta)^2 (\gamma - \delta)^2 (\alpha - \gamma)(\beta - \delta)\right]^2,$$

$$P = (ag - 6bf + 15ce - 10d^2)^2 = \left[-\frac{1}{2^4 \cdot 3 \cdot 5} \Sigma (\alpha - \beta)^2 (\gamma - \delta)^2 (\epsilon - \phi)^2\right]^2,$$

$$Q^* = \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} = \left\{\begin{array}{l} aceg - acf^2 - ad^2g + 2adef \\ - ae^3 - b^2eg + b^2f^2 + 2bcdg \\ - 2bcef - 2bd^2f + 2bde^2 - c^3g \\ + 2c^2df + c^2e^2 - 3cd^2e + d^4 \end{array}\right.$$

$$= \frac{1}{2^5 \cdot 3^3 \cdot 5^3} \Sigma (\alpha - \beta)^4 (\gamma - \delta)^4 (\epsilon - \phi)^4 - \frac{71}{2^{10} \cdot 3^4 \cdot 5^4} \left[\Sigma (\alpha - \beta)^2 (\gamma - \delta)^2 (\epsilon - \phi)^2\right]^2.$$

* M. Faà de Bruno, in the tables at the end of his *Théorie des Formes Binaires*, designates Q and $\Sigma (\alpha - \beta)^4 (\gamma - \delta)^4 (\epsilon - \phi)^4$ by the same symbol I_4; a misleading circumstance which gave rise in this instance, and might in others to a large amount of useless labour. As can easily be seen from the above, the true value of $\Sigma (\alpha - \beta)^4 (\gamma - \delta)^4 (\epsilon - \phi)^4$ is

$$120\,(71P + 900Q) = 120\,(71a^2g^2 - 852abfg + 3030aceg - 900b^2eg - 2320ad^2g + 1800bcdg - 900c^3g - 900acf^2 + 3456b^2f^2 + 1800adef - 14580bcef + 6720bd^2f + 1800c^2df - 900ae^3 + 1800bde^2 + 16875c^2e^2 - 24000cd^2e + 8000d^4).$$

It should also be observed that in the expression for Q (the catalecticant) given in the same table, the signs of the terms $-2bd^2f + 2bde^2$ have been interchanged.

Then
$$IL = \frac{P-6Q}{2^5 \cdot 3^2}, \qquad IM = \frac{P-33Q}{6^5},$$
$$IP = \frac{L+2M}{2^4 \cdot 5}, \qquad IQ = \frac{9L-142M}{2^6 \cdot 3^2 \cdot 5^3},$$
$$I^2L = \frac{7614L+23868M}{2^{11} \cdot 3^6 \cdot 5^3}, \qquad I^2M = \frac{201L+2162M}{2^{11} \cdot 3^6 \cdot 5^3}.$$

In order that $\lambda L + \mu M$ shall be a principal form we must have
$$(7614 - 2^{11} \cdot 3^6 \cdot 5^3 \rho)\lambda + 201\mu = 0,$$
$$23868\lambda + (2162 - 2^{11} \cdot 3^6 \cdot 5^3 \rho)\mu = 0,$$
$$\begin{vmatrix} 7614 - 2^{11} \cdot 3^6 \cdot 5^3 \rho, & 201 \\ 23868 \quad , & 2162 - 2^{11} \cdot 3^6 \cdot 5^3 \rho \end{vmatrix} = 0,$$
or, putting $\sigma = 2^8 \cdot 3^6 \cdot 5^3 \rho$,
$$\sigma^2 - 1222\sigma + 182250 = 0,$$
where it may perhaps be worth noticing that the last term is $2 \cdot 3^6 \cdot 5^3$ and the coefficient of the second term $2 \cdot 13 \cdot 47$. We obtain from this equation
$$\rho = \frac{611 \pm \sqrt{(191071)}}{2^8 \cdot 3^6 \cdot 5^3}*.$$

The principal forms in L and M will then be found to be
$$201L + \{-2726 + 8\sqrt{(191071)}\}M, \qquad 201L + \{-2726 - 8\sqrt{(191071)}\}M;$$
and those in P and Q
$$101P + \{-11436 + 24\sqrt{(191071)}\}Q, \qquad 101P + \{-11436 - 24\sqrt{(191071)}\}Q.$$
Or, if we please, the principal forms in the two cases may be taken as the factors of
$$201L^2 - 5452LM - 23868M^2 \text{ and } 101P^2 - 22872PQ + 205200Q^2$$
respectively†. The question, what reduced quadratic forms can appear in the theory of diadelphic types, may one day or another become the subject of *à priori* investigation and form a new connecting link between the Calculus of Invariants and the Theory of Numbers. The linear functions of L and M and of P and Q, corresponding to the reduced forms of the above expressions might perhaps be termed the principal *rational* forms of the two types respectively.

* The number under the radical sign is, I believe, a prime number, but I have not within reach the tables necessary for verifying this. Professor Newcomb, by an exceedingly ingenious combination of a table of squares with Crelle's table of multipliers (a real stroke of genius), was able to ascertain by an inspection (the work of a few minutes) that 191071, if not a prime number, must contain a factor not greater than a certain moderate sized integer (137 if my memory serves me right) which reduces the trials necessary to be made to a very small compass.

† These are reducible to
$(201, 68, -60800 \between L', M)^2$, $(101, -23, -1089667 \between P', Q)^2$, where $L' = L - 14M$, $P' = P - 113Q$.

It may be well to notice that if $I^2U = \rho U$, then $I^2 \cdot IU = I \cdot I^2U = \rho IU$, and consequently the principal forms for two reciprocal types are images respectively of one another, and the principal multipliers are the same for the two systems.

Note D to Appendix 2.

On the Probable Relation of the Skew Invariants of Binary Quintics and Sextics to one another and to the Skew Invariant of the same Weight of the Binary Nonic.

The law of reciprocity extended, as it has already been in these pages, to systems of quantics, admits of an additional important generalization.

We know that Regnault's law of substitution holds good for algebraical forms, and in fact when transferred to the algebraical sphere becomes identical with the method which I believe I was the first to employ (now familiar to algebraists through the use made of it by Professors Clebsch and Gordan) to which I gave the name of emanation (Faà de Bruno, p. 198).

The principle, stated in chemico-algebraical language, is that in algebraical compounds any number of atoms of a given valence may be replaced by the same number of *new* equi-valent atoms. [In algebra it is essential to lay a peculiar stress on the word *new*; for if the substituted atoms should be homonymous with the remaining atoms, there is a possibility of the transformed compound reducing to zero. As for instance in the algebraical compound $ab' - a'b$ (the representative, say, of potassic iodide), if the atom of potassium should be changed into another of iodine (or *vice versa*), the compound, viewed algebraically, would disappear.]

The law of reciprocity as I have previously given it, translated into chemico-algebraical language amounts to saying that the total number of atoms of one kind (say m n-valent of one kind) may be replaced by n m-valent atoms of another kind; but by applying the rule of substitution first and then that of reciprocity we may see that the condition of *totality* may be done away with and the proposition reduced to the simplified form that in any algebraical compound *m n-valent atoms may be replaced by n m-valent ones.* Whether this law has any application in the chemical sphere, I must leave to chemists to determine.

In addition to the well known fact that a quintic possesses an invariant of the 18th order, and a sextic one of the 15th order, having obtained a complete scheme of the irreducible invariants for the binary quantic of the 10th degree, I was put in possession of the new fact that this last form

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possesses an invariant of the 9th order and consequently that the nonic possesses an invariant of the 10th order*.

Now the weight of each of these skew invariants is the same number 45, and I was thus led to suspect that they coexisted in virtue of some secret connexion. What that connexion is I think that I am now (very unexpectedly) in a position to explain and to show (with a high degree of probability) how the values of these three invariants may be actually deduced and calculated from one another. This follows as a consequence of the combined laws of reciprocity and substitution otherwise called emanation. For suppose we have an invariant of a quantic of the mth degree, of the order np in the coefficients. By the principle of emanation we may transform this into an invariant to a system of n quantics, each of the degree m and of the order p in each set of coefficients, and by the generalized law of reciprocity this may be again transformed into an invariant to a system of n quantics, each of degree p and of the order m in each set of coefficients. If now finally these n quantics be all made identical with one another, then the transformed invariant, *provided it does not vanish*, becomes an invariant of the order mn to a single quantic of the degree p, and accordingly we may pass in certain

* I have calculated, with the kind assistance of Mr Halsted, the expression in its canonical form of the generating function to a binary quantic of the 10th degree. The coefficient of t^m in this fraction developed, represents the number of parameters in the general invariant of the mth order of the given decadic. Its denominator is

$$(1-t^2)(1-t^4)(1-t^6)^2(1-t^8)(1-t^9)(1-t^{10})(1-t^{14})$$

and its numerator is the rational integer function

$$1+2t^6+\ldots+2t^{42}+t^{48},$$

the successive coefficients being

1, 0, 0, 0, 0, 0, 2, 0, 4, 2, 7, 6, 15, 13, 16, 25, 22, 31, 34, 40, 41, 47, 46, 49, 48, 49, 46, 47, 41, 40, 34, 31, 22, 25, 16, 13, 15, 6, 7, 2, 4, 0, 2, 0, 0, 0, 0, 0, 1,

showing that the primary fundamental invariants are of the orders 2, 4, 6, 6, 8, 9, 10, 14, and that (by the law of "Tamisage" *anglice* siftage) the secondary (or as they might be better termed the auxiliary) ones are of the orders 6, 8, 9, 10, 11, 12, 13, 14, 15, 17 taken 2, 4, 2, 7, 6, 12, 13, 18, 21, 11 times respectively. Any other invariant of the decadic can be represented as a linear function of a limited number of combinations of the secondaries, having for its coefficients some combination of powers of the primaries.

Suppose that the same numerical order occurs among the primaries and secondaries, as for example 6, which occurs twice among the former and twice among the latter. This will indicate in the first place that, calling A and B the quadric and quartic invariants, the general sextic one will be of the form

$$\lambda A^3+\mu AB+\nu_1Q_1+\nu_2Q_2+\nu_3Q_3+\nu_4Q_4$$

and that any two independent special values of $\nu_1Q_1+\nu_2Q_2+\nu_3Q_3+\nu_4Q_4$ may be taken as primaries and any other independent two as secondaries, and so in general; I mention this to prevent the false suggestion, which might otherwise arise, that the secondaries and primaries are different in internal constitution. This remark receives a beautiful illustration in an algebraical theory (recently developed by me) of chemical isomerism, which gives rise to a generating function precisely similar in character to that applicable to in- and co-variants and is subject to a similar law of interpretation, graphs taking the place of algebraical forms, and atomicules and the numbers of grouped atoms, of degrees and orders.

cases from the type $[m, np: 0]$ to the type $[p, mn: 0]$. So in all probability we may pass from the type $[5, 18: 0]$ to the type $[6, 15: 0]$ and to the type $[9, 10: 0]$. As there is only one invariant of the type $[6, 15: 0]$, or of the type $[9, 10: 0]$, it follows that, if the passage from type to type is real and not nugatory, the three invariants of these second types may be deduced, any one from any other, by the explicit processes above described. There is nothing at all doubtful in the course of the transformation except what arises from the possibility that in the last step of it the effect of rendering identical the different sets of coefficients—that is of finding the counter-emanant, so to say, of the invariant containing n sets of variables—may be to render the whole expression null. This of course would happen if we attempted to pass from the type $[5, 18: 0]$ to the type $[3, 30: 0]$, or to the type $[2, 45: 0]$, which we know are void of forms. But there is no reason why we should expect this to happen when we pass from the given type to other types known to contain one or more forms. It would require no impracticable amount of labour to actually verify the fact of the transformation being effectual between the skew invariants of the sextic and quintic forms. The survival of a *single* known term in either of them, in the process of attempting to deduce it from the other, would be sufficient to establish the effectualness of the method, and that it will be found to be effectual, for reasons too long to dwell upon here, I scarcely entertain a doubt. The process to be employed may be summarily comprehended under the three rubrics of diversification, reciprocation and unification. The first is one of differentiation alone; the second involves the expansion of functions of the coefficients of an equation in terms of roots and the substitution of η_i for α^i; the third consists merely in replacing distinct sets of letters (η) by a single set. In practice the two latter processes would be of course combined into one. It will be instructive to consider some simple example of this method of transformation of types.

Let us take $(ac-b^2)^3$ regarded as belonging to the type $[2, 6: 0]$. I shall show how to pass from this to a form of the type $[3, 4: 0]$. Taking a third emanant of the given form, that is the result of the operation upon it of $\dfrac{1}{1\cdot 2\cdot 3}(a'\delta_a + b'\delta_b)^3$, we obtain

$$(ac' + a'c - 2bb')^3 + 2(ac-b^2)(a'c'-b'^2)(ac'+a'c-2bb').$$

Let us call $\alpha, \beta, \alpha', \beta'$ the roots of the two forms $[1, b, c]$, $[1, b', c']$ respectively; then the emanant last found (multiplied by 8) becomes

$$(2\alpha\beta + 2\alpha'\beta' - \alpha\alpha' - \alpha\beta' - \beta\alpha' - \beta\beta')$$
$$\{(2\alpha\beta + 2\alpha'\beta' - \alpha\alpha' - \alpha\beta' - \beta\alpha' - \beta\beta')^2 + (\alpha-\beta)^2\cdot(\alpha'-\beta')^2\}.$$

After performing all the multiplications and introducing the zero powers of $\alpha, \alpha', \beta, \beta'$ in such terms as do not contain one or more of these letters, all that remains is to substitute

$$\begin{aligned}
\alpha^0 &= \alpha'^0 = \beta^0 = \beta'^0 = \quad a,\\
\alpha &= \alpha' = \beta = \beta' = -b,\\
\alpha^2 &= \alpha'^2 = \beta^2 = \beta'^2 = \quad c,\\
\alpha^3 &= \alpha'^3 = \beta^3 = \beta'^3 = -d,
\end{aligned}$$

the letters a, b, c, d for greater simplicity being used instead of $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3$, that is $\eta_0, -\eta_1, \eta_2, -\eta_3$. The result will not vanish. To show this consider the group of terms which change into a^2d^2. These are the binary combinations of $\alpha^3, \alpha'^3, \beta^3, \beta'^3$. $2\alpha\beta$ and $2\alpha'\beta'$ in the first factor give rise to $8\alpha^3\beta^3$, $8\alpha'^3\beta'^3$ and the remaining four terms to $-2\alpha^3\alpha'^3, -2\alpha^3\beta'^3, -2\beta^3\alpha'^3, -2\beta^3\beta'^3$ respectively. Hence the term a^2d^2 will survive with the multiplier $8+8-2-2-2-2$, that is, 8. So again the only terms introducing ac^3 will be the ternary combinations of $\alpha^2, \alpha'^2, \beta^2, \beta'^2$. $2\alpha\beta$ and $2\alpha'\beta'$ will be found to produce as many positive as negative terms of this kind, but $-\alpha\alpha'$ will produce $4\alpha^2\alpha'^2\beta^2 + 4\alpha^2\beta^2\beta'^2$, giving rise to $8ac^3$, and as the same will be true for $-\alpha\beta', -\beta\alpha', -\beta\beta'$, we see that $32ac^3$ will emerge in the result. Hence the given invariant becomes converted into

$$(a^2d^2 + 4ac^3 + \ldots),$$

that is the discriminant of the cubic whose type is [3, 4: 0] as was to be shown.

I think it is little doubtful that wherever there exist forms contained under each of two types, the product of whose rank and order is identical, we may pass from the one to the other by means of the combined processes of emanation and reciprocation, as in the foregoing example*. [The case is much the same as with transvection. That process may produce a null form, but any actually existent form may be produced by it and exhibited as a transvect.] To pass from Hermite's to Cayley's skew form, we must first by emanation change [5, 18: 0] into [5, 6; 5, 6; 5, 6: 0] and then this latter into [6, 15: 0]; by means of the process last exemplified.

* Call

$$(b^2-ac)^3=A,\ a^2d^2+4ac^3+\ldots=B,\ a'\delta_a+b'\delta_b+c'\delta_c=E,\ a\delta_{a'}+b\delta_{b'}+c\delta_{c'}+d\delta_{d'}=H^{-1}.$$

Then it follows from the text that

$$B=\tfrac{1}{12}H^{-2}IE^3A,$$

where it may be observed that E^3A is diadelphic, for it will be proved that (6: 3, 2; 3, 2) = 16, and (5: 3, 2; 3, 2) = 14, so that any form whatever coming under the same type as E^3A is a linear function of $(ac'+a'c-2bb')^3$ and $(ac'+a'c-2bb')(ac-b^2)(a'c'-b'^2)$, say L and M (whose difference, $L-M$, is $\frac{1}{6}E^3A$), and operated on by $H^{-2}I$ would produce a multiple of B (whose type is monadelphic) with the sole exception of $\lambda L - 2\mu M$, the result of operating upon which would be zero. Similarly we may see that in any given case the chances are infinitely in favour of the expectation that the process will *not* be nugatory by which it has been shown we may pass from one known type [m, np: 0] to another known one [p, nm: 0].

APPENDIX 3.

On Clebsch's Theory of the "Einfachstes System associirter Formen" (*vide Binären Formen, p.* 330) and its Generalization.

Let $(a, b, c, \ldots k, l \between x, y)^n$ be any binary quantic. Let the provector symbol $(l\delta_k + 2k\delta_h + 3h\delta_g + \ldots)$ be denoted by Ω, and the revector symbol $(a\delta_b + 2b\delta_c + 3c\delta_d + \ldots)$ by $\mho$. Let Q_{2i} represent the quadrinvariant of the above form when $n = 2i$. Now let Ω and $\mho$ be made to comprise the $2i+1$ letters $a, b, c, \ldots l, m$; then $a\Omega Q_{2i} - 2bQ_{2i}$* will be nullified by the operation of $\mho$ and will therefore be a cubinvariant for the case of $n = 2i + 1$, which we may call Q_{2i+1}. Also let $Q_0 = a$; then $Q_0, Q_1, Q_2, \ldots Q_\mu$ will be differentiants to all binary quantics of degree equal to or greater than μ. The above I call basic differentiants. Their distinguishing characteristic is that the highest letter in each of them enters into it only in the first degree multiplied by a or by a^2 and by no other letter. Now let D be any given differentiant of degree μ and for the moment make $a = 1$. Then it is obvious that D may be expressed—by means of successive substitutions of its ultimate, its penultimate, its antepenultimate, etc. letters up to c inclusive, in terms of the corresponding basic differentiants and the anterior letters,—as a rational integer function of $Q_1, Q_2, \ldots Q_\mu, b$; or, restoring to a its general value, will be a rational integer function of $Q_0, Q_1, Q_2, \ldots Q_\mu, b$, say F, divided by a power of a. But I say that b will have disappeared in the process. For $\mho D = 0$; and $\mho Q_0 = 0$, $\mho Q_1 = 0 \ldots \mho Q_\mu = 0$. Hence, regarding each Q as a constant, $\left(a\dfrac{d}{db}\right) F = 0$, or F does not contain b.

Again, suppose we take a system of two quantics and let $Q_0, Q_1, \ldots Q_\mu$ be the basic differentiants of the one, $Q_0', Q_1', \ldots Q_\nu'$ of the other, and let D be any differentiant of the system. Then by the same method as before we shall find

$$D = \frac{F(Q_0, Q_2 \ldots Q_\mu : Q_0', Q_2' \ldots Q_\nu' : b, b')}{a^m \cdot a'^n}.$$

Also each Q will be nullified by $\mho$, and each Q' by $\mho'$, and therefore each Q and Q' as well as D will be nullified by the operator $\mho + \mho'$. Hence we shall have

$$\left(a\frac{d}{db} + a'\frac{d}{db'}\right) F = 0,$$

or

$$F = \phi(ab' - a'b),$$

* For by a well-known formula if D is a differentiant in x of the type $[w : i, j]$,

$$\mho\Omega D = (ij - 2w) D.$$

Consequently when Q_{2i} is regarded as a differentiant in x of the type $[2i : 2i+1, 2]$

$$\mho\Omega Q_{2i} = Q_{2i} \text{ also } \mho Q_{2i} = 0 \text{ and } \mho b = a.$$

Hence $\mho(a\Omega Q_{2i} - 2bQ_{2i}) = 0.$

ϕ being a rational integral form of function. In like manner for a system of three quantics, regarding the several sets of its basic differentiants as constant, we shall have

$$F = \phi\,(ab' - a'b : ac' - a'c : bc' - b'c),$$

where ϕ is a rational integral form of function, or

$$F = \psi\,(ab' - a'b : ac' - a'c : a, a'),$$

and so in general. Hence, remembering that any relation between differentiants must continue to subsist between the covariants of which they are the roots, and now, understanding by base forms the complete covariants of which the basic coefficients are the roots, we may pass from differentiants to in- or co-variants and obtain the following theorems.

(1) For a single quantic of degree i, any in- or co-variant is expressible by a fraction whose numerator is a rational integer function of its i base forms and whose denominator is a power of the quantic. This is Clebsch's theorem.

(2) For a system of quantics, any in- or co-variant is expressible by a fraction whose numerator is a rational integer function of the separate base forms of its several quantics and of any complete system of $(\mu - 1)$ independent Jacobians of the quantics taken in pairs, and whose denominator is a product of powers of the quantics of the system.

Also it will be observed that these theorems will continue to subsist when the base forms have for their roots in lieu of the basic differentiants, as above defined, any ascending scale of differentiants in which the letters enter successively one at a time and each letter on its first appearance figures only in the first degree and combined exclusively with powers of a.

On the theory of basic forms may be grounded a method for obtaining, *in propriâ personâ*, the fundamental in- and co-variants to a quantic or system of quantics in regular succession, by a process which continues so long as there are many more to be elicited and comes to a self-manifesting end as soon as the last irreducible form has been obtained, like an air pump that refuses to act as soon as the exhaustion has become complete. In a word, the cataloguing of the irreducible in- or co-variants is transferred to the province of, and becomes a problem in, ordinary algebra.

I have previously observed that any expression which represents a differentiant in regard to a quantic of a given degree necessarily does the same for quantics of all higher degrees. And I may take this occasion to remark, or to repeat, that a differentiant may be irreducible in respect to the quantic of minimum degree to which it can be referred, and yet not so for quantics of higher degrees. Thus, if we take the expression

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd,$$

this referred to a cubic is irreducible (as is well known), but regarded as a differentiant of a quartic or higher degreed quantic, is reducible, being in fact identical with

$$(ac-b^2)(ae-4bd+3c^2)-a\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}.$$

Let us suppose a linear function $yu-xv$ combined with a quantic into a system. Then it follows as a corollary from (2) at [p. 200], that if the quantic belongs to the form $(a, b, c, \ldots l \between u, v)^i$, or say more simply to the form $[a, b, c, \ldots l]$ any covariant of such quantic multiplied by a suitable power of a will be a function of y, $ax+by$ and of the differentiants, or in a word, every covariant of the quantic expressed as a function of x and $ax+by$ will have no coefficients but what are differentiants, or to use Professor Cayley's term, semi-invariants. Thus, for example, the Hessian of the cubic $(a, b, c, d \between x, y)^3$ may be put under the form

$$\frac{1}{a^2}\left\{(ac-b^2)(ax+by)^2+(a^2d-3abc+2b^3)(ax+by)y+(ac-b^2)^2y^2\right\}.$$

So it will be found that the Hessian of the quintic, namely

$$(ae-4bc+3c^2)x^2+(af-3be+2cd)xy+(bf-4cd+3d^2)y^2$$

on writing $ax+by=X$, becomes

$$\frac{1}{a^2}\Big\{(ae-4bc+3c^2)X^2+(a^2f-5abe+2acd+8b^2d-6bc^2)Xy \\ -\left[(ac-b^2)(ae-4bd+3c^2)+3a(ace+2bcd-ad^2-b^2e-c^3)\right]y^2\Big\},$$

where all the coefficients are semi-invariants-in-x, the second coefficient being one of the basic differentiants and the latter part of the third coefficient, the catalecticant

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix},$$

and so more generally, it may be shown to follow from (2), that if there be any number of binary quantics

$$[a, b, c \ldots], \quad [a', b', c', \ldots], \quad [a'', b'', c'', \ldots],$$

every covariant of such system, expressed as a function of y and of *any one* of the quantics

$$ax+by, \; a'x+b'y, \ldots$$

chosen at will, has differentiants-in-x exclusively for its coefficients.

It is easy to express the base-covariants in terms of the roots. Those of weight $2n$ and order 2 will be of the form

$$\Sigma F(a_1, a_2, a_3, \ldots a_{2n})(x-a_{2n+1})^2(x-a_{2n+2})^2 \ldots$$

where F may be expressed as

$$(a_1 - a_2)^2 (a_3 - a_4)^2 \dots (a_{2n-1} - a_{2n})^2,$$

or,
$$(a_1 - a_2)(a_2 - a_3)(a_3 - a_4) \dots (a_{2n-1} - a_{2n})(a_{2n} - a_1),$$

or under a variety of other forms all equal to a numerical factor près; for the type $[2n: 2n, 2]$ and the more general one $[2n: 2n + \nu, 2]$ are monadelphic. And again those of the weight $2n + 1$ and order 3 may take, or at all events be replaced by, the form

$$\Sigma \{(a_1 - a_2)(a_2 - a_3) \dots (a_{2n-1} - a_{2n})(a_{2n} - a_1)(a_1 - a_{2n+1})(x - a_1)(x - a_2) \dots \\ (x - a_{2n+1})(x - a_{2n+2})^3 (x - a_{2n+3})^3 \dots\}.$$

It is proper to notice that the type $[2n + 1 : 2n + 1 + \nu\,;\ 3]$ is only monadelphic so long as $2n + 1$ is less than 9, so that we cannot, without an investigation which might be tedious, determine whether the above representation coincides with the basic forms of the third order in the coefficients adopted in [p. 199]; but such investigation would be a work of supererogation, for the only *material* character for any of the base-covariants in question to possess is, that its root differentiant-in-x shall be not higher than of the third order in the coefficients and shall contain the element ϵ_{2n+1}. Any formula having this property (which is enjoyed by the root function above given) is just as good as any other for the purposes of this theory*.

It will be seen to follow from the theorem I have given for differentiants from which Clebsch's follows as an immediate consequence, that all the permutation-sums of any rational integer function of the differences of the roots of an algebraical equation of the nth degree are rational integer functions of $(n-1)$ of them of the second and third order alternately; so, for example, all the coefficients in Lagrange's equations to the squares of the differences of the roots of an algebraical equation in its ordinary form are rational integer

* Writing the type under the form $[2n+1: 2n+1+\nu,\ 3]$, the degree of the corresponding covariant in the variables is $2n+1+3\nu$, which is the degree in x of the symmetrical function assumed in the text; also each letter in this function occurs 3 times agreeing with the order 3 of the type, and the number of factors in the coefficient of the highest power of x is $2n+1$, which is right for the weight. It is obvious also by inspection that the product $a_1 . a_2 \dots a_{2n+1}$ will arise from each term of the assumed symbolical function affected always with the same sign, so that ϵ_{2n+1} will occur (as required) in its expression in terms of the coefficients. Of course all the same conclusions will apply if in the formula

$$(a_1 - a_2)^2 (a_3 - a_4)^2 \dots (a_{2n-1} - a_{2n})^2$$

is substituted in lieu of

$$(a_1 - a_2)(a_2 - a_3) \dots (a_{2n-1} - a_{2n})(a_{2n} - a_1).$$

That the type to which Q_{2n+1} belongs is non-monadelphic from and after $2n+1=9$ is obvious from the fact that that type, when the degree of the parent quantic is made a minimum, is of the form $[2n+1: 2n+1,\ 3]$, the multiplicity of which is the same as that of $[2n+1: 3,\ 2n+1]$, or set out in full $[2n+1: 3,\ 2n+1: 2n+1]$; but cubics include covariants of orders and degrees $2: 2$ and $3: 3$ among their fundamental forms, and $9: 9$ can be formed either by taking a triplication of $3: 3$, or by combining $3: 3$ with a triplication of $2: 2$, so that when $2n+1=9$ the type is diadelphic, and *a fortiori*, it is non-monadelphic for values of $2n+1$ superior to 9.

functions of $(n-1)$ known quantities. Thus, for instance, the equation to the squares of the differences of a cubic equation will be

$$\rho^3 + 18(b^2 - ac)\rho^2 + 81(b^2 - ac)^2 + 27\Delta = 0,$$

where the coefficients are given in terms of two differentiants $(b^2 - ac)$ and Δ.

Throughout this paper the perspicuity of expression has been considerably marred by want of a complete nomenclature which the theory of graphs and types necessarily calls for and which I shall hereafter employ whenever I may have occasion to revert to the subject. It is as follows:

In the first place, w, the weight in respect to the selected variable, and j, the order in the coefficients, are terms well understood and need no change or further illustration; i, the degree of the parent quantic, I shall hereafter call the *rank* of the type, $ij - 2w$ which becomes the degree of a covariant got by expanding the differentiant of type $[w: i, j]$ may be called the *grade*. The order and rank may be termed collectively the *permutable indices*.

When a differentiant is given algebraically its weight and order are given but *not* its rank; in addition to the weight and order a third number which may be called the *range* (and which I shall denote by a Greek ϵ) is given, being the number less 1 of the letters which enter into it. The relation between *rank* and *range* is one of inequality. The former may be equal to, or greater than, but not less than the latter.

The multiplicity of the type to which a given differentiant belongs is a function of the *weight*, *order* and *rank* and is consequently not known until the *rank* is assigned. Thus, for example $(ac - b^2)^2$, considered as having the lowest possible rank, namely 2 (the *range*) is monadelphic; its type is then $[2: 2, 4]$, but if the rank 4 be assigned to it so that its type is $[2: 4, 4]$, it becomes diadelphic. We have then, in general, 6 characters (not all independent) appertaining to a differentiant, namely, *weight*, *rank*, *order*, *grade*, *range* and *multiplicity*. The theory of types has never hitherto formed the subject of distinct contemplation, and that is why the necessity for the use of some of the above terms has not been previously felt. But it will have been observed that throughout the preceding memoir it has forced itself upon our notice, and in particular, that it is impossible to go to the bottom of the so-called law of reciprocity or that of the radical representation of forms without keeping in view the question of type and multiplicity.

I have also to remark that since the preceding matter was completed I have been surprised to learn that recent chemical research favours the notion of simple elements (hydrogen atoms in special) being distinguishable from each other in chemical composition. If this view is confirmed, the discrepancy, which I have pointed to, between the known conditions for the existence of algebraical graphs and the unknown natural laws which govern the production of chemical substances may become partially or wholly obliterated, so that, for example, the hydrogen molecule and the extended derivatives from marsh gas may exist in accordance with, and not in contradiction to, algebraical law, and thus it is possible to conceive that all the phenomena of chemistry and algebra may ultimately be shown to be identical.

Since the above matter was sent to press I have been led to study algebraically what may be termed the direct problem of isomerism, that is to say the determination of the number of combinations subject to given conditions that can be formed between the constituents of groups each containing a given number of equivalent chemical atoms, the valences of the several groups being either independent or given linear functions of a certain number of independent parameters. In this problem the numbers of atoms are given and the valences left indeterminate. In the inverse problem the valences are given and the numbers left indeterminate.

The problem of the enumeration of the saturated hydro-carbons, investigated by Professor Cayley, is a simple example of the inverse problem. The direct problem admits of a uniform and unfailing method of solution by generating functions, the exposition of which may probably form the subject of an additional Appendix in the following number*. This method is

* The principle employed in this method leads to the following theorem only a particular case of which comes into play in the general partition problem which covers the ground occupied by the allied invariantive and isomeric theories. Let there be given a product of a limited number of rational functions of

$$u_1^{a_1} . u_2^{a_2} \dots u_i^{a_i}; \quad u_1^{a_1'} . u_2^{a_2'} \dots u_i^{a_i'} ; \text{ etc., etc.,}$$

where all the indices are *positive or negative* integers, and let $\mu_1, \mu_2, \dots \mu_i$ be given linear functions of $\nu_1, \nu_2, \dots \nu_j$ (j being not greater than i), then it is always possible to find a limited product of rational functions of

$$v_1^{\beta_1} . v_2^{\beta_2} \dots v_j^{\beta_j}; \quad v_1^{\beta_1'} . v_2^{\beta_2'} \dots v_j^{\beta_j'} ; \text{ etc., etc.,}$$

where the indices are exclusively *positive*, such that the coefficient of $v_1^{\nu_1} . v_2^{\nu_2} \dots v_j^{\nu_j}$, in their product developed according to ascending powers of $v_1, v_2, \dots v_j$, shall be the same as the coefficient of $u_1^{\mu_1} u_2^{\mu_2} \dots u_i^{\mu_i}$ in the original product developed according to ascending powers of $u_1, u_2, \dots u_i$. Previous to the discovery of this principle the problem of isomerism, now completely solved potentially for the direct case, must have remained unattackable by any existing methods, such for example as were known to Euler, the inventor of the application of the method of generating functions to the theory of partitions. It renders supererogatory a large part of the methods devised by myself for the treatment of the problem of compound partitions contained in the printed notes of my lectures on Partitions, delivered at King's College, London, in the year 1859†. As an example of the direct problem of isomerism, suppose that three atoms of the same valence j are to combine with ϵ atoms of hydrogen which do not combine *inter se*; then the number of combinations which can be so formed is the coefficient of $a^j x^\epsilon$ in the development of the generating function

$$\frac{1 + ax + a^2x^2}{(1 - a^2)(1 - ax)^2 (1 - ax^3)}$$

if the three atoms are all unlike, and of the generating function

$$\frac{1}{(1 - a^2)(1 - ax)(1 - a^2x^2)(1 - ax^3)}$$

if they are all alike.

[† Volume II of this Reprint, p. 119.]

substantially the same as that which I have described* in general terms in the *Comptes Rendus* as applicable to the theory of ternary and other higher varieties of quantics but less difficult of application to the Isomeric Problem on account of the greater simplicity of the crude forms subject to reduction, which appear in it. Appendix 4 will contain the application of the theory of "Associirter Formen" to the algebraical deduction of the irreducible forms of the quintic and certain other cases which but for the press of matter awaiting publication in the *Journal* would have formed part (as announced) of the present Appendix.

As already stated in a previous footnote, the theory of irreducible forms reappears in the isomeric investigation, the general character of the reduced generating function to be interpreted in it being precisely the same as in the invariantive theory, which constitutes an additional and a closer and more real bond of connexion between the chemical and algebraical theories than any which I had in view when I commenced the subject of this memoir.

Note on the Ladenburg Carbon-Graph.

The reasoning by which I have† established, in the preceding number of the *Journal*, the validity of the Ladenburg graph (and the invalidity of the Kekulean one) as a representative of the root differentiant to a covariant of the 6th degree in the variables and of the 6th order in the coefficients to a quartic, is so peculiar and it may seem to some of my readers so far-fetched, that it appears highly desirable to confirm it by a direct demonstration founded on the principle, that the permutation-sum of the product of the bonds in a valid graph interpreted as differences between the letters which they connect, shall not vanish. Previous to applying this principle to Ladenburg's graph we must convert it into an invariant by attaching hydrogen atoms to the six apices. Let these apices be called a, b, c, d, e, f, and the hydrogen atoms $\alpha, \beta, \gamma, \delta, \epsilon, \phi$: then the permutation-sum under consideration is

$$\Sigma (a-b)(a-c)(b-c)(d-e)(d-f)(e-f)(a-d)(b-e)(e-f)(a-\alpha)(b-\beta)(c-\gamma)$$
$$(d-\delta)(e-\epsilon)(f-\phi)$$

where the 6 letters a, b, c, d, e, f are interpermutable, as are also the 6 letters α, β, γ, δ, ϵ, ϕ.

It may be well to observe at this point that if we struck off the hydrogen atoms and treated the graph as representing an invariant to a cubic form, the permutation-sum

$$\Sigma (a-b)(a-c)(b-c)(d-e)(d-f)(e-f)(a-d)(d-c)(c-f)$$

would be found to vanish, as may easily be shown and as it ought to do, because there exists no invariant of the 6th order in the coefficients to a cubic form. Let a and d be interchanged in the term given under the sign of summation in the permutation-sum formed from the Ladenburg graph ; then the sum of this together with the original term becomes

$$(a-d)(b-e)(c-f)(b-c)(e-f)(b-\beta)(c-\gamma)(e-\epsilon)(f-\phi)$$

[* p. 100 above.] [† p. 155 above.]

multiplied by

$$(a\delta - da)\{a^2-(b+c)a+bc\}\{d^2-(e+f)d+ef\}-(d\delta-aa)\{d^2-(b+c)d+bc\}\{a^2-(e+f)a+ef\},$$

which last named multiplier will be found to contain the quantity $(a^3d^2-a^2d^3)(a+\delta)$. Again, in the multiplicand, let b and c be interchanged; then, since

$$(b-e)(c-f)-(c-e)(b-f)=(b-c)(e-f),$$

the sum of the original and permuted multiplicand will contain a term

$$(a-d)(b-c)^2(e-f)^2bc(e-\epsilon)(f-\phi),$$

and accordingly the entire permutation-sum will contain the terms

$$(a+\delta)(a-d)(a^3d^2-a^2d^3)(b-c)^2(e-f)^2bc\Sigma(e-\epsilon)(f-\phi).$$

The partial sum last written is

$$4ef+4\epsilon\phi-2(e+f)(\epsilon+\phi).$$

Hence we may readily see that the total permutation-sum will contain *inter alia* a positive multiple of the combination $a^4b^3e^3d^2cfa$ and will not vanish, and consequently the graph is valid and not illusory; I presume that the same method applied to Kekulé's graph regarded as a representation of the covariant to the type [9 : 4, 6 : 6], which is the same thing (except that the hydrogen atoms are suppressed) as the graph to the invariant [15 : 4, 6; 1, 6 : 0], would serve to show it to be illusory as previously inferred from other considerations.