## BRIEF NOTES

# Preliminary note on an underdetermined impingement problem of Helmholtz jets 

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The steady plane potential flow due to symmetrical impingement of two jets on both sides of a central jet is studied and it is found that the available upstream parameters are not sufficient to determine the flow uniquely.

## 1. Introduction

Liquid jets in air, impinging against each other or against a wall, behave very much like inviscid flows subject to a condition of constant pressure along their free boundary surface. The theory of such flows relies largely on balances of mass and momentum. As a more refined tool of investigation the Helmholtz hodograph method [1,2] becomes available if attention is restricted to steady two-dimensional (plane) problems. That will be done in this note.

General principles of cause and effect, as well as common experience suggest that even a complicated steady flow, composed of more than one jet (but without any wall where separation could occur) should be determined completely by the upstream conditions, i.e. by the shape and position of the participating nozzles and by the fluxes emerging from them. However, there seem to be exceptions, at least within the framework of inviscid theory. Below will be discussed a steady, plane, inviscid and incompressible jet interaction problem, which admits a continuous one-parameter-set of different flow configurations after the upstream conditions have already been fixed. An example is schematically shown in Fig. 1.

## 2. Statement of the problem

A two-dimensional jet of inviscid incompressible fluid is approaching from infinity, where its width is $2 a$ (say) and the magnitude of its flow velocity is $V$ (say). Upon both sides of this "first" jet are impinging two "secondary" jets of the same fluid, having each the width $b$ (say) at upstream infinity and also the velocity magnitude $V$. Their direction relative to the first jet is specified by the angle $\beta$ (say) as defined in Figs. 2, 3, 4.

The irrotational, plane flow resulting from the impingement is supposed to be steady, to contain no discontinuities in its interior and to remain symmetrical with respect to the straight centerline of the first jet (which could be replaced by a plane wall in inviscid theory). All streamline plots of this note show the flow or hodograph on one side of the centerline only. The pressure outside of the flow has the same value everywhere. Hence, by Bernoulli's equation, the flow speed along the free flow boundaries is constant and everywhere equal to $V$.

From the interaction zone the fluid is escaping either via a jet along the centerline, which assumes the width 2d (say) at downstream infinity, and via two symmetrical jets of the asymptotic width $c$ (say) flowing in the directions given by angle $\gamma$ (say) which is also defined in Figs. 2, 3, 4. The values of $a$ and $V$ may be chosen as units of length and of velocity, respectively, without loss of generality.

Now, this problem may be posed: given arbitrary values $b / a, \beta$, which are the resulting values of the unknowns $c / a, d / a, \gamma$ ?



## 3. Balances of mass and momentum

We choose a closed control surface which intersects the participating jets sufficiently far from their region of mutual interaction. There the pressure is undisturbed and each jet is a uniform parallel stream, the mass and momentum-flux of which is proportional to its respective width.

Conservation of mass within the control volume yields:

$$
\begin{equation*}
a+b-c-d=0 \tag{3.1}
\end{equation*}
$$

Conservation of the momentum component parallel to the centerline renders:

$$
\begin{equation*}
a-b \cos \beta+c \cos \gamma-d=0 . \tag{3.2}
\end{equation*}
$$

No useful equation can be obtained from the normal momentum component, nor from the moment of momentum because of the symmetry of the flow.

From Eqs. (3.1) and (3.2) follows by subtraction:

$$
\begin{equation*}
b(1+\cos \beta)-c(1+\cos \gamma)=0 \tag{3.3}
\end{equation*}
$$

This means: With fixed parameters $b / a, \beta$ the conservation equations admit different solutions $c / a, \gamma$. Provided the parameter $\gamma$ can also be prescribed then the jet width $c(\gamma)$ is an increasing function of angle $\gamma$, the ratio $c / b$ being independent of $b / a$. In consequence of Eq. (3.1), the jet width $d(\gamma)$ is a decreasing function of $\gamma$.

Some additional, restrictive conditions for the flow emerge from simple geometry. The angles $\beta, \gamma$ must not be negative, $\beta \geqslant 0, \gamma \geqslant 0$. Their sum must not exceed two
right angles, $\beta+\gamma \leqslant \pi$. All jets must have a positive width, i.e. $b / a \geqslant 0, d / a \geqslant 0$. Substituting $d$ from Eq. (3.1) and $c$ from Eq. (3.3), the latter condition is equivalent to

$$
\begin{equation*}
b_{l} a \leqslant \frac{1+\cos \gamma}{\cos \beta-\cos \gamma}, \quad d / a \geqslant 0 \tag{3.4}
\end{equation*}
$$

No restriction is imposed by the condition $c / a \geqslant 0$, since this value is always positive if $\beta<\pi$ by Eq. (3.3).

Figure 6 illustrates: there is a compact domain of the three-dimensional (b/a, $\beta, \gamma$ )parameterspace, bounded by the planes $b / a=0, \beta=0, \gamma=0, \beta+\gamma=\pi$ and by the

curved surface $d / a=0$ of Eq. (3.4). Every point within this domain represents a flow which is compatible with the conservation equations.

## 4. Streamline patterns

Consider the topology of the streamline patterns, both in the physical plane and in the hodograph, occurring at different parameter triplets $(b / a, \beta, \gamma)$ in the region of jet interaction. Two types of patterns will readily be distinguished, type $A$ and $B$ (say) according to their stagnation points' position off or on the centerline.

The hodograph, a conformal map of the physical streamlines (or rather: of their mirror image on the other side of the centerline) fills a circle of radius $V$ with the exception of a straight notch, intersecting the periphery perpendicularly and penetrating less or more into the circle's interior. The notch is the image of the centerline. Sources of strengths $V a, V b$ and sinks of strengths $V c, V d$ represent the approaching and retreating jets of the widths $a, b, c, d$, respectively. Their position on the periphery is determined by the radius vector pointing into the flow direction at infinity of the particular jet. The arcs of the periphery between these singular points are the images of the three free streamlines bounding the flow in the physical plane.

For any flow ( $b / a, \beta, \gamma$ ), the values $c / a$ and $d / a$ are determined by the conservation laws, Eqs. (3.1), (3.2), but the depth of the hodograph notch, i.e. the variation of flow velocity along the centerline, is not yet known. It can eventually be evaluated from the condition that the image of the stagnation point (or points) must fit to the center of the hodograph. The mathematical formulation of this condition will not be given here, except for the limiting case of Eq. (4.1) below.

In flows of type $A$, illustrated by Fig. 2, the stagnation streamline splits the volume flux $V a$ of the (half) first jet into one part to be deflected into direction $\gamma$ and another one to continue along the centerline. If parameters $b / a, \beta$ are fixed, the partition of the first jet changes with $\gamma$ and, obviously, the deflected part increases with $\gamma$.

In flows of type $B$, Fig. 3, the first jet is completely deflected, only the secondary jet suffers a partition of its flux $V b$. Between the two stagnation points on the centerline, the flow direction along the centerline is inverted. Accordingly, the hodograph notch is longer than the radius V .

The limit between flow types $A$ and $B$ is formed by those exceptional flows featuring a single stagnation point on the centerline, where three streamlines intersect (under equal angles $\pi / 3$, instead of the common perpendicular intersection of two streamlines) as shown in Fig. 4. The hodograph notch penetrates exactly to the center of the hodograph circle. In consequence of this particular form, the hodograph transforms by a simple square root operation into the interior of a half circle, Fig. 5. The sources and sinks remain on the periphery, their polar angles are halved, the hodograph center, the image of the exceptional stagnation point, remains in the center point. The half circle may then be completed to a circle and to the full plane using the mirror principle. All singularities in that plane are then known. One component of the "flow velocity" induced in that plane by the sources and sinks vanishes in the center point because of the symmetry. The other component vanishes there too, and that center is the image of the stagnation point indeed, if

$$
\begin{equation*}
a-b \sin \frac{\beta}{2}-c \sin \frac{\gamma}{2}+d=0 \tag{4.1}
\end{equation*}
$$

This is the condition for the limiting case mentioned above. Eliminating $d$ and $c$ by Eqs. (3.1), (3.3), it can be written

$$
\begin{equation*}
b / a=\frac{2(1+\cos \gamma)}{(1+\cos \beta)\left(1+\sin \frac{\gamma}{2}\right)-(1+\cos \beta)\left(1-\sin \frac{\beta}{2}\right)} \tag{4.2}
\end{equation*}
$$

In the available domain of the ( $b / a, \beta, \gamma$ )-parameterspace, Eq. (4.2) is an interface which separates the flows of type $A$ from those of type $B$. That interface is indicated in Fig. 6.

It should be mentioned that flows of twofold symmetry: $\gamma=\beta, d=a, c=b$ are possible at any value $b / a$, if $\beta<\pi / 2$. They are of either type $A$ or $B$, as may be judged from Fig. 6. The most symmetric case, having straight stagnation streamlines, is $\gamma=\beta=\pi / 3, b / a=2$ which is included in the sketches of Fig. 1.

## 5. Conclusion

During the above investigation of an inviscid jet impingement problem, a three-dimensional ( $b / a, \beta, \gamma$ )-parameterspace was introduced and a compact domain of this space was found to contain steady flows, either of type $A$ or $B$, which seem to be reasonable solutions of the problem. The author is unable to imagine an experiment which could prescribe more than the two upstream parameters $(b / a, \beta)$. Since the experimental results should be reproducible, it must be expected that a unique selection $\gamma(b / a, \beta)$ will be effected by some unknown mechanism among the inviscid solutions, if a steady flow can be obtained at all.

Experiments with water jets in air are planned for the future. Therefore, and because no detailed computations have yet been made of the streamline patterns and hodographs which are shown here as schematic sketches only, this note has been termed a preliminary one.

## 6. References

1. A. Betz, Konforme Abbildung, 2. Aufl., Berlin Göttingen, Heidelberg 1964.
2. G. Birkhoff, and E. H. Zarantonello, Jets, wakes and cavities, New York 1957.
