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ON THE GEODESIC CURVATURE OF A CURVE ON A SURFACE.

[From the *Proceedings of the London Mathematical Society*, vol. XII. (1881), pp. 110—117. Read April 14, 1881.]

THERE is contained in Liouville's Note II. to his edition of Monge's *Application de l'Analyse à la Géométrie* (Paris, 1850), see pp. 574 and 575, the following formula,

$$\begin{aligned}\frac{1}{\rho} &= -\frac{di}{ds} + \frac{1}{2G\sqrt{E}}\frac{dG}{du}\cos i - \frac{1}{2E\sqrt{G}}\frac{dE}{dv}\sin i,\\ &= -\frac{di}{ds} + \frac{\cos i}{\rho_2} + \frac{\sin i}{\rho_1},\end{aligned}$$

which gives the radius of geodesic curvature of a curve upon a surface when the position of a point on the surface is defined by the parameters u, v, belonging to a system of orthotomic curves; or, what is the same thing, such that

$$ds^2 = Edu^2 + Gdv^2.$$

Writing with Gauss p, q instead of u, v, I propose to obtain the corresponding formula in the general case where the parameters p, q are such that

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2.$$

I call to mind that, if PQ, PQ' are equal infinitesimal arcs on the given curve and on its tangent geodesic, then the radius of geodesic curvature ρ is, by definition, a length ρ such that $2\rho\,.\,QQ' = \overline{PQ}^2$. More generally, if the curves on the surface are any two curves which touch each other, then ρ as thus determined is the radius of relative curvature of the two curves.

The notation is that of the Memoir, "Disquisitiones generales circa superficies curvas" (1827), Gauss, *Werke*, t. III.; see also my paper "On geodesic lines, in particular those of a quadric surface," *Proc. Lond. Math. Society*, t. IV. (1872), pp. 191—211, [508]; and Salmon's *Solid Geometry*, 3rd ed., 1874, pp. 251 *et seq.* The coordinates (x, y, z) of a point on the surface are taken to be functions of two independent parameters p, q; and we then write

$$dx + \tfrac{1}{2}d^2x = a\,dp + a'dq + \tfrac{1}{2}(\alpha\,dp^2 + 2\alpha'\,dp\,dq + \alpha''\,dq^2),$$

$$dy + \tfrac{1}{2}d^2y = b\,dp + b'dq + \tfrac{1}{2}(\beta\,dp^2 + 2\beta'\,dp\,dq + \beta''\,dq^2),$$

$$dz + \tfrac{1}{2}d^2z = c\,dp + c'dq + \tfrac{1}{2}(\gamma\,dp^2 + 2\gamma'\,dp\,dq + \gamma''\,dq^2):$$

$$E,\ F,\ G = a^2+b^2+c^2,\quad aa'+bb'+cc',\quad a'^2+b'^2+c'^2;\quad V^2 = EG - F^2;$$

and therefore

$$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2,$$

where E, F, G are regarded as given functions of p and q.

To determine a curve on the surface, we establish a relation between the two parameters p, q, or, what is the same thing, take p, q to be functions of a single parameter θ; and we write as usual p', p'', q', etc., to denote the differential coefficients of p, q, etc., in regard to θ; we write also E_1, E_2, etc., to denote the differential coefficients $\dfrac{dE}{dp}, \dfrac{dE}{dq}$, etc. In the first instance, θ is taken to be an arbitrary parameter, but we afterwards take it to be the length s of the curve from a fixed point thereof.

First formula for the radius of relative curvature.

Consider any two curves touching at the point P, coordinates (x, y, z) which are regarded as given functions of (p, q); where (p, q) are for the one curve given functions, and for the other curve other given functions, of θ.

The coordinates of a consecutive point for the one curve are then

$$x + dx + \tfrac{1}{2}d^2x,\quad y + dy + \tfrac{1}{2}d^2y,\quad z + dz + \tfrac{1}{2}d^2z,$$

where

$$dp = p'd\theta + \tfrac{1}{2}p''d\theta^2,\quad dq = q'd\theta + \tfrac{1}{2}q''d\theta^2;$$

hence these coordinates are

$$\begin{array}{l} x + (ap' + a'q')\,d\theta + \tfrac{1}{2}(\alpha p'^2 + 2\alpha'p'q' + \alpha''q'^2)\,d\theta^2 + \tfrac{1}{2}(ap'' + a'q'')\,d\theta^2, \\ \vdots \end{array}$$

and for the other curve they are in like manner

$$\begin{array}{l} x + (ap' + a'q')\,d\theta + \tfrac{1}{2}(\alpha p'^2 + 2\alpha'p'q' + \alpha''q'^2)\,d\theta^2 + \tfrac{1}{2}(aP'' + a'Q'')\,d\theta^2, \\ \vdots \end{array}$$

the only difference being in the terms which contain the second differential coefficients, p'', q'' for the first curve, and P'', Q'' for the second curve. Hence the differences of the coordinates are

$$\tfrac{1}{2}\{a(p''-P'')+a'(q''-Q'')\}\,d\theta^2,\quad \tfrac{1}{2}\{b(p''-P'')+b'(q''-Q'')\}\,d\theta^2,$$

$$\tfrac{1}{2}\{c(p''-P'')+c'(q''-Q'')\}\,d\theta^2,$$

and consequently the distance QQ' of the two consecutive points Q, Q' is

$$=\tfrac{1}{2}\sqrt{(E,\ F,\ G\between p''-P'',\ q''-Q'')^2}\,d\theta^2.$$

The squared arc $\overline{PQ}^2$ is

$$=(E,\ F,\ G\between p',\ q')^2\,d\theta^2;$$

and hence, if as before $2\rho\,.\,QQ'=\overline{PQ}^2$, that is, $\dfrac{1}{\rho}=2QQ'\div\overline{PQ}^2$, then

$$\frac{1}{\rho}=\frac{\sqrt{(E,\ F,\ G\between p''-P'',\ q''-Q'')^2}}{(E,\ F,\ G\between p',\ q')^2},$$

the required formula for ρ.

Second formula for the radius of relative curvature.

We now take the variable θ to be the length s of the curve measured from a fixed point thereof, so that p', p'', etc. denote $\dfrac{dp}{ds}$, $\dfrac{d^2p}{ds^2}$, etc. We have therefore

$$1=(E,\ F,\ G\between p',\ q')^2,$$

and the formula becomes

$$\frac{1}{\rho}=\sqrt{(E,\ F,\ G\between p''-P'',\ q''-Q'')^2}.$$

But, differentiating the above equation as regards the curve, we find

$$0=2(E,\ F,\ G\between p',\ q'\between p'',\ q'')+(\dot{E},\ \dot{F},\ \dot{G}\between p',\ q')^2,$$

where $\dot{E}$, $\dot{F}$, $\dot{G}$ are used to denote the complete differential coefficients $E_1p'+E_2q'$, etc. And similarly, differentiating in regard to the tangent geodesic, we obtain

$$0=2(E,\ F,\ G\between p',\ q'\between P'',\ Q'')+(\dot{E},\ \dot{F},\ \dot{G}\between p',\ q')^2;$$

and hence, taking the difference of the two equations,

$$0=(E,\ F,\ G\between p',\ q'\between p''-P'',\ q''-Q'').$$

Hence, in the equation for $\dfrac{1}{\rho}$, the function under the radical sign may be written

$$(E,\ F,\ G\between p',\ q')^2\,.\,(E,\ F,\ G\between p''-P'',\ q''-Q'')^2-\{(E,\ F,\ G\between p',\ q'\between p''-P'',\ q''-Q'')\}^2,$$

which is identically

$$=(EG-F^2)\{p'(q''-Q'')-q'(p''-P'')\}^2.$$

Hence, extracting the square root, and for $\sqrt{EG-F^2}$ writing V, we have

$$\frac{1}{\rho}=V\{p'(q''-Q'')-q'(p''-P'')\},$$

or say

$$\frac{1}{\rho}=V(p'q''-q'p'')-V(p'Q''-q'P''),$$

which is the new formula for the radius of relative curvature.

Formula for the radius of geodesic curvature.

In the paper "On Geodesic Lines, etc.," p. 195, [vol. VIII. of this Collection, p. 160], writing $EG-F^2=V^2$, and P'', Q'' in place of p'', q'', the differential equation of the geodesic line is obtained in the form

$$\begin{aligned}&(Ep'+Fq')\{(2F_1-E_2)p'^2+2G_1p'q'+G_2q'^2\}\\ &-(Fp'+Gq')\{E_1p'^2+2E_2p'q'+(2F_2-G_1)q'^2\}\\ &+2V^2(p'Q''-q'P'')=0\,;\end{aligned}$$

or, denoting by Ω the first two lines of this equation, we have

$$V(p'Q''-q'P'')=-\frac{\frac{1}{2}}{V}\Omega.$$

The foregoing equation gives therefore, for the radius of geodesic curvature,

$$\frac{1}{\rho}=V(p'q''-p''q')+\frac{\frac{1}{2}}{V}\Omega,$$

which is an expression depending only upon p', q', the first differential coefficients (common to the curve and geodesic), and on p'', q'', the second differential coefficients belonging to the curve.

Observe that Ω is a cubic function of p', q': we have

$$\Omega=(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{D}\between p',\ q')^3,$$

the values of the coefficients being

$$\begin{aligned}\mathfrak{A}&=2EF_1-\ \ EE_2-\ \ FE_1,\\ \mathfrak{B}&=2EG_1+2FF_1-3FE_2-\ \ GE_1,\\ \mathfrak{C}&=\ \ EG_2+3FG_1-2FF_2-2GE_2,\\ \mathfrak{D}&=\ \ FG_2-2GF_2+\ \ GG_1.\end{aligned}$$

The Special Curves, $p=$ constant and $q=$ constant.

Consider the curve $p=\text{const.}$ For this curve $p'=0$, $p''=0$; therefore also $Gq'^2=1$, and, if R be the radius of geodesic curvature, then

$$\frac{1}{R}=\frac{\frac{1}{2}}{V}\mathfrak{D}q'^3,\quad =\frac{\frac{1}{2}}{V}\frac{\mathfrak{D}}{G\sqrt{G}}.$$

Similarly for the curve $q=\text{const.}$ Here $q'=0$, $q''=0$; therefore $Ep'^2=1$, and, if S be the radius of geodesic curvature, then

$$\frac{1}{S}=\frac{\frac{1}{2}}{V}\mathfrak{A}p'^3,\quad =\frac{\frac{1}{2}}{V}\frac{\mathfrak{A}}{E\sqrt{E}}.$$

These values of R and S are interesting for their own sakes, and they will be introduced into the expression for the radius of geodesic curvature ρ of the general curve.

Transformed Formula for the Radius of Geodesic Curvature.

From the values of $\frac{1}{R}$, $\frac{1}{S}$, we have

$$\frac{1}{\rho}-\frac{q'\sqrt{G}}{R}-\frac{p'\sqrt{E}}{S}=V(p'q''-p''q')+\frac{\frac{1}{2}}{V}\left\{\Omega-\frac{\mathfrak{A}}{E}p'-\frac{\mathfrak{D}}{G}q'\right\},$$

where the term in $\{\ \}$ is

$$=\mathfrak{A}p'^3-\frac{\mathfrak{A}}{E}p'+\mathfrak{B}p'^2q'+\mathfrak{C}p'q'^2+\mathfrak{D}q'^3-\frac{\mathfrak{D}}{G}q'.$$

The terms in $\mathfrak{A}$ are

$$=-\frac{\mathfrak{A}}{E}p'(1-Ep'^2),\quad =-\frac{\mathfrak{A}}{E}p'(2Fp'q'+Gq'^2),$$

and those in $\mathfrak{D}$ are

$$=-\frac{\mathfrak{D}}{G}q'(1-Gq'^2),\quad =-\frac{\mathfrak{D}}{G}q'(Ep'^2+2Fp'q').$$

Hence the whole expression contains the factor $p'q'$, and is, in fact,

$$=p'q'\left\{p'\left(\mathfrak{B}-\frac{2\mathfrak{A}F}{E}-\frac{\mathfrak{D}E}{G}\right)+q'\left(\mathfrak{B}-\frac{\mathfrak{A}G}{E}-\frac{2\mathfrak{D}F}{G}\right)\right\};$$

or substituting for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ their values, this is

$$\begin{aligned}=p'q'\Big\{&p'\left(-GE_1+EG_1+\frac{2F^2E_1}{E}-2FF_1-FE_2+2EF_2-\frac{EFG_2}{G}\right)\\&+q'\left(-GE_2+EG_2-\frac{2F^2G_2}{G}+2FF_2+FG_1-2GF_1+\frac{GFE_1}{E}\right)\Big\},\end{aligned}$$

say this is

$$=p'q'(Lp'+Mq');$$

and the formula thus is

$$\frac{1}{\rho}-\frac{q'\sqrt{G}}{R}-\frac{p'\sqrt{E}}{S}=V(p'q''-p''q')+\frac{\frac{1}{2}}{V}p'q'(Lp'+Mq').$$

Taking ϕ, θ to be the inclination of the curve to the curves $q=\text{const.}$, $p=\text{const.}$, respectively, and $\omega\,(=\phi+\theta)$ the inclination of these two curves to each other, then

$$\cos\phi=\frac{Fp'+Gq'}{\sqrt{G}},\quad \cos\theta=\frac{Ep'+Fq'}{\sqrt{E}},\quad \cos\omega=\frac{F}{\sqrt{EG}},$$

$$\sin\phi=\frac{Vp'}{\sqrt{G}},\qquad \sin\theta=\frac{Vq'}{\sqrt{E}},\qquad \sin\omega=\frac{V}{\sqrt{EG}};$$

hence $\dfrac{\sin\phi}{\sin\omega}=p'\sqrt{E}$, $\dfrac{\sin\theta}{\sin\omega}=q'\sqrt{G}$, and the formula may also be written

$$\frac{1}{\rho}-\frac{\sin\theta}{\sin\omega}\frac{1}{R}-\frac{\sin\phi}{\sin\omega}\frac{1}{S}=V(p'q''-p''q')+\frac{\frac{1}{2}}{V}p'q'(Lp'+Mq').$$

The Orthotomic Case $F=0$, or $ds^2=Edp^2+Gdq^2$.

The formula becomes in this case much more simple. We have

$$1=Ep'^2+Gq'^2,\quad V=\sqrt{EG},\quad \omega=90^\circ,\quad \sin\theta=\cos\phi;$$

and the term $Lp'+Mq'$ becomes $=E\dot{G}-\dot{E}G$, if, as before, $\dot{E}$, $\dot{G}$ denote the complete differential coefficients $E_1p'+E_2q'$ and $G_1p'+G_2q'$. The formula then is

$$\frac{1}{\rho}-\frac{\cos\phi}{R}-\frac{\sin\phi}{S}=V(p'q''-p''q')+\frac{\frac{1}{2}}{V}(E\dot{G}-\dot{E}G),$$

where the values $\dfrac{1}{R}$ and $\dfrac{1}{S}$ are now $=\dfrac{\frac{1}{2}G_1}{G\sqrt{E}}$ and $\dfrac{-\frac{1}{2}E_2}{E\sqrt{G}}$, respectively. But we have moreover $\phi=\tan^{-1}\dfrac{p'\sqrt{E}}{q'\sqrt{G}}$, and thence

$$\phi'=q'\sqrt{G}\left(p''\sqrt{E}+\frac{\frac{1}{2}p'\dot{E}}{\sqrt{E}}\right)-p'\sqrt{E}\left(q''\sqrt{G}+\frac{\frac{1}{2}q'\dot{G}}{\sqrt{G}}\right),$$

$$=-V(p'q''-p''q')-\frac{\frac{1}{2}}{V}p'q'(E\dot{G}-\dot{E}G);$$

or the formula finally is

$$\frac{1}{\rho}-\frac{\cos\phi}{R}-\frac{\sin\phi}{S}+\phi'=0,$$

which is Liouville's formula referred to at the beginning of the present paper. It will be recollected that ϕ' is the differential coefficient $\dfrac{d\phi}{ds}$ with respect to the arc s of the curve.

Addition.—Since the foregoing paper was written, I have succeeded in obtaining a like interpretation of the term

$$V(p'q''-p''q')+\frac{\frac{1}{2}}{V}p'q'(Lp'+Mq'),$$

which belongs to the general case. I find that these terms are, in fact, $=-\dot{\phi}+\omega_1 p'$; or, what is the same thing (since $\omega=\phi+\theta$ and therefore $\omega_1 p'+\omega_2 q'=\dot{\phi}+\dot{\theta}$), are $=\dot{\theta}-\omega_2 q'$. It will be recollected that ϕ is the inclination of the curve to the curve $q=c$, which passes through a given point of the curve, $\dot{\phi}$ is the variation of ϕ corresponding to the passage to the consecutive point of the curve, viz., $\phi+\dot{\phi}ds$ is the inclination at this consecutive point to the curve $q=c+dc$, which passes through the consecutive point; ω is the inclination to each other of the curves $p=b$, $q=c$, which pass through the given point of the curve, ω_1 the variation corresponding to the passage along the curve $q=c$, viz., $\omega+\omega_1 ds$ is the inclination to each other of the curves $p=b+db$, $q=c$; and the like as regards $\dot{\theta}$ and ω_2.

For the demonstration, we have, as above,

$$\phi=\tan^{-1}\frac{Vp'}{Fp'+Gq'},\quad \omega=\tan^{-1}\frac{V}{F},$$

where

$$V=\sqrt{EG-F^2};$$

and moreover $Ep'^2+2Fp'q'+Gq'^2=1$. In virtue of this last equation,

$$V^2p'^2+(Fp'+Gq')^2=G;$$

and we have

$$\dot{\phi}=-V(p'q''-p''q')+\frac{1}{G}\square,$$

where

$$\square=(Fp'+Gq')\,p'\dot{V}-Vp'(\dot{F}p'+\dot{G}q');$$

or, since $V^2=EG-F^2$, and thence $2V\dot{V}=G\dot{E}-2F\dot{F}+E\dot{G}$, we have

$$\square=\frac{\frac{1}{2}p'}{V}\{(Fp'+Gq')(G\dot{E}-2F\dot{F}+E\dot{G})-2(EG-F^2)(\dot{F}p'+\dot{G}q')\}.$$

Substituting herein for $\dot{E}$, $\dot{F}$, $\dot{G}$ their values $E_1p'+E_2q'$, $F_1p'+F_2q'$, $G_1p'+G_2q'$, the term in $\{\ \}$ becomes

$$=Ip'^2+Jp'q'+Kq'^2,$$

where

$$\begin{aligned}I&=FGE_1-2EGF_1+EFG_1,\\ J&=G^2E_1-2FGF_1+(-EG+2F^2)\,G_1+FGE_2-2EGF_2+EFG_2,\\ K&=G^2E_2-2FGF_2+(-EG+2F^2)\,G_2.\end{aligned}$$

But from the equation $\omega=\tan^{-1}\frac{V}{F}$, differentiating in regard to p, we obtain

$$\omega_1=\frac{\frac{1}{2}}{EGV}(FG\dot{E}-2EG\dot{F}+EF\dot{G})=\frac{\frac{1}{2}}{EGV}I;$$

or, for p writing

$$p'(Ep'^2+2Fp'q'+Gq'^2),\quad =Ep'\left(p'^2+2\frac{F}{E}p'q'+\frac{G}{E}q'^2\right),$$

we have

$$\dot{\phi}-\omega_1 p'=-V(p'q''-p''q')+\frac{\frac{1}{2}p'}{GV}(Ip'^2+Jp'q'+Kq'^2)$$
$$-\frac{\frac{1}{2}p'}{GV}I\left(p'^2+2\frac{F}{E}p'q'+\frac{G}{E}q'^2\right).$$

The terms in p'^3 destroy each other, and the form thus is

$$\dot{\phi}-\omega_1 p'=-V(p'q''-p''q')-\frac{\frac{1}{2}}{V}p'q'(Lp'+Mq'),$$

where

$$L=-\frac{J}{G}+\frac{2IF}{GE},$$

$$M=-\frac{K}{G}+\frac{I}{E};$$

and, upon substituting herein for I, J, K their values, we find

$$L=-GE_1+EG_1+\frac{2F^2E_1}{F}-2FF_1-FE_2+2EF_2-\frac{EFG_2}{G},$$

$$M=-GE_2+EG_2-\frac{2F^2G_2}{G}+2FF_2+FG_1-2GF_1+\frac{GFE_1}{E};$$

viz., these are the values denoted above by the same letters L, M. The final result thus is

$$\frac{1}{\rho}-\frac{q'\sqrt{G}}{R}-\frac{p'\sqrt{E}}{S}=-\dot{\phi}+\omega_1 p',$$
$$=\quad\dot{\theta}-\omega_2 q',$$

where the meanings of the symbols have been already explained. A formula substantially equivalent to this, but in a different (and scarcely properly explained) notation, is given, Aoust, "Théorie des coordonnées curvilignes quelconques," *Annali di Matem.*, t. II. (1868), pp. 39—64; and I was, in fact, led thereby to the foregoing further investigation.

As to the definition of the radius of geodesic curvature, I remark that, for a curve on a given surface, if PQ be an infinitesimal arc of the curve, then if from Q we let fall the perpendicular QM on the tangent plane at P (the point M being thus a point on the tangent PT of the curve), and if from M, in the tangent plane and at right angles to the tangent, we draw MN to meet the osculating plane of the curve in N, then MN is in fact equal to the infinitesimal arc QQ' mentioned near the beginning of the present paper, and the radius of geodesic curvature ρ is thus a length such that $2\rho\,.\,MN=\overline{PQ}^2$.