960.

ON THE CIRCLE OF CURVATURE AT ANY POINT OF AN ELLIPSE.

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LET
$$u=\frac{x}{a},\quad v=\frac{y}{b},\quad u^2+v^2=1.$$

The equation of the circle of curvature at the point (x, y) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is
$$X^2+Y^2+2\frac{a^2-b^2}{ab}(-bXu^3+aYv^3)+u^2(a^2-2b^2)+v^2(b^2-2a^2)=0.$$

Write $X=a\xi$, $Y=b\eta$, then this becomes
$$a^2\xi^2+b^2\eta^2+2(a^2-b^2)(-\xi u^3+\eta v^3)+u^2(a^2-2b^2)+v^2(b^2-2a^2)=0.$$

To find where this meets the ellipse, we must write $\xi^2+\eta^2=1$; eliminating η, we have
$$a^2\xi^2+b^2(1-\xi^2)-2(a^2-b^2)\xi u^3+u^2(a^2-2b^2)+v^2(b^2-2a^2)+2(a^2-b^2)v^3\sqrt{(1-\xi^2)}=0,$$
or putting for shortness
$$a^2-b^2=A,\quad u^2(a^2-2b^2)+v^2(b^2-2a^2)=B,$$
the equation for ξ is
$$A\xi^2-2A\xi u^3+b^2+B+2Av^3\sqrt{(1-\xi^2)}=0,$$
but
$$b^2+B=b^2(u^2+v^2)+u^2(a^2-2b^2)+v^2(b^2-2a^2)=u^2(a^2-b^2)+v^2(2b^2-2a^2)=A(u^2-2v^2),$$
viz.
$$\xi^2-2\xi u^3+u^2-2v^2+2v^3\sqrt{(1-\xi^2)}=0,$$
that is,
$$(\xi^2-2u^3\xi+u^2-2v^2)^2-4v^6(1-\xi^2)=0,$$
which is without difficulty reduced to the form
$$(\xi-u)^3\{\xi-(u^3-3uv^2)\}=0,$$
that is,
$$\xi=u^3-3uv^2,$$
and hence
$$\eta=v^3-3vu^2,$$
viz. writing $u, v=\cos\theta, \sin\theta$, then we have
$$\xi=\cos^3\theta-3\cos\theta\sin^2\theta=\cos 3\theta,$$
$$\eta=\sin^3\theta-3\sin\theta\cos^2\theta=-\sin 3\theta,$$
or the circle of curvature at $(a\cos\theta, b\sin\theta)$ cuts the ellipse in $(a\cos 3\theta, -b\sin 3\theta)$, as is known.