755.

ON THE MATRIX $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$, AND IN CONNEXION THEREWITH THE FUNCTION $\dfrac{ax+b}{cx+d}$.

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In the preceding paper, [due to Prof. W. W. Johnson,] the theory of the symbolic powers and roots of the function $\dfrac{ax+b}{cx+d}$ is developed in a complete and satisfactory manner; the results in the main agreeing with those obtained in the original memoir, Babbage, "On Trigonometrical Series," *Memoirs of the Analytical Society* (1813), Note I. pp. 47—50, and which are to some extent reproduced in my "Memoir on the Theory of Matrices," *Phil. Trans.*, t. CXLVIII. (1858), pp. 17—37, [152]. I had recently occasion to reconsider the question, and have obtained for the nth function $\phi^n x$, where $\phi x = \dfrac{ax+b}{cx+d}$, a form which, although substantially identical with Babbage's, is a more compact and convenient one; viz. taking λ to be determined by the quadric equation

$$\frac{(\lambda+1)^2}{\lambda} = \frac{(a+d)^2}{ad-bc},$$

the form is

$$\phi^n(x) = \frac{(\lambda^{n+1}-1)(ax+b)+(\lambda^n-\lambda)(-dx+b)}{(\lambda^{n+1}-1)(cx+d)+(\lambda^n-\lambda)(\ \ cx-a)}.$$

The question is, in effect, that of the determination of the nth power of the matrix $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$; viz. in the notation of matrices

$$(x_1, y_1) = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}(x, y),$$

means the two equations $x_1 = ax + by$, $y_1 = cx + dy$; and then if x_2, y_2 are derived in like manner from x_1, y_1, that is, if $x_2 = ax_1 + by_1$, $y_2 = cx_1 + dy_1$, and so on, x_n, y_n will be linear functions of x, y; say we have $x_n = a_n x + b_n y$, $y_n = c_n x + d_n y$: and the nth power of $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$ is, in fact, the matrix $\begin{pmatrix} a_n, & b_n \\ c_n, & d_n \end{pmatrix}$.

In particular, we have

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}^2, = \begin{pmatrix} a_2, & b_2 \\ c_2, & d_2 \end{pmatrix}, = \begin{pmatrix} a^2+bc, & b(a+d) \\ c(a+d), & d^2+bc \end{pmatrix},$$

and hence the identity

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} = 0;$$

viz. this means that the matrix

$$\begin{pmatrix} a_2-(a+d)\,a+ad-bc, & b_2-(a+d)\,b \\ c_2-(a+d)\,c \quad , & d_2-(a+d)\,d+ad-bc \end{pmatrix} = \begin{pmatrix} 0, & 0 \\ 0, & 0 \end{pmatrix},$$

or, what is the same thing, that each term of the left-hand matrix is $=0$; which is at once verified by substituting for a_2, b_2, c_2, d_2 their foregoing values.

The explanation just given will make the notation intelligible and show in a general way how a matrix may be worked in like manner with a single quantity: the theory is more fully developed in my Memoir above referred to. I proceed with the solution in the algorithm of matrices. Writing for shortness $M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$, the identity is

$$M^2 - (a+d)\,M + (ad-bc) = 0,$$

the matrix $\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$ being in the theory regarded as $=1$; viz. M is determined by a quadric equation; and we have consequently $M^n =$ a linear function of M. Writing this in the form

$$M^n - AM + B = 0,$$

the unknown coefficients A, B can be at once obtained in terms of α, β, the roots of the equation

$$u^2 - (a+d)\,u + ad - bc = 0,$$

viz. we have

$$\alpha^n - A\alpha + B = 0,$$

$$\beta^n - A\beta + B = 0;$$

or more simply from these equations, and the equation for M^n, eliminating α, β, we have

$$\begin{vmatrix} M^n, & M, & 1 \\ \alpha^n, & \alpha, & 1 \\ \beta^n, & \beta, & 1 \end{vmatrix} = 0;$$

that is,

$$M^n(\alpha-\beta)-M(\alpha^n-\beta^n)+\alpha\beta(\alpha^{n-1}-\beta^{n-1})=0.$$

But instead of α, β, it is convenient to introduce the ratio λ of the two roots, say we have $\alpha=\lambda\beta$; we thence find

$$(\lambda+1)\beta=a+d,$$

$$\lambda\beta^2=ad-bc,$$

giving

$$\frac{(\lambda+1)^2}{\lambda}=\frac{(a+d)^2}{ad-bc}$$

for the determination of λ, and then

$$\beta=\frac{a+d}{\lambda+1},$$

$$\alpha=\frac{(a+d)\lambda}{\lambda+1}.$$

The equation thus becomes

$$M^n(\lambda-1)\beta-M(\lambda^n-1)\beta^n+(\lambda^n-\lambda)\beta^{n+1}=0,$$

or we have

$$M^n=\frac{\beta^{n-1}}{\lambda-1}\{(\lambda^n-1)M-(\lambda^n-\lambda)\beta\}.$$

It is convenient to multiply the numerator and denominator by $\lambda+1$, viz. we thus have

$$M^n=\frac{\beta^{n-1}}{\lambda^2-1}[(\lambda^{n+1}-1)M+(\lambda^n-\lambda)\{M-(\lambda+1)\beta\}].$$

The exterior factor is here

$$=\frac{1}{\lambda^2-1}\left(\frac{a+d}{\lambda+1}\right)^{n-1},$$

moreover $(\lambda+1)\beta$ is $=a+d$: hence

$$M=\begin{pmatrix} a, & b \\ c, & d \end{pmatrix},$$

and

$$M-(\lambda+1)\beta=\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}-\begin{pmatrix} a+d, & 0 \\ 0, & a+d \end{pmatrix},=\begin{pmatrix} -d, & b \\ c, & -a \end{pmatrix};$$

the formula thus is

$$M^n=\frac{1}{\lambda^2-1}\left(\frac{a+d}{\lambda+1}\right)^{n-1}\left\{(\lambda^{n+1}-1)\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}+(\lambda^n-\lambda)\begin{pmatrix} -d, & b \\ c, & -a \end{pmatrix}\right\},$$

viz. we have thus the values of the several terms of the nth matrix

$$M^n=\begin{pmatrix} a_n, & b_n \\ c_n, & d_n \end{pmatrix};$$

and, if instead of these we consider the combinations a_nx+b_n and c_nx+d_n, we then obtain

$$a_nx+b_n=\frac{1}{\lambda^2-1}\left(\frac{a+d}{\lambda+1}\right)^{n-1}\{(\lambda^{n+1}-1)(ax+b)+(\lambda^n-\lambda)(-dx+b)\},$$

$$c_nx+d_n=\quad,, \qquad ,, \qquad \{(\lambda^{n+1}-1)(cx+d)+(\lambda^n-\lambda)(\ \ cx-a)\};$$

and in dividing the first of these by the second, the exterior factor disappears.

It is to be remarked that, if $n=0$, the formulæ become as they should do $a_0x+b_0=x$, $c_0x+d_0=1$; and if $n=1$, they become $a_1x+b_1=ax+b$, $c_1x+d_1=cx+d$.

If $\lambda^m-1=0$, where m, the least exponent for which this equation is satisfied, is for the moment taken to be greater than 2, the terms in { } are

$$(\lambda-1)(ax+b)+(1-\lambda)(-dx+b),$$

and

$$(\lambda-1)(cx+d)+(1-\lambda)(\ \ cx-a);$$

viz. these are $(\lambda-1)(a+d)x$, and $(\lambda-1)(a+d)$, or if for $(\lambda-1)(a+d)$ we write $(\lambda^2-1)\dfrac{a+d}{\lambda+1}$, the formulæ become for $n=m$

$$a_mx+b_m=\left(\frac{a+d}{\lambda+1}\right)^m x,$$

$$c_mx+d_m=\left(\frac{a+d}{\lambda+1}\right)^m;$$

viz. we have here

$$\frac{a_mx+b_m}{c_mx+d_m}=x,$$

or the function is periodic of the mth order. Writing for shortness $\vartheta=\dfrac{s\pi}{n}$, s being any integer not $=0$, and prime to n, we have $\lambda=\cos 2\vartheta+i\sin 2\vartheta$, hence

$$1+\lambda=2\cos\vartheta(\cos\vartheta+i\sin\vartheta),$$

or $\dfrac{(1+\lambda)^2}{\lambda}=4\cos^2\vartheta$; consequently, in order to the function being periodic of the nth order, the relation between the coefficients is

$$4\cos^2\frac{s\pi}{n}=\frac{(a+d)^2}{ad-bc}.$$

The formula extends to the case $m=2$, viz. $\cos\frac{1}{2}(s\pi)=0$, or the condition is $a+d=0$. But here $\lambda+1=0$, and the case requires to be separately verified. Recurring to the original expression for M^2, we see that, for $a+d=0$, this becomes

$$\begin{vmatrix} a^2+bc, & 0 \\ 0\ \ , & d^2+bc \end{vmatrix},\ =(a^2+bc)\begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix};$$

that is,

$$\frac{a_2x+b_2}{c_2x+d_2}=x,$$

or the result is thus verified.

But the case $m=1$ is a very remarkable one; we have here $\lambda=1$, and the relation between the coefficients is thus $(a+d)^2=4(ad-bc)$, or what is the same thing $(a-d)^2+4bc=0$. And then determining the values for $\lambda=1$ of the vanishing fractions which enter into the formulæ, we find

$$a_nx+b_n=\frac{1}{2^n}(a+d)^{n-1}\{(n+1)(ax+b)+(n-1)(-dx+b)\},$$

$$c_nx+d_n=\frac{1}{2^n}(a+d)^{n-1}\{(n+1)(cx+d)+(n-1)(\quad cx-a)\},$$

or as these may also be written

$$a_nx+b_n=\frac{1}{2^n}(a+d)^{n-1}\{x[n(a-d)+(a+d)]+2nb\},$$

$$c_nx+d_n=\frac{1}{2^n}(a+d)^{n-1}\{x\,.\,2nc+[-n(a-d)+a+d]\},$$

which for $n=0$, become as they should do $a_0x+b_0=x$, $c_0x+d_0=1$, and for $n=1$ they become $a_1x+b_1=ax+b$, $c_1x+d_1=cx+d$. We thus do *not* have $\dfrac{a_1x+b_1}{c_1x+d_1}=x$, and the function is *not* periodic of any order. This remarkable case is noticed by Mr Moulton in his edition (2nd edition, 1872) of Boole's *Finite Differences*.

If to satisfy the given relation $(a-d)^2+4bc=0$, we write $2b=k(a-d)$, $2c=-\dfrac{1}{k}(a-d)$. then the function of x is

$$\frac{ax+\frac{1}{2}k(a-d)}{-\frac{1}{2}k^{-1}(a-d)x+d},$$

and the formulæ for the nth function are

$$a_nx+b_n=\frac{1}{2^n}(a+d)^{n-1}\{(a+d)x+n(a-d)(x+k)\},$$

$$c_nx+d_n=\frac{1}{2^n}(a+d)^{n-1}\left\{(a+d)\quad-n(a-d)\left(\frac{x}{k}+1\right)\right\};$$

which may be verified successively for the different values of n.

Reverting to the general case, suppose $n=\infty$, and let u be the value of $\phi^\infty(x)$. Supposing that the modulus of λ is not $=1$, we have λ^n indefinitely large or indefinitely small. In the former case, we obtain

$$u=\frac{\lambda(ax+b)+(-dx+b)}{\lambda(cx+d)+(\quad cx-a)},\quad=\frac{(\lambda a-d)x+b(\lambda+1)}{c(\lambda+1)x+\lambda d-a};$$

which, observing that the equation in λ may be written

$$\frac{\lambda a-d}{c(\lambda+1)}=\frac{b(\lambda+1)}{\lambda d-a},$$

is independent of x, and equal to either of these equal quantities; and if from these two values of u we eliminate λ, we obtain for u the quadric equation

$$cu^2-(a-d)\,u-b=0,$$

that is,

$$u=\frac{au+b}{cu+d},$$

as is, in fact, obvious from the consideration that n being indefinitely large the nth and $(n+1)$th functions must be equal to each other. In the latter case, as λ^n is indefinitely small, we have the like formulæ, and we obtain for u the same quadric equation: the two values of u are however not the same, but (as is easily shown) their product is $=-b\div c$; u is therefore the other root of the quadric equation. Hence, as n increases, the function $\phi^n x$ continually approximates to one or the other of the roots of this quadric equation. The equation has equal roots if $(a-d)^2+4bc=0$, which is the relation existing in the above-mentioned special case; and here $u=\frac{1}{2c}(a-d), =\frac{-2b}{a-d}$, which result is also given by the formulæ of the special case on writing therein $n=\infty$.