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ON THE JACOBIAN SEXTIC EQUATION.

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THE Jacobian sextic equation has been discussed under the form

$$(z-a)^6 - 4a(z-a)^5 + 10b(z-a)^3 - 4c(z-a) + 5b^2 - 4ac = 0,$$

(see references at end of paper), but the connexion of this form with the general sextic equation has not, so far as I am aware, been considered. And although this is probably known, I do not find it to have been explicitly stated that the group of the equation is the positive half-group, or group of the 60 positive substitutions out of the 120 substitutions, which leave unaltered Serret's 6-valued function of six letters.

Invariantive Property of the Jacobian Sextic.

Taking $z-a$ as the variable, and comparing the equation with the general sextic equation

$$(\mathrm{a, b, c, d, e, f, g} \between z-a,\ 1)^6 = 0,$$

we have

$$\begin{array}{l} \mathrm{a,\quad b,\ c,\ \ d,\ \ e,\quad f\ ,\quad g} \\ = 1,\ -\tfrac{2}{3}a,\ 0,\ \tfrac{1}{2}b,\ 0,\ -\tfrac{2}{3}c,\ 5b^2-4ac; \end{array}$$

the Jacobian equation is thus an equation

$$(\mathrm{a, b, c, d, e, f, g} \between x,\ y)^6 = 0,$$

for which $\mathrm{c}=0$, $\mathrm{e}=0$, $\mathrm{ag}+9\mathrm{bf}-20\mathrm{d}^2=0$; but of course any equation, which can be by a linear transformation upon the variables brought into this form, may be regarded as a Jacobian equation.

Hence, using henceforward the small italic in place of the small roman letters, the Jacobian sextic may be regarded as an equation

$$(a, b, c, d, e, f, g \between x,\ y)^6 = 0,$$

linearly transformable into the form

$$(a, b, 0, d, 0, f, g \between x,\ y)^6 = 0,$$

where $ag + 9bf - 20d^2 = 0$. It is to be shown, that this implies a single relation between the four invariants A, B, C, and Δ of the sextic function.

I call to mind that the general sextic has five invariants A, B, C, D, E of the orders 2, 4, 6, 10, 15 respectively; the last of them E is not independent, but its square is equal to a rational and integral function of A, B, C, D; and instead of D, we consider the discriminant Δ which is an invariant of the same order 10. The values of A, B, C are given, Table Nos. 31, 34, and 35 of my Third Memoir on Quantics, *Phil. Trans.*, vol. CXLVI. (1856), pp. 627—647, [144]; those of D, Δ, E were obtained by Dr Salmon, see his *Higher Algebra*, second ed. 1866, where the values of A, B, C, D, Δ, E are all given; only those of A, B, C, Δ are reproduced in the third edition, 1876.

It may be remarked, that for the general form we have $A = ag - 6bf + 15ce - 10d^2$, and that B is the determinant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} :$$

C and Δ are complicated forms, the latter of them containing 246 terms. But writing $c = 0$, $e = 0$, there is a great reduction; we have

$A =$	$B =$	$C =$	$D =$
$ag \ + 1$	$ad^2g - 1$	$a^2d^2g^2 \ + 1$	$a^5g^5 \ + 1$
$bf \ - 6$	$b^2f^2 \ + 1$	,, $df^3 \ + 4$	$a^4bfg^4 \ - 30$
$d^2 \ - 10$	$bd^2f - 2$	$a\, bd^2fg + 12$	,, $d^2g^2 \ - 300$
	$d^4 \ + 1$	,, $d^4g \ - 20$	,, $df^2g \ - 2500$
		$a^0b^3dg^2 + 4$	,, $f^6 \ - 3125$
		,, $b^3f^3 \ + 8$	$a^3b^2f^2g^3 \ - 15$
		,, $b^2d^2f^2 - 24$	,, $bd^2fg^2 \ - 4800$
		,, $bd^4f + 24$	,, $bdf^4g \ - 7500$
		,, $d^6 \ - 8$	,, $d^4g^3 \ + 30000$
			,, $d^3f^3g \ + 50000$
			$a^2b^3dg^4 \ - 2500$
			,, $b^3f^3g^2 \ - 410$
			,, $b^2d^2f^2g^2 - 171300$
			,, $b^2df^5 \ - 240000$
			,, $bd^4fg^2 \ + 780000$
			,, $bd^3f^4 \ + 1200000$
			,, $d^6g^2 \ - 1000000$
			,, $d^5f^2 \ - 1600000$
			$a\, b^4dfg^3 \ - 7500$
			,, $b^4f^4g \ - 11520$
			,, $b^3d^3g^3 \ + 50000$
			,, $b^3d^2f^3g \ + 83200$
			$a^0b^6g^4 \ - 3125$
			,, $b^5df^2g^2 \ - 240000$
			,, $b^5f^5 \ - 331776$
			,, $b^4d^3fg^2 \ + 1200000$
			,, $b^4d^2f^4 \ + 1843200$
			,, $b^3d^5g^2 \ - 1600000$
			,, $b^3d^4f^3 \ - 2560000$

It is clear that these are all functions of ag, bf, d^2 and $a^2f^3+b^3g^2$, say of α, β, δ and ϕ. In fact, A and B are functions of α, β, δ; C contains two terms, coefficient 4, which are $=4\sqrt{\delta}\,.\,\phi$; Δ contains two terms

$$-3125\,(a^4f^6+b^6g^4),$$

which are $=-3125\,(\phi^2-2\alpha^2\beta^3)$; and also several pairs of terms, each which pair contains the factor ϕ. We thus have

$A=$	$B=$	$C=$	$\Delta=$
$\alpha+\ \ 1$	$\alpha\delta\ -\ 1$	$\alpha^2\delta\quad +\ 1$	$\alpha^5\quad +\quad 1$
$\beta-\ \ 6$	$\beta^2\ +\ 1$	$\alpha\beta\delta\quad +12$	$\alpha^4\beta\quad -\quad 30$
$\delta-10$	$\beta\delta\ -\ 2$	$\alpha\delta^2\quad -20$	$\alpha^4\delta\quad -\quad 300$
	$\delta^2\ +\ 1$	$\beta^3\quad +\ 8$	$\alpha^2\beta^3\quad +\quad 5840\ (=6250-410)$
		$\beta^2\delta\quad -24$	$\alpha^3\beta^3\quad -\quad 15$
		$\beta\delta^2\quad +24$	$\alpha^3\beta\delta\quad -\quad 4800$
		$\delta^3\quad -\ 8$	$\alpha^3\delta^2\quad +\quad 3000$
		$\phi\sqrt{\delta}\ +\ 4$	$\alpha^2\beta^2\delta\quad -\ 171300$
			$\alpha^2\beta\delta^2\quad +\ 780000$
			$\alpha^2\delta^3\quad -\ 1000000$
			$\alpha\beta^4\quad -\quad 11520$
			$\alpha\beta^3\delta\quad +\quad 83200$
			$\beta^5\quad -\ 331776$
			$\beta^4\delta\quad +\ 1843200$
			$\beta^3\delta^2\quad -\ 2560000$
			$\phi\sqrt{\delta}\,.\,\alpha^3\ -\quad 2500$
			,, $\alpha\beta\ -\quad 7500$
			,, $\alpha\delta\ +\quad 50000$
			,, $\beta^2\ -\ 240000$
			,, $\beta\delta\ +\ 1200000$
			,, $\delta^2\ -\ 1600000$
			$\phi^2.\quad -\quad 3125$

We have *ante*, the relation $\alpha+9\beta-20\delta=0$, and using this to eliminate α, we have A, B, C, Δ as functions of β, δ, ϕ (that is, of bf, d^2 and $a^2f^3+b^3g^2$). Effecting the substitution, we find the values of A, B, C without difficulty. As regards the value of Δ, this is

$$=-3125\phi^2+2\phi K\sqrt{\delta}+\text{terms without }\phi,$$

where

$$\begin{aligned} 2K = &- \quad 2500\,(\;\; 81\beta^2 - 360\beta\delta + 400\delta^2) \\ &- \quad 7500\,(-\;9\beta^2 + \;20\beta\delta \qquad\qquad) \\ &+ \;\; 50000\,(\qquad\quad -\;9\beta\delta + \;20\delta^2) \\ &- 240000\,(\;\;\beta^2 \qquad\qquad\qquad\quad) \\ &+ 1200000\,(\qquad\qquad \beta\delta \qquad\quad) \\ &- 1600000\,(\qquad\qquad\qquad\qquad \delta^2), \end{aligned}$$

or, reducing and dividing by 2,

$$K = -\,3125\,(60\beta^2 - 240\beta\delta + 256\delta^2).$$

The calculation of the terms without ϕ is much more laborious, but they come out

$$= -\,3125\,(60\beta^2 - 240\beta\delta + 256\delta^2)^2\,\delta.$$

Hence the value of Δ is

$$\begin{aligned} \Delta = -\,3125\,\{ &\quad \phi^2 \\ &+ 2\phi\;(60\beta^2 - 240\beta\delta + 256\delta^2)\sqrt{\delta} \\ &+ \quad (60\beta^2 - 240\beta\delta + 256\delta^2)^2\;\delta\}, \end{aligned}$$

say this is

$$\Delta = -\,3125h^2,$$

where

$$h = \phi + (60\beta^2 - 240\beta\delta + 256\delta^2)\sqrt{\delta},$$

that is,

$$= a^2f^3 + b^3g^2 + (60b^2df^2 - 240bd^3f + 256d^5).$$

The values of A, B, C, and the foregoing value of h then are

$A =$	$B =$	$C =$	$h =$
$\beta - 15$	$\beta^2 + 1$	$\beta^3 + 8$	$\beta^2\sqrt{\delta} + 60$
$\delta + 10$	$\beta\delta + 7$	$\beta^2\delta + 51$	$\beta\delta\sqrt{\delta} - 240$
	$\delta^2 - 19$	$\beta\delta^2 + 84$	$\delta^2\sqrt{\delta} + 256$
		$\delta^3 - 8$	$\phi + 1$
		$\phi\sqrt{\delta} + 4$	

We may, if we please, regard β, δ, ϕ as irrational invariants of the sextic, viz. A, B, C being rational and integral functions of β, δ, ϕ, we have conversely β, δ, ϕ irrational functions of A, B, C; and then the equation for h, say

$$\frac{1}{25\sqrt{5}}\sqrt{(-\Delta)} = \phi + \sqrt{\delta}\,(60\beta^2 - 240\beta\delta + 256\delta^2)$$

is the invariantive relation which characterises the Jacobian sextic.

The expression for Δ in terms of A, B, C, D is

$$\Delta = A^5 - 375A^3B - 625A^2C + 3125D,$$

and it was in the foregoing investigation proper to use Δ in place of D. But I annex the value of D for the case in question $b=0$, $f=0$; and also its value in terms of $\alpha, \beta, \delta, \phi$. These are

$D=$		$D=$	
a^4f^6	-1	ϕ^2	-1
a^3bdf^4g	-12	$\alpha^2\beta^3$	$+2$
,, d^4g^3	$+5$	$\alpha\beta^4$	-12
$a^2b^2d^2f^2g^2$	-90	β^5	-72
,, b^2df^5	-48	$\delta\,\alpha^2\beta^2$	-90
,, bd^4fg^2	$+246$	,, $\alpha\beta^3$	$+168$
,, bd^3f^4	$+480$	,, β^4	$+552$
,, d^6g^2	-258	$\delta^2\alpha^3$	$+5$
,, d^5f^3	-432	,, $\alpha^2\beta$	$+246$
$a\,b^4dfg^3$	-12	,, $\alpha\beta^2$	$+240$
,, b^4f^4g	-12	,, β^3	-976
,, $b^3d^2f^3g$	$+168$	$\delta^3\alpha^2$	-258
,, $b^2d^4f^2g$	$+240$	,, $\alpha\beta$	-168
,, bd^6fg	-168	,, β^2	$+336$
,, d^8g	-228	$\delta^4\alpha$	-228
$a^0b^6g^4$	-1	,, β	-408
,, $b^5df^2g^2$	-48	δ^5	-240
,, b^5f^5	-72	$\phi\,\sqrt{\delta}\,\alpha\beta$	-12
,, $b^4d^3fg^2$	$+480$	,, β^2	-48
,, $b^4d^2f^3$	$+552$	$\phi\delta\,\sqrt{\delta}\,\beta$	-480
,, $b^3d^5g^2$	-432	$\phi\delta^2\,\sqrt{\delta}$	-432
,, $b^3d^4f^3$	-976		
,, $b^2d^6f^2$	$+336$		
,, bd^8f	$+408$		
,, d^{10}	-248		

The Group of the Jacobian Sextic.

The solution of the Jacobian sextic equation depends upon that of a quintic; in fact, calling the roots $z_\infty, z_0, z_1, z_2, z_3, z_4$, then there exists a quintic having the roots

$$\sqrt{(z_\infty - z_0 \,.\, z_2 - z_3 \,.\, z_4 - z_1)},$$
$$\sqrt{(z_\infty - z_1 \,.\, z_3 - z_4 \,.\, z_0 - z_2)},$$
$$\sqrt{(z_\infty - z_2 \,.\, z_4 - z_0 \,.\, z_1 - z_3)},$$
$$\sqrt{(z_\infty - z_3 \,.\, z_0 - z_1 \,.\, z_2 - z_4)},$$
$$\sqrt{(z_\infty - z_4 \,.\, z_1 - z_2 \,.\, z_3 - z_0)},$$

the coefficients of which are rational functions of the coefficients a, b, d, f, g, and of the fourth root of the discriminant, i.e., $\sqrt{h}$. But the meaning of this has not, so far as I am aware, been noticed. Passing to the quintic whose roots are the squares of the foregoing values, i.e., $z_\infty - z_0 . z_2 - z_3 . z_4 - z_1$, &c., the coefficients are here rational functions of a, b, d, f, g and h; that is, they are rational functions of a, b, d, f, g. The symmetrical functions of these roots $z_\infty - z_0 . z_2 - z_3 . z_4 - z_1$, &c., are thus rational functions of the coefficients of the sextic; each such rational function is a 12-valued function of z_∞, z_0, z_1, z_2, z_3, z_4, invariable by all the substitutions of a group of 60 substitutions; and therefore also every like 12-valued function of the roots z_∞, z_0, z_1, z_2, z_3, z_4 is invariable by the substitutions of this group of 60; or, in other words, this group of 60 is the group of the Jacobian sextic equation.

I write for convenience, in this section only,

$$z_\infty,\ z_0,\ z_1,\ z_2,\ z_3,\ z_4 = f,\ a,\ b,\ c,\ d,\ e;$$

and writing further ab for shortness instead of $a-b$, &c., (so that of course $ba=-ab$), and putting B, C, D, E, $F = -ab . cd . ef$, $-ac . bf . de$, $ad . bc . ef$, $ae . bd . cf$, $af . be . cd$, then the five functions are B, C, D, E, F, and the group of 60 which leaves unaltered every symmetrical function of these functions is made up of the substitutions

1.				1
ab . ce,	*ab . df*,	*ce . df*,		15
ac . bf,	*ac . de*,	*bf . de*,		
ad . bc,	*ad . ef*,	*bc . ef*,		
ae . bd,	*ae . cf*,	*bd . cf*,		
af . be,	*af . cd*,	*be . cd*.		
abcde,	*acebd*,	*adbec*,	*aedbc*,	24
afbce,	*abefc*,	*acfeb*,	*aecbf*,	
abdef,	*adfbe*,	*aebfd*,	*afedb*,	
afced,	*acdfe*,	*aefdc*,	*adecf*,	
afdbc,	*adcfb*,	*abfcd*,	*acbdf*,	
bdcef,	*bcfde*,	*bedfc*,	*bfecd*.	
abc . dfe,	*acb . def*,			20
abd . cfe,	*adb . cef*,			
abe . cfd,	*aeb . cdf*,			
abf . ced,	*afb . cde*,			
acd . bef,	*adc . bfe*,			
ace . bfd,	*aec . bdf*,			
acf . bed,	*afc . bde*,			
ade . bfc,	*aed . bcf*,			
adf . bce,	*afd . bec*,			
aef . bcd,	*afe . bdc*,			60

where the symbols, *ab, abcde, abc*, &c. denote cyclical substitutions. It is easy to verify that each of these substitutions does in fact merely permute B, C, D, E, F; thus

$$\begin{array}{llllll} & B & C & D & E & F \\ abcde \text{ on} & -ab\,.\,ce\,.\,df, & -ac\,.\,bf\,.\,de, & ad\,.\,bc\,.\,ef, & ae\,.\,bd\,.\,cf, & af\,.\,be\,.\,cd \\ & =-bc\,.\,da\,.\,ef, & -bd\,.\,cf\,.\,ea, & be\,.\,cd\,.\,af, & ba\,.\,ce\,.\,df, & bf\,.\,ca\,.\,de \\ & =+ad\,.\,bc\,.\,ef, & ae\,.\,bd\,.\,cf, & af\,.\,be\,.\,cd, & -ab\,.\,ce\,.\,df, & -ac\,.\,bf\,.\,de \\ & = \quad D & E & F & B & C, \end{array}$$

which (expressed as a cyclical substitution) is $= BDFCE$, and so in other cases.

We may to the foregoing 60 substitutions join the 60 other substitutions:

cdef,	*cfed*,	30
bdfe,	*befd*,	
becf,	*bfce*,	
bcdf,	*bfdc*,	
bced,	*bdec*,	
aedf,	*afde*,	
acef,	*afec*,	
acfd,	*adfc*,	
adce,	*aecd*,	
abfe,	*aefb*,	
adbf,	*afbd*,	
abed,	*adeb*,	
abcf,	*afcb*,	
acbe,	*aebc*,	
abdc,	*acdb*.	
ab . cd . ef,	*ab . cf . de*,	10
ac . bd . ef,	*ac . be . df*,	
ad . be . cf,	*ad . bf . ce*,	
ae . bc . df,	*ae . bf . cd*,	
af . bc . de,	*af . bd . ce*.	
abcefd,	*adfecb*,	20
abfdec,	*acedfb*,	
abecdf,	*afdceb*,	
abdfce,	*aecfdb*,	
acfbde,	*aedbfc*,	
acbfed,	*adefbc*,	
acdebf,	*afbedc*,	
adbcfe,	*aefcbd*,	
adcbef,	*afebcd*,	
aebdcf,	*afcdbe*,	60

each of which changes B, C, D, E, F into a permutation of $-B, -C, -D, -E, -F$.

The 60 and 60 substitutions form together a group of 120 substitutions, which leave unaltered any even symmetrical function of B, C, D, E, F, or say any symmetrical function of B^2, C^2, D^2, E^2, F^2; such a function is thus a 6-valued function of a, b, c, d, e, f, viz. it is Serret's 6-valued function of 6 letters.

Transformation of the Jacobian Sextic into the Resolvent Sextic of a special quintic equation.

Starting from the Jacobian Sextic Equation

$$(a,\ b,\ 0,\ d,\ 0,\ f,\ g \between z,\ 1)^6 = 0,$$

$ag + 9bf - 20d^2 = 0$, I effect upon it the Tschirnhausen transformation

$$X = -az^3 - 6bz^2 - 10d;$$

which, it may be remarked, is a particular case of the Tschirnhausen-Hermite form

$$\begin{aligned} X\ (az+b)\,B + (az^2+6bz+5c)\,C + (az^3+6bz^2+15cz+10d)\,D \\ + (az^4+6bz^3+15cz^2+20dz+10e)\,E + (az^5+6bz^4+15cz^3+20dz^2+15ez+5f)\,F. \end{aligned}$$

Writing for convenience $Y = X + 10d$, $Z = X - 10d$, this is

$$\begin{array}{rrrrrrl} . & . & az^3 & +\,6bz^2 & - & . & +\,Y = 0, \end{array}$$

and we thence have

$$\begin{array}{rrrrrrl} . & az^4 & +\,6bz^3 & . & +\,Yz & . & = 0, \\ az^5 & +\,6bz^4 & . & +\,Yz^2 & . & . & = 0, \\ & & -\,Zz^3 & . & -\,6fz & -\,g & = 0, \\ & -\,Zz^4 & . & -\,6fz^2 & -\,gz & . & = 0, \\ -\,Zz^5 & . & -\,6fz^3 & -\,gz^2 & . & . & = 0, \end{array}$$

or, eliminating, the resulting equation is

$$\left|\begin{array}{cccccc} . & . & a, & 6b, & . & Y \\ . & a, & 6b, & . & Y, & . \\ a, & 6b, & . & Y, & . & . \\ . & . & Z, & ., & 6f, & g \\ . & Z, & . & 6f, & g, & . \\ Z, & . & 6f, & g, & . & . \end{array}\right| = 0.$$

The developed form is most easily obtained by expanding the determinant in the form

$$123\,.\,\overline{456} - 456\,.\,\overline{123},\ \&c.,$$

where the terms 123, &c., belong to the matrix

$$\left|\begin{array}{cccccc} . & . & a, & 6b, & . & Y \\ . & a, & 6b, & . & Y, & \\ a, & 6b, & . & Y, & & \end{array}\right|,$$

and those of $\overline{123}$, &c., to the matrix

$$\left|\begin{array}{cccccc} . & . & Z, & . & 6f, & g \\ . & Z, & . & 6f, & g, & . \\ Z, & . & 6f, & g, & . & . \end{array}\right| .$$

The several terms are

$123 \,.\, \overline{456}$	$+ - a^3$	$. - g^3$
$-124 \,.\, 356$	$- - 6a^2b$	$. - 6fg^2$
$+125 \,.\, 346$	$+ \quad 0$	$. - 36f^2g$
$-126 \,.\, 345$	$- - a^2Y$	$. \quad g^2Z - 216f^3$
$+134 \,.\, 256$	$+ - 36ab^2$	$. \quad 0$
$-135 \,.\, 246$	$- \quad a^2Y$	$. - g^2Z$
$+136 \,.\, 245$	$+ - 6ab\,Y$	$. - 6fgZ$
$+145 \,.\, 236$	$+ \quad 6ab\,Y$	$. - 6fgZ$
$-146 \,.\, 235$	$- \quad 0$	$. - 36f^2Z$
$+156 \,.\, 234$	$+ - a\,Y^2$	$. - gZ^2$
$-234 \,.\, 156$	$- - a^2Y - 216b^3$	$. \quad g^2Z$
$+235 \,.\, 146$	$+ \quad 6ab\,Y$	$. \quad 6fgZ$
$-236 \,.\, 145$	$- - 36b^2Y$	$. \quad 36f^2Z$
$-245 \,.\, 136$	$- \quad 36b^2Y$	$. \quad 0$
$+246 \,.\, 135$	$+ \quad aY^2$	$. \quad g\,Z^2$
$-256 \,.\, 134$	$- - 6b\,Y^2$	$. \quad 6fZ^2$
$+345 \,.\, 126$	$+ - a\,Y^2$	$. - g\,Z^2$
$-346 \,.\, 125$	$- \quad 6b\,Y^2$	$. - 6fZ^2$
$+356 \,.\, 124$	$+ \quad 0$	$. \quad 0$
$-456 \,.\, 123$	$- - Y^3$	$. \quad Z^3.$

Hence, collecting and reducing, the equation is

$$\begin{aligned} 0 = \; & Y^3Z^3 \,. \\ & + Y^2Z^2 \,.\, (3ag + 72bf) \\ & + YZ \;\;.\, (3a^2g^2 + 36agbf + 1296b^2f^2) \\ & + Y \quad .\;\; 216a^2f^3 \\ & + Z \quad . - 216b^3g^2 \\ & + \qquad\;\; a^3g^3 - 36a^2g^2bf, \end{aligned}$$

where Y, Z denote $X+10d$, $X-10d$ respectively, and consequently $YZ = X^2 - 100d^2$. Hence, writing as before α, β, δ, ϕ to denote ag, bf, d^2 and $a^2f^3 + b^3g^2$ respectively, the result finally is

(	1	0	$\alpha + 3$	0	$\alpha^2 + 3$	$a^2f^3 + 216$	$\phi\sqrt{\delta} + 2160$	$\big)(X, 1)^6 = 0,$
			$\beta + 72$		$\alpha\beta + 36$	$b^3g^2 - 216$	$\alpha^3 + 1$	
			$\delta - 300$		$\alpha\delta - 600$		$\alpha^2\beta - 36$	
					$\beta^2 + 1296$		$\alpha^2\delta - 30$	
					$\beta\delta - 14400$		$\alpha\beta\delta - 360$	
					$\delta^2 + 30000$		$\alpha\delta^2 + 30000$	
							$\beta^2\delta + 12960$	
							$\beta\delta^2 + 720000$	
							$\delta^3 - 1000000$	

where observe that the coefficient of the term in X is $216(a^2f^3 - b^3g^2)$, $= 216\sqrt{(\phi^2 - 4\alpha^2\beta^3)}$. We have as before $ag + 9bf - 20d^2 = 0$, that is, $\alpha + 9\beta - 20\delta = 0$; and using this equation to eliminate α, also in the constant term writing its value for ϕ in terms of h,

$$\phi = h + (-60\beta^2 + 240\beta\delta - 256\delta^2)\sqrt{\delta},$$

the new equation is

			$-5\times$		$5\times$		$5\times$	
(	1	0	$\beta - 9$	0	$\beta^2 - 243$	$-216\sqrt{\Lambda}$	$h\sqrt{\delta} + 432$	$\big)(X, 1)^6 = 0,$
			$\delta + 48$		$\beta\delta - 1872$		$\beta^3 - 729$	
					$\delta^2 + 3840$		$\beta^2\delta + 4184$	
							$\beta\delta^2 - 11520$	
							$\delta^3 + 8292$	

where

$$\Lambda = \{h + (-60\beta^2 + 240\beta\delta - 256\delta^2)\sqrt{\delta}\}^2 - 4(-9\beta + 20\delta)^2\beta^3$$

$$= h^2 + 2h\sqrt{\delta}\,.\left|\begin{array}{l} \beta^2 - 60 \\ \beta\delta + 240 \\ \delta^2 - 256 \end{array}\right| - 4(\beta - 4\delta)^3(9\beta - 16\delta)^2.$$

It is to be shown that this Tschirnhausen-transformation of the Jacobian sextic is, in fact, the resolvent sextic of the quintic equation

$$(\mathrm{a},\ 0,\ \mathrm{c},\ 0,\ \mathrm{e},\ \mathrm{f}\big)(x, 1)^5 = 0,$$

where

$$\mathrm{a} = 1, \quad \mathrm{c} = 2d, \quad \mathrm{e} = -9bf + 36d^2, \quad \mathrm{f}^2 = 216h.$$

I consider the general quintic $(a, b, c, d, e, f \between x, 1)^5 = 0$; taking the roots to be x_1, x_2, x_3, x_4, x_5, and writing

$$\phi_1 = 12345 - 24135,$$
$$\phi_2 = 13425 - 32145,$$
$$\phi_3 = 14235 - 43125,$$
$$\phi_4 = 21435 - 13245,$$
$$\phi_5 = 31245 - 14325,$$
$$\phi_6 = 41325 - 12435,$$

where 12345 is used to denote the function

$$= (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1)\sqrt{(20)},$$

(this numerical factor $\sqrt{(20)}$ being inserted for greater convenience), then the equation whose roots are $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6$, which equation may be regarded as the resolvent sextic of the given quintic equation, is

$a^6 \times$		$-5a^4 \times$	$5a^2 \times$	$-\sqrt{(\square)} \cdot a^2$	$+5$
(1	0	ae	$-2a^2df$	$+1$	$+1a^3cf^3$
		$-4bd$	$+3a^2e^2$		$-2a^2def$
		$+3c^2$	&c.		$+$ &c.

$\between \phi, 1)^6 = 0,$

$\square = a^4f^4 + \&c.$, the discriminant of the quintic: see p. 274* of my paper "On a new auxiliary equation in the theory of equations of the fifth order," *Phil. Trans.* t. CLI. (1861), pp. 263—276, [268].

I now write $b = 0$, $d = 0$, but, to avoid confusion again, write roman instead of italic letters, viz. I consider the resolvent sextic of the quintic equation

$$(\mathrm{a}, 0, \mathrm{c}, 0, \mathrm{e}, \mathrm{f} \between x, 1)^5.$$

Many of the terms thus vanish, and the equation assumes the form

$\mathrm{a}^6 \times$		$-5\mathrm{a}^4$	$5\mathrm{a}^2$	$-\mathrm{a}^2\sqrt{\square}$	$+5$
(1	0	$\mathrm{ae} + 1$	$\mathrm{a^2e^2} + 3$	$+1$	$\mathrm{a^3cf^2} + 1$
		$\mathrm{c^2} + 3$	$\mathrm{ac^2e} - 2$		$\mathrm{a^3e^3} + 1$
			$\mathrm{c^4} + 15$		$\mathrm{a^2c^2e^2} - 11$
					$\mathrm{ac^4e} + 35$
					$\mathrm{c^6} - 25$

$\between X, 1)^6 = 0,$

and then if, as before,

$$\mathrm{a} = 1,\ \mathrm{c} = 2d,\quad \mathrm{e} = -9bf + 36d^2,\ \mathrm{f}^2 = 216h,$$

or say

$$\mathrm{a} = 1,\ \mathrm{c} = 2\sqrt{\delta},\ \mathrm{e} = -9\beta + 36\delta,\ \mathrm{f}^2 = 216h,$$

* [This Collection, vol. IV., p. 321.]

this becomes identical with the foregoing Tschirnhausen-transformation equation; thus

$$\begin{aligned}\text{ae} + 3\text{c}^2 = -9\beta + 36\delta + 12\delta, &= \beta - 9\\ &\quad\ \delta + 48;\end{aligned}$$

and similarly

$$\begin{aligned}3\text{a}^2\text{e}^2 - 2\text{ac}^2\text{e} + 15\text{c}^4 = &\ \beta^2 + 243,\\ &\ \beta\delta - 1872,\\ &\ \delta^2 + 3840.\end{aligned}$$

So for the constant term, $+1\text{a}^3\text{cf}^2$ gives the term $432h\sqrt{\delta}$, and $+1\text{a}^3\text{e}^3$, &c., give the remaining terms $-729\beta^3$, &c. of the value in question.

It only remains to verify the equality of the coefficients of X,

$$216\sqrt{\Lambda} = \sqrt{\square} \text{ or } 46656\Lambda = \square.$$

Here $\square$, the discriminant of the quintic (a, 0, c, 0, e, f$\between x, 1)^5$, from the general form (see my Second Memoir on Quantics, [141], or Salmon's *Higher Algebra,* third edition, p. 209) putting therein b = 0, d = 0, is

$$\begin{aligned}\square = &\ \text{a}^4\text{f}^4 &+1,\\ &\ \text{a}^3\text{ce}^2\text{f}^2 &+160,\\ &\ \text{a}^3\text{e}^5 &+256,\\ &\ \text{a}^2\text{c}^3\text{ef}^2 &-1440,\\ &\ \text{a}^2\text{c}^2\text{e}^4 &-2560,\\ &\ \text{ac}^5\text{f}^2 &+3456,\\ &\ \text{ac}^4\text{e}^3 &+6400,\end{aligned}$$

and writing for a, c, e, f their values 1, $2\sqrt{\delta}$, $9(-\beta+4\delta)$, $216h$, the value becomes

$$\begin{aligned}\square = &\ (216)^2 \,.\, h^2 \,.\\ &+432h\sqrt{\delta} \quad . \quad 12960\,(\beta-4\delta)^2\\ &\qquad\qquad\qquad +34560\,(\beta-4\delta)\,\delta\\ &\qquad\qquad\qquad +55296\ \delta^2\\ &-256 \quad .\ 9^5\,.\ \ (\beta-4\delta)^5\\ &-10240 \ .\ 9^4\,.\ \ (\beta-4\delta)^4\,\delta\\ &-102400\,.\ 9^3\,.\ \ (\beta-4\delta)^3\,\delta^3.\end{aligned}$$

The whole divides by $(216)^2$, and we thus obtain

$$\begin{aligned}\Lambda = h^2 + 2h\sqrt{\delta}\,. \quad &60\ \ (-\beta+4\delta)^2\,. + (-\beta-4\delta)^3\,. \quad &324\,(\beta-4\delta)^2\\ &-240\,(-\beta+4\delta)\,\delta &+1440\,(\beta-4\delta)\,\delta\\ &+256 \quad \delta^2 &+1600\ \delta^2,\end{aligned}$$

which is, in fact, equal to the foregoing value of Λ.

The conclusion is that, starting from the Jacobian sextic

$$(a,\ b,\ 0,\ d,\ 0,\ f,\ g \between z,\ 1)^6 = 0,$$

where $ag + 9bf - 20d^2 = 0$, and effecting upon it the Tschirnhausen-transformation

$$X = -az^3 - 6bz^2 - 10d,$$

so as to obtain from it a sextic equation in X, this sextic equation in X is the resolvent sextic of the quintic equation

$$(1,\ 0,\ \mathrm{c},\ 0,\ \mathrm{e},\ \mathrm{f} \between x,\ 1)^5 = 0,$$

where

$$\mathrm{c} = 2d,\ \mathrm{e} = -9bf + 36d^2,\ \mathrm{f} = \sqrt{(216h)},$$

and, Δ being the discriminant of the Jacobian sextic, then

$$h = \frac{1}{5^2\sqrt{5}}\sqrt{(-\Delta)},\ = a^2f^3 + b^3g^2 + 60b^2df^2 - 240bd^3f + 256d^5.$$

As to the subject of the present paper, see in particular Brioschi, "Ueber die Auflösung der Gleichungen vom fünften Grade," *Math. Annalen*, t. XIII. (1878), pp. 109—160, and the third Appendix to his translation of my *Elliptic Functions*, Milan, 1880, each containing references to the earlier papers.