739.

NOTE ON THE OCTAHEDRON FUNCTION.

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A SEXTIC function

$$U=(a,\ b,\ c,\ d,\ e,\ f,\ g \between x,\ y)^6,$$

such that its fourth derivative

$$\begin{aligned}(U,\ U)^4,\ = \quad & (ae-4bd+3c^2)\,x^4\\ &+2\,(af-3be+2cd)\,x^3y\\ &+\ (ag-9ce+8d^2)\,x^2y^2\\ &+2\,(bg-3cf+2de)\,xy^3\\ &+\ (cg-4df+3e^2)\,y^4\end{aligned}$$

is identically $=0$, is considered by Dr Klein, and is called by him the octahedron function. Supposing that by a linear transformation the function is made to contain the factors x, y, or what is the same thing assuming $a=0$, $g=0$, then the equations to be satisfied become

$$-4bd+3c^2=0,\quad -3be+2cd=0,\quad -9ce+8d^2=0,\quad -3cf+2de=0,\quad -4df+3e^2=0,$$

which are all satisfied if only $c=d=e=0$; and then assuming, as is allowable,

$$b=-f=1,$$

we have his canonical form $xy(x^4-y^4)$ of the octahedron function.

But the equations may be satisfied in a different manner; viz. the first and last equations give

$$b=\frac{3c^2}{4d},\quad f=\frac{3e^2}{4d},$$

and, substituting these in the remaining equations, they become

$$\frac{c}{4d}(-9ce+8d^2)=0,\ -9ce+8d^2=0,\ \frac{e}{4d}(-9ce+8d^2)=0,$$

all satisfied if only $-9ce+8d^2=0$. Assuming $b=f=2$, the values are

$$b,\ c,\ d,\ e,\ f=2,\ 2\sqrt{(2)},\ 3,\ 2\sqrt{(2)},\ 2,$$

and the form is

$$xy\left(x^4+\frac{5}{\sqrt{(2)}}x^3y+5x^2y^2+\frac{5}{\sqrt{(2)}}xy^3+y^4\right),$$

$$=xy\left(x^2+\frac{3}{\sqrt{(2)}}xy+y^2\right)\{x^2+\sqrt{(2)}\,xy+y^2\},$$

$$=xy\left(x+\frac{1+i}{\sqrt{(2)}}y\right)\left(x+\frac{1-i}{\sqrt{(2)}}y\right)\{x+y\sqrt{(2)}\}\left(x+\frac{y}{\sqrt{(2)}}\right).$$

This is, in fact, a linear transformation of the foregoing form $XY(X^4-Y^4)$; for writing

$$X=\left(x+\frac{1+i}{\sqrt{(2)}}y\right),$$

$$Y=\left(x+\frac{1-i}{\sqrt{(2)}}y\right),$$

we have

$$X^2=x^2+(1+i)\sqrt{(2)}\,xy+iy^2,$$
$$Y^2=x^2+(1-i)\sqrt{(2)}\,xy-iy^2;$$

and therefore

$$X^2+Y^2=\quad 2x\{x+\sqrt{(2)}\,y\},$$

$$X^2-Y^2=2i\sqrt{(2)}\,y\left(x+\frac{y}{\sqrt{(2)}}\right),$$

or finally

$$XY(X^4-Y^4)=4i\sqrt{(2)}\,xy\left(x+\frac{1+i}{\sqrt{(2)}}y\right)\left(x+\frac{1-i}{\sqrt{(2)}}y\right)\{x+y\sqrt{(2)}\}\left(x+\frac{y}{\sqrt{(2)}}\right);$$

and the two forms are thus identical.