955.

THE NUMERICAL VALUE OF Πi, $=\Gamma(1+i)$.

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I do not know whether the numerical value of Πx for an imaginary value of x has ever been calculated; and I wish to calculate it for a simple case $x=i$.

We have

$$\begin{aligned}\frac{1}{\Pi z} = &\left(1+\frac{z}{1}\right)\\ &\left(1+\frac{z}{2}\right)e^{z\,\mathrm{hl}\,\frac{1}{2}}\\ &\left(1+\frac{z}{3}\right)e^{z\,\mathrm{hl}\,\frac{2}{3}}\\ &\quad\vdots\\ &\left(1+\frac{z}{s}\right)e^{z\,\mathrm{hl}\,\frac{s-1}{s}}\\ &\quad\vdots\end{aligned}$$

where hl denotes the hyperbolic logarithm. Hence, in particular, when $z=i$, we have

$$\begin{aligned}\frac{1}{\Pi i} = &\,1+\frac{i}{1}\\ &\,1+\frac{i}{2}\,.\cos\mathrm{hl}\,\tfrac{1}{2}+i\sin\mathrm{hl}\,\tfrac{1}{2}.\\ &\,1+\frac{i}{3}\,.\cos\mathrm{hl}\,\tfrac{2}{3}+i\sin\mathrm{hl}\,\tfrac{2}{3}.\\ &\,1+\frac{i}{4}\,.\cos\mathrm{hl}\,\tfrac{3}{4}+i\sin\mathrm{hl}\,\tfrac{3}{4}.\\ &\quad\vdots\\ = &\,\sqrt{(1+1)}\,.\cos\theta_1+i\sin\theta_1\,.\cos\phi_1-i\sin\phi_1.\\ &\,\sqrt{(1+\tfrac{1}{4})}\,.\cos\theta_2+i\sin\theta_2\,.\cos\phi_2-i\sin\phi_2.\\ &\,\sqrt{(1+\tfrac{1}{9})}\,.\cos\theta_3+i\sin\theta_3\,.\cos\phi_3-i\sin\phi_3.\\ &\quad\vdots\end{aligned}$$

($\phi_1 = 0$, and in the subsequent terms the imaginary part is taken with a negative sign in order to obtain positive values for ϕ_2, ϕ_3, &c.), $= \Omega(\cos\Theta + i\sin\Theta)$, if Ω be the modulus and Θ the sum

$$(\theta_1 - \phi_1) + (\theta_2 - \phi_2) + (\theta_3 - \phi_3) + \ldots.$$

We have

$$\Omega_1 = \sqrt{(1+1)} \cdot \sqrt{(1+\tfrac{1}{4})} \cdot \sqrt{(1+\tfrac{1}{9})} \ldots,$$

which may be calculated directly. The value of Ω admits, however, of a finite expression, viz. we have

$$\Omega^2 = \frac{1}{\Pi i\, \Pi(-i)} = \frac{\sin \pi i}{\pi i} = \frac{e^{\pi} - e^{-\pi}}{2\pi},$$

the approximate numerical value is $\Omega = 1{\cdot}9173$, viz. we have

$$e^{\pi} - e^{-\pi} = 23{\cdot}141 - {\cdot}043 = 23{\cdot}098: \ \log = 1{\cdot}3635744, \quad -\log 2\pi = \bar{1}{\cdot}201819,$$

whence

$$\log \Omega^2 = {\cdot}5653935, \quad \log \Omega = {\cdot}2826967, \text{ or } \Omega = 1{\cdot}9173.$$

We have

$$\tan\theta_1 = 1, \quad \tan\theta_2 = \tfrac{1}{2}, \quad \tan\theta_3 = \tfrac{1}{3}, \text{ \&c.},$$

also

$$\phi_1 = 0, \quad \phi_2 = \frac{180^\circ}{M\pi}\log\tfrac{1}{2}, \quad \phi_3 = \frac{180^\circ}{M\pi}\log\tfrac{2}{3}, \text{ \&c.},$$

where M is the modulus for the Briggian logarithms,

$$M = {\cdot}4342944 \log \quad = \bar{1}{\cdot}6377843,$$
$$\pi = 3{\cdot}1415926 \quad " \quad = {\cdot}4971499,$$
$$180 \quad " \quad = 2{\cdot}2552755,$$

whence

$$\log\frac{180}{M\pi} = 2{\cdot}1203383, \quad \frac{180^\circ}{M\pi} = 131^\circ{\cdot}9284.$$

We hence calculate the succession of values of θ and ϕ as follows:

θ	tan	arc
1	1	45°
2	·5	26 34′
3	·3333333	18 26
4	·25	14 2
5	·2	11 19
6	·1666666	9 28
7	·1428571	8 8
8	·125	7 8
9	·1111111	6 20
10	·1	5 43

	ϕ	$131^\circ{\cdot}93 \times$				$\theta - \phi =$	
1		$= 0$			1	45°	
2	log 1/2	$= {\cdot}3010300 =$	39°	$43'$	2	-13°	$9'$
3	2/3	·1760913	23	14	3	4	48
4	3/4	·1249387	16	29	4	2	27
5	4/5	·0969100	12	47	5	1	28
6	5/6	·0791813	10	26	6	0	58
7	6/7	·0669467	8	50	7	0	42
8	7/8	·0579920	7	39	8	0	31
9	8/9	·0511525	6	44	9	0	24
10	9/10	·0457575	6	2	10	0	19

The sum of all the negative arcs $\theta_2 - \phi_2$, $\theta_3 - \phi_3$, ... as far as calculated, that is, up to $\theta_{10} - \phi_{10}$ is $= 24^\circ\, 46'$, or, writing x for the sum of the remaining arcs $\theta_{11} - \phi_{11}$ to infinity, we have

$$\frac{1}{\Pi i} = 1{\cdot}9173\,(\cos\Theta + i\sin\Theta),$$

where

$$\Theta = 45^\circ - 24^\circ\, 46' - x, \quad = 20^\circ\, 14' - x.$$

It would not be difficult to calculate a limit to the value of x.