751.

NOTE ON RIEMANN'S PAPER "VERSUCH EINER ALLGEMEINEN AUFFASSUNG DER INTEGRATION UND DIFFERENTIATION*."

[From the *Mathematische Annalen*, t. XVI. (1880), pp. 81, 82.]

THE Editors of Riemann's works remark that the paper in question was contained in a MS. of his student time (dated 14 Jan. 1847) and was probably never intended for publication: indeed that he would not in later years have recognised the validity of the principles upon which it is founded. The idea is however a noticeable one: Riemann considers z_{x+h}, a function of $x+h$, expanded in a doubly infinite, necessarily divergent, series of integer or fractional powers of h, according to the law

$$z_{x+h} = \sum_{\nu=-\infty}^{\nu=+\infty} k_\nu \partial^\nu{}_x z \,.\, h^\nu, \tag{2}$$

where the meaning is explained to be that the exponents differ from each other by integer values, in effect, that ν has all the values $\alpha+p$, α a given integer or fractional value, and p any integer number from $-\infty$ to $+\infty$, zero included.

Riemann deduces a theory of fractional differentiation: but without considering the question which has always appeared to me to be the great difficulty in such a theory: what is the real meaning of a complementary function containing an infinity of arbitrary constants? or, in other words, what is the arbitrariness of the complementary function of this nature which presents itself in the theory?

I wish to point out the relation between the paper referred to, and a short paper of my own "On a doubly infinite Series," *Quart. Math. Journ.* t. VI. (1851), pp. 45—47, [102]: this commences with the remark "The following completely paradoxical investigation of the properties of the function Γ (which I have been in possession

* *Werke*, pp. 331—344.

of for some years) may perhaps be found interesting from its connexion with the theories of expansion and divergent series." And I then give the expansion

$$C_n e^x = \Sigma^r [n-r]^r x^{n-1-r},$$

where n is any integer or fractional number whatever, and the summation extends to all positive and negative integer values (zero included) of r. And I remark that, n being an integer, we have $C_n = \Gamma(n)$, and hence that assuming that this is so in general, or writing

$$\Gamma(n) . e^x = \Sigma^r [n-1]^r x^{n-1-r},$$

we have this equation as a definition of $\Gamma(n)$. The point of resemblance of course is that we have a doubly infinite expansion of e^x in a series of integer or fractional powers of x, corresponding to Riemann's like expansion of z_{x+h} in powers of h.

Cambridge, 10 *Sept.* 1879.