732.

A THEOREM IN SPHERICAL TRIGONOMETRY.

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IN a spherical triangle, where a, b, c are the sides, and A, B, C the opposite angles, we have

$$-\tan\tfrac{1}{2}c\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\sin(A-B)=\tan\tfrac{1}{2}b\sin A-\tan\tfrac{1}{2}a\sin B,$$

$$\tan\tfrac{1}{2}c\,\{1-\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\cos(A-B)\}=\tan\tfrac{1}{2}b\cos A+\tan\tfrac{1}{2}a\cos B;$$

which are both included in the form

$$\tan\tfrac{1}{2}a\,(\cos B-i\sin B)=\frac{\tan\frac{1}{2}c-\tan\frac{1}{2}b\,(\cos A+i\sin A)}{1+\tan\frac{1}{2}c\tan\frac{1}{2}b\,(\cos A+i\sin A)}.$$

For the first of the two identities: from

$$\cos a=\frac{\cos A+\cos B\cos C}{\sin B\sin C},$$

$$\cos b=\frac{\cos B+\cos A\cos C}{\sin A\sin C},$$

we deduce

$$\begin{aligned}\cos a-\cos b&=\frac{1}{\sin C}\left(\frac{\cos A}{\sin B}-\frac{\cos B}{\sin A}\right)+\frac{\cos C}{\sin C}\left(\frac{\cos B}{\sin B}-\frac{\cos A}{\sin A}\right)\\&=\frac{1}{\sin C}\,\frac{\frac{1}{2}(\sin 2A-\sin 2B)}{\sin A\sin B}+\frac{\cos C\sin(A-B)}{\sin C\sin A\sin B}\\&=\frac{\sin(A-B)}{\sin C\sin A\sin B}\{\cos(A+B)+\cos C\}\\&=\frac{\sin(A-B)}{\sin C}(\cos c-1);\end{aligned}$$

that is,

$$-\sin(A-B)=\frac{\sin C}{1-\cos c}(\cos a-\cos b)$$

$$=\frac{\sin C}{\sin c}\,\frac{\sin c}{1-\cos c}(\cos a-\cos b);$$

or, what is the same thing,

$$-\tan\tfrac{1}{2}c\sin(A-B)=\frac{\sin C}{\sin c}(\cos a-\cos b).$$

Here $\cos a-\cos b$ is $=(1+\cos a)-(1+\cos b)$; substituting for $\frac{\sin C}{\sin c}$ successively $\frac{\sin A}{\sin a}$ and $\frac{\sin B}{\sin b}$, the right-hand side is

$$=\frac{1+\cos a}{\sin a}\sin A-\frac{1+\cos b}{\sin b}\sin B,$$

$$=\cot\tfrac{1}{2}a\sin A-\cot\tfrac{1}{2}b\sin B;$$

whence, multiplying each side by $\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b$, we have the relation in question.

For the second identity which is

$$\tan\tfrac{1}{2}c\,\{1-\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\cos(A-B)\}=\tan\tfrac{1}{2}b\cos A+\tan\tfrac{1}{2}a\cos B;$$

if on the right-hand side we substitute for $\cos A$, $\cos B$ their values

$$\frac{\cos a-\cos b\cos c}{\sin b\sin c}\quad\text{and}\quad\frac{\cos b-\cos a\cos c}{\sin a\sin c},$$

the right-hand side becomes

$$\frac{1}{\sin c}\left\{\frac{\cos a-\cos b\cos c}{1+\cos b}+\frac{\cos b-\cos a\cos c}{1+\cos a}\right\};$$

whence, multiplying the whole equation by $\sin c\,(1+\cos a)(1+\cos b)$, it becomes

$$(1-\cos c)\{(1+\cos a)(1+\cos b)-\sin a\sin b\cos(A-B)\}$$

$$=(1+\cos a)(\cos a-\cos b\cos c)+(1+\cos b)(\cos b-\cos c\cos a).$$

We have here

$$\cos(A-B)=\cos A\cos B+\sin A\sin B=\frac{(\cos a-\cos b\cos c)(\cos b-\cos c\cos a)+\square}{\sin^2 c\sin a\sin b},$$

by substituting for $\cos A$, $\cos B$ their foregoing values, and for $\sin A$, $\sin B$ their values $\frac{\sqrt{\square}}{\sin b\sin c}$, $\frac{\sqrt{\square}}{\sin a\sin c}$, where

$$\square=1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c.$$

The numerator is

$$\cos a \cos b - \cos c\,(\cos^2 a + \cos^2 b) + \cos a \cos b \cos^2 c$$
$$+\,1 - \cos^2 c \;-\qquad (\cos^2 a + \cos^2 b) + \cos a \cos b \,.\, 2\cos c\,;$$

viz. this is

$$= \cos a \cos b\,(1 + \cos c)^2 - (\cos^2 a + \cos^2 b)\,(1 + \cos c) + 1 - \cos^2 c,$$

having the factor $1 + \cos c$, which is also a factor of $\sin^2 c$, $= 1 - \cos^2 c$, in the denominator. We have, therefore,

$$\cos(A - B) = \frac{\cos a \cos b\,(1 + \cos c) - (\cos^2 a + \cos^2 b) + 1 - \cos c}{(1 - \cos c)\sin a \sin b}\,;$$

and the equation thus is

$$(1 - \cos c)\,(1 + \cos a)\,(1 + \cos b) - \{\cos a \cos b\,(1 + \cos c) - (\cos^2 a + \cos^2 b) + 1 - \cos c\}$$
$$= (1 + \cos a)\,(\cos a - \cos b \cos c) + (1 + \cos b)\,(\cos b - \cos c \cos a),$$

where each side is in fact

$$= \cos a + \cos^2 a + \cos b + \cos^2 b - \cos c\,(\cos a + \cos b) - 2\cos a \cos b \cos c\,;$$

and the second identity is thus proved.