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Statistical irreversibility of turbulence

E. A. NOVIKOV

THE PROBLEM of statistical irreversibility of incompressible turbulent flows is formulated. The group of incompressible (i.e. preserving the elementary volume) transformations of the region in itself is chosen as the phase space of flow. A statistical principle which is close to the ideas of Boltzmann and Gibbs is suggested. The calculations based on this principle explain the general irreversible tendencies in the spectra of turbulent flows: in the two-dimensional flows the energy is transferred from the smaller scale to larger scale motions, in the three-dimensional—from larger to small scale motions. The equilibrium probability distribution is suggested.

1. IN THREE-DIMENSIONAL turbulent flows of incompressible fluid the energy is transferred from large scale to the small scale motions. The influence of the viscosity is important only in the range of very small scales [1], when the spectrum of energy decreases sharply (exponentially) [2]. The main parameter determining the internal structure of a turbulent flow, the energy flux is not dependent on the viscosity [1]. Then we may suppose that this tendency of energy transfer in three-dimensional flows takes place in an ideal non-dissipative fluid as well.

At the same time, this motion of an ideal fluid is dynamically reversible. If at some instant we change the sign of the velocity field, then all the processes will be reversed. Therefore, the tendency of the energy transfer has a statistical character.

The problem in question becomes particularly interesting if we recall that in two-dimensional turbulent flows the reverse tendency takes place, i.e. the energy is transferred from the smaller scale to larger scale motions. Such a possibility was pointed out in some theoretical studies [3–6]. The recent numerical experiments on the simulation of the two-dimensional turbulence [7] and laboratory experiments with a conductive liquid in a strong magnetic field [8] confirm this effect. Finally, the data on the atmospheric general circulation also give evidence for the energy transfer from the smaller to larger scales which is sometimes called "the negative viscosity" [9].

Irreversible tendencies appear not only in the energy spectrum but in all probability distributions and they produce in particular the phenomenon of intermittency of turbulent flows (cf. [10], where you can find references on earlier works).

This set of problems might naturally be called the problem of statistical irreversibility of turbulent flows. Here we suggest a certain approach to the solution of this problem which is close to the ideas of Boltzmann and Gibbs. The main objective is to choose the phase space and the averaging procedure in such a way that a standard turbulent flow regime would correspond to an "equilibrium distribution" in the phase space.

2. For the construction of a general theory of turbulence, statistical characteristics of vorticity field are of major interest [11, 2, 12]. Velocity and pressure fields are expressed in an incompressible fluid through the vorticity field.

Let $\mathbf{\Omega}^0(r)$ be an initial vorticity, $\mathbf{x}^t(\mathbf{a})$—a position of liquid particles at the time t, $\mathbf{a}$—initial coordinates. In the two-dimensional flow of ideal fluid, the vorticity of a liquid particle is conserved,

$$\Omega^t(\mathbf{x}^t(\mathbf{a})) = \Omega^0(\mathbf{a}), \tag{1}$$

but in the three-dimensional flows, the vorticity lines are stretching as [13]:

$$\Omega^t_i(\mathbf{x}(\mathbf{a}) = \frac{\partial x^t_i(\mathbf{a})}{\partial a_e}\Omega^0_e(\mathbf{a}). \tag{2}$$

As the phase space of flow at the fixed $\mathbf{\Omega}^0$ we choose the group G of incompressible, i.e., preserving the elementary volume, transformations of the region into itself $\mathbf{x}(\mathbf{a})$. On this group (more accurately on its finite approximation—see p. 3) we give the probability distribution μ. We require μ to be invariant in relation to shifts on the group which is an analog of the Liouville condition of phase volume conservation.

The condition of the energy conservation determines an energy surface in the phase space. The area of the energy surface $\sigma(E)$ determined by an integral over μ does not depend on the time moment at which the initial vorticity field was taken, i.e. it is the integral of motion.

We assume that the fluid moves the more free the quicker the phase trajectory fills (with some spreading) the whole available region of the phase space. The question what this available region at a fixed $\mathbf{\Omega}^0$ is requires a special study. Basing on the known ideas of Boltzmann and Gibbs and on some intuitive considerations, we formulate the following statistical principle: the statistical characteristics of flow are readjusted on an average over a large time in such a way that the liquid may move with a maximum freedom. Starting from the averaging procedure over $\mathbf{x}(\mathbf{a})$ and taking into account (1) and (2), one may obtain statistical characteristics of the "equilibrium" state to which the system is approaching.

Here we shall make the averaging over the whole phase space. Therefore the energy of the equilibrium state corresponds to the mean energy calculated for the distribution $\sigma(E)$. The statistical characteristics of the equilibrium state must be invariant relative to transformation of $\mathbf{\Omega}^0$ by formulae (1) and (2) with an arbitrary $\mathbf{x}(\mathbf{a})$. In particular, when replacing $\mathbf{\Omega}^0$ by $\mathbf{\Omega}^t$ these formulae must be invariant. This corresponds to the exfoliation of the functional space of the flows on the so-called "sheets of fields with equal circulation" [14]. Together with averaging over $\mathbf{x}(\mathbf{a})$ which is an averaging over the sheet, one may perform a complementary averaging over $\mathbf{\Omega}^0$.

3. The calculations are always performed with the help of some finite-dimensional approximation. In this work, we choose the averaging procedure to be as simple as possible. A finite region of the fluid-filled space is divided into n cells with equal volumes although possibly different shapes. Consider various transferences of fluid from certain cells to other ones. These transferences (transpositions) form a non-commutative group G_n. We prescribe to each transference an equal probability $1/n!$ The probability distribution μ_n so constructed in this way is evidently invariant with respect to shifts over the group.

If the field $\mathbf{\Omega}^0$ is smooth, then choosing n sufficiently large may approximate with any accuracy the fluid motion over a finite time interval using the indicated transferences. In this sense, μ_n is invariant relating to the shifts along the phase trajectories of the roughened system. On the other hand, any discrete transference (e.g. transposition of two volumes) may be performed by continuous incompressible transformation constructing, if necessary, circulations for the exchange of fluids between neighbouring cells.

4. Let us consider two-dimensional flow. Expand the vorticity field into series using the system of eigenfunctions of the Laplace operator

$$\Omega^t(\mathbf{r}) = \sum_k \Omega_k^t S_k(\mathbf{r}), \quad \Delta S_k = -k^2 S_k, \tag{3}$$

which satisfy the condition of orthogonal normalization

$$\overline{S_k(\mathbf{r})\, S_{k'}(\mathbf{r})} = \delta_{kk'}. \tag{4}$$

Here and below the overbar denotes averaging over the volume occupied by the fluid, $\delta_{kk'}$ is Kronecker's symbol.

From (3) and (4), making use of the change of variables $\mathbf{r} = \mathbf{x}^t(\mathbf{a})$ and accounting for (1) and incompressibility condition, we get

$$\Omega_k^t = \overline{\Omega^t(\mathbf{r})\, S_k(\mathbf{r})} = \overline{\Omega^0(\mathbf{a})\, S_k\big(\mathbf{x}^t(\mathbf{a})\big)}.$$

The averaging over μ_n we shall denote by $\langle\ \rangle_n$. The vorticity spectrum is expressed by the value

$$\langle S_k\big(\mathbf{x}(\mathbf{a})\big)\, S_k\big(\mathbf{x}(\mathbf{b})\big)\rangle_n$$

and when determining this value it is convenient to consider separately the cases when the points a and b are in the same cell and in different cells. After simple calculations of combinatorial type one gets that

$$\langle \Omega_k^2 \rangle_n = \frac{n}{n-1}(\overline{\Omega^0})^2(\overline{S_k})^2 + \frac{1}{n-1}\big[\overline{(\omega_{(n)}^0)^2}\,\overline{S_{k,(n)}^2} - \overline{(\Omega_{(n)}^0)^2}\,\overline{(S_k)^2}\big], \tag{5}$$

where $\omega_{(n)}^0 = \Omega_{(n)}^0 - \overline{\Omega}$, index (n) shows that the preliminary averaging inside each cell was performed. The higher order moments are calculated in a similar way. In particular,

$$\langle \omega(\mathbf{r})\omega(\mathbf{r}')\,\omega(\mathbf{r}'')\rangle_n = \frac{n^2\overline{(\omega_{(n)}^0)^3}}{(n-1)(n-2)}\left[\delta_{\mathbf{r},\mathbf{r}',\mathbf{r}''}^{(0)} - \frac{1}{n}\big(\delta_{\mathbf{r},\mathbf{r}'}^{(n)} + \delta_{\mathbf{r},\mathbf{r}''}^{(n)} + \delta_{\mathbf{r}',\mathbf{r}''}^{(n)}\big) + \frac{2}{n^2}\right].$$

Here $\omega = \Omega - \langle\Omega\rangle$, $\langle\Omega\rangle = \overline{\Omega^0}$, symbols $\delta^{(n)}$ are equal to one when all the indicated points are in one cell and equal to zero in the opposite case.

Functions $\overline{S_k}$ entering (5) are quickly decreasing with increasing of K and values of $\overline{S_{k(n)}^2}$ are close to one due to (4). After filtration of large scale components which is denoted by tilda, we get for the mean square of vorticity gradient, vorticity and velocity that at $n \to \infty$

$$\langle(\nabla\tilde{\Omega})^2\rangle_n \to \infty,\ \langle(\tilde{\Omega})^2\rangle_n \to \text{const},\ \langle(\tilde{v})^2\rangle_n \to 0.$$

Thus in the equilibrium state of two-dimensional flow the energy is concentrated in the range of the small wave numbers and vorticity spreads into the range of large wave numbers (compare [5, 6]). The second statement follows also from the first and from an easily provable inequality

$$K_v K_\Omega \geqslant K_0^2,$$

where K_v and K_Ω are wave numbers averaged correspondingly over velocity and vorticity spectra, K_0 is a constant of motion.

5. In the case of three-dimensional flows, by virtue of (2), it is necessary to average quantities dependent on the derivatives of the transformation $\mathbf{x}(\mathbf{a})$. In the averaging procedure suggested in p. 3 there are discontinuous functions, therefore this procedure is inadmissible in the three-dimensional case (the invariance relative to the pointed out in p. 2 transformations of $\mathbf{\Omega}^0$ is violated here)(1). Nevertheless, the calculations were carried out by two methods: using approximations of derivatives by finite differences and using Fourier transformations. The vorticity spectrum is expressed then through the quantities of the type

$$\langle x_i(\mathbf{a}')x_i(\mathbf{b}')S_k(\mathbf{x}(\mathbf{a}))S_k(\mathbf{x}(\mathbf{b}))\rangle_n,$$

which are calculated taking into account the fact that some points may occur in the same cell. Results obtained by both methods prove to be close to each other.

We present here the basic fact: at $n \to \infty$

$$\langle(\tilde{\Omega})^2\rangle_n \to \infty, \qquad \langle(\tilde{v})^2\rangle_n \to \text{const}.$$

Therefore, according to these calculations, the equilibrium state of three-dimensional flows is characterized by the growth of vorticity and by the transfer of energy into the large wave-number range. The difference between the three-dimensional case and the two-dimensional one is naturally related to the effect of stretching of vortex lines.

6. The proposed statistical principle allows to explain the general irreversible tendencies in the spectra of two-dimensional and three-dimensional flows. In order to perform detailed calculations of statistical characteristics of particular flows on the base of this principle it is necessary to specify the averaging procedure over the phase space. In particular, one should take into account the conservation law or the balance of energy. The simplest presumption is that in case of isotropic flows the phase trajectory fills uniformly the region of intersection of the sheet, having equal circulation, with the energy surface. The condition of the maximum of entropy for the distribution on the sheet at a fixed mean energy leads to a factor $\exp\{-\beta H[\Omega^0, x)]\}$, where H is kinetic energy. This factor is to be introduced into the averaging procedure over $\mathbf{x}(\mathbf{a})$, described in p. 3, whereby a corresponding statistical sum will appear instead of $n!$

It is possible to make a generalization of the described approach for the case of compressible conducting fluid.

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(1) In order not to disrupt the vortex lines in the three-dimensional case at transferences of fluid it is useful to approximate $\boldsymbol{\omega}^0$ by superpositions of vortex rings. This question will be considered in detail in another paper.

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