DYNAMIC STABILITY ANALYSIS IN NON-LOCAL FRACTIONAL THERMODYNAMICS

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1. Introduction

In material instability problems, spatial non-locality plays an important role in nonlinear bifurcation (post-bifurcation) investigations. By using dynamical systems, stability analysis of thermo-mechanical continua can easily be studied, even when fractional derivatives are used. Such case may be obtained, when non-locality is described by a generalized, fractional strain. In constitutive formulation, two types of constitutive equations are used, thermodynamical constitutive equations should also be added to the classical (mechanical) constitutive equations. In such a way, a closed system of equations is obtained to determine the motion of the thermo-mechanical continuum.

2. Fractional thermo-mechanics

First, the basic equations for a solid body with thermal stresses should be formed. In addition to the basic equations of continuum mechanics, heat propagation should also be taken into account by using Vernotte-Cattaneo equation instead of Fourier law. The set of basic equations consists of the kinematic equation

$$\dot{\varepsilon} = \frac{\partial^\alpha v}{\partial x^\alpha}, \tag{1}$$

the equation of motion

$$\dot{v} = \frac{1}{\rho}\frac{\partial \sigma}{\partial x}, \tag{2}$$

and the constitutive equation, which in rate form reads

$$\dot{\sigma} = B(\dot{\varepsilon} - \theta\dot{\vartheta}) + \chi\dot{h}. \tag{3}$$

In equations (1), (2), and (3) the notations are: strain (for uniaxial small deformations) ε, velocity v, space coordinate x, mass density ρ, temperature ϑ. Overdot denotes derivative with respect to time t, B is the tangent stiffness, while χ is a material constant. In (1) a generalized strain is used, where $0 < \alpha < 1$ denotes the order of the fractional derivative. Such generalization of strain is generally used in fractional continuum mechanics. Symbol $\frac{\partial^\alpha}{\partial x^\alpha}$ refers to all kinds of symmetric fractional derivatives. For example, it may denote symmetric Caputo's derivative from Atanackovic [1], where left and right Caputo derivatives are defined:

$${}^{C}D^{\alpha}_{a+}u(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(\int_a^x \frac{u(\xi)-u(a)}{(x-\xi)^\alpha}d\xi\right)$$

and

$${}^{C}D^{\alpha}_{L-}u(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(\int_x^L \frac{u(\xi)-u(a)}{(\xi-x)^\alpha}d\xi\right).$$

Then a generalized strain is defined by the symmetric fractional derivative of the displacement field

$$\varepsilon = \frac{1}{2}\left({}^{C}D^{\alpha}_{a+}u(x) - {}^{C}D^{\alpha}_{L-}u(x)\right).$$

Several types of definitions are used for such derivatives [3], [4]. All of them use a fractional integral operator, which adds non-local effects to the formulation. A detailed overview of symmetric fractional derivatives applied in non-local mechanics can be found in [2].

In the following, symmetric fractional derivative will not be specified, a generalized formula is used instead. Instead of Fourier law, heat conduction is given by the Vernotte-Cattaneo equation

$$\tau\dot{h} + a\frac{\partial}{\partial x}\vartheta + h = 0, \tag{4}$$

where the relaxation time of heat flux is denoted by τ, the heat flux by h, and heat conductivity by a. The reason of such selection is that Fourier heat conduction equation results infinite wave speeds, which is a non-generic dynamic behavior. For the constitutive variables $\varepsilon, \sigma, \vartheta$, and h, two types of constitutive equations are given. The one in form (3) could be referred as mechanical constitutive equation, while the other

$$\Theta_1\dot{\sigma} + \Theta_2\dot{\varepsilon} + \Theta_3\dot{h} = \dot{\vartheta} \tag{5}$$

may be called the thermodynamic constitutive equation [9], with material constants $\Theta_1, \Theta_2, \Theta_3$. From (4)

$$\dot{h} = -\frac{a}{\tau}\frac{\partial}{\partial x}\vartheta - \frac{h}{\tau}. \tag{6}$$

By substituting (1) and (6) into the mechanical and thermodynamic constitutive equations (3) and (5), we have

$$\dot{\sigma} = c_1\frac{\partial^\alpha v}{\partial x^\alpha} + c_2\,\frac{a}{\tau}\frac{\partial}{\partial x}\vartheta + c_2\frac{h}{\tau}\ , \tag{7}$$

$$\dot{\vartheta} = d_1\frac{\partial^\alpha v}{\partial x^\alpha} - d_2\frac{a}{\tau}\frac{\partial}{\partial x}\vartheta - d_2\frac{h}{\tau}\ . \tag{8}$$

Now equations (2), (6), (7), and (8) for variables v, σ, ϑ, and h can be used to describe the motion of the thermodynamic continuum.

3. Stability and bifurcaton investigation

By using the system of equations presented in the previous part, a dynamical system can be defined and the requirements of generic static and dynamic bifurcations can be studied similarly to [5]. Unfortunately, in the general case, the evaluation of such conditions is a difficult problem and simplifications are necessary. When the investigation is restricted to homogeneous periodic perturbations, general necessary conditions are formulated for both static and dynamic bifurcations. For the conventional setting (small deformations, linearized constitutive equations, Vernotte-Cattaneo equation), no generic static bifurcations are found. For dynamic bifurcation, there are possibilities to have generic behavior. A general formula is derived for such case. Moreover, having done a few simplifying restrictions, conditions are presented for the material constants of the constitutive equations to ensure generic dynamic bifurcation.

References

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