434.

ON CERTAIN SKEW SURFACES, OTHERWISE SCROLLS.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XI. Part II. (1869), pp. 277—289. Read Nov. 11, 1867.]

THE investigations contained in the present Memoir were suggested to me by the Memoirs of M. De la Gournerie, presented by him to the Academy of Sciences in 1865 and 1866, published in extract in the *Comptes Rendus*, and reproduced in his work "*Recherches sur les surfaces réglées tétraédrales symétriques*, par Jules De la Gournerie, avec des notes par Arthur Cayley," 8vo. Paris, 1867. Although the results or the greater part of them, agree with those in the work just referred to, the mode of treatment is different, and more general, the orders, &c. of the different scrolls being obtained by considerations founded on the theory of Correspondence, and I have thought it not improper to submit to geometers in this altered form the theory of the very interesting class of Scrolls for which they are indebted to M. De la Gournerie's researches.

Article Nos. 1 to 10. *Geometrical Construction of a Class of Scrolls.*

1. Consider any two curves (plane or of double curvature) U, U', of the orders m, m' respectively, and let the points of U have with those of U' an (α, α') correspondence; viz. let the points of the two curves be so related that to each point of U correspond α' points of U', and to each point of U' correspond α points of U: then the lines joining the corresponding points of U, U' form a scroll the order of which is $= m\alpha' + m'\alpha$.

2. In particular let U, U' be plane curves in the planes Π, Π' respectively; and let the correspondence between the points of the two curves be established as follows; viz. consider in the plane Π a curve Ω of the class μ, and in the plane Π' a curve Ω' of the class μ'; and (to avoid useless generality) let the tangents of these two curves Ω, Ω' have to each other a (1, 1) correspondence; that is, to each tangent of

Ω there corresponds a single tangent of Ω', and to each tangent of Ω' a single tangent of Ω (this assumes that the curves Ω, Ω' are rational transformations one of the other, and that they have consequently the same Deficiency). This being so, let the points of U which lie on any tangent of Ω and the points of U' which lie on the corresponding tangent of Ω' be taken to be corresponding points of U, U'. The correspondence is then $(\mu m', \mu' m)$: in fact through a given point of U there pass μ tangents of Ω, and the corresponding μ tangents of Ω' meet U' in $\mu m'$ points, that is, to a given point of U correspond $\mu m'$ points of U'; and similarly to a given point of U' correspond $\mu' m$ points of U. And hence the order of the scroll formed by the lines joining the corresponding points of U, U' is $= (\mu + \mu') mm'$.

3. This conclusion may be otherwise established as follows; let K, K' be any two corresponding points of U, U', so that the scroll we are concerned with is that generated by the series of lines KK'; and let I denote the line of intersection of the planes Π, Π'. The line I meets the curve U in m points, and taking one of these points for a point K we may from this point draw μ tangents to the curve Ω, that is, the point in question is a point K in respect of μ different tangents of the curve Ω; to each of these tangents there corresponds a single tangent of Ω', and such tangent of Ω' meets the curve U' in m' points, that is, to the point K in question there correspond $\mu m'$ points K' and consequently $\mu m'$ lines KK' in the plane Π'; hence to each of the m points K on the line I there correspond $\mu m'$ lines KK' in the plane Π'; and we have thus $\mu mm'$ generating lines in the plane Π'; there are in like manner $\mu' mm'$ generating lines in the plane Π.

Take K an arbitrary point on the curve U; there are $\mu m'$ corresponding points K', and consequently $\mu m'$ generating lines through K, that is, through each point of the curve U; or the curve U (which is of the order m) is a $\mu m'$-tuple line on the scroll; similarly the curve U' (which is of the order m') is a $\mu' m$-tuple line on the scroll.

The complete section of the scroll by the plane Π consists of the curve U taken $\mu m'$ times (order $\mu mm'$) and of the $\mu' mm'$ generating lines in the plane Π; that is, the order of the section is $= (\mu + \mu') mm'$; and we thus see that the order of the scroll is $= (\mu + \mu') mm'$. Of course in like manner the complete section of the scroll by the plane Π' consists of the curve U' taken $\mu' m$ times (order $\mu' mm'$) and of the $\mu mm'$ generating lines in the plane Π', the order of the section being thus $= (\mu + \mu') mm'$.

4. There are on the scroll certain singular tangent planes; viz. if we have two corresponding tangents of Ω, Ω' meeting the line I in the same point, then we have m points K and m' points K' all lying in the plane of the two tangents; and of course the mm' lines KK' will all lie in the plane of the two tangents; that is, the intersection of the scroll by the plane in question will be made up of the mm' lines, and of a curve of the order $(\mu + \mu' - 1) mm'$; and the plane in question is thus a singular tangent plane.

5. The number of these singular tangent planes is $= \mu + \mu'$; in fact considering as corresponding points on the line I, the intersection of this line by any tangent of Ω and the intersection by the corresponding tangent of Ω', the correspondence is

obviously (μ, μ'); viz. through a given point P considered as belonging to the first system there pass μ tangents of Ω, and corresponding thereto we have μ tangents of Ω' each intersecting I in a single point P'; that is, to a given point P correspond μ points P'; and similarly to a given point P' correspond μ' points P'. And this being so, the number of united points, that is, points of I through which pass corresponding tangents of Ω, Ω', is $= \mu + \mu'$.

6. In particular the curves Ω, Ω' may reduce themselves each to a point: the tangents to the two curves are here the lines passing through the points Ω, Ω' respectively: and the condition for the (1, 1) correspondence of the two tangents is that the pencils of lines shall be homographically related; or, what is the same thing, that these two pencils shall determine on the line I two ranges which are homographically related; the entire construction is then as follows:

Given in the plane Π a curve U and a point Ω, and in the plane Π' a curve U' and a point Ω'; and taking in the plane Π a pencil of lines through Ω, and in the plane Π' a pencil of lines through Ω', in such wise that the two pencils correspond homographically; then if a line of the first pencil meets the curve U in the m points K, and the corresponding line of the second pencil meets the curve U' in the m' points K', the scroll in question is that generated by the mm' lines KK'.

7. By what precedes, the scroll is of the order $2mm'$; the curve U is a m'-tuple line, and the complete section by the plane Π is made up of this curve taken m' times and of mm' generating lines; similarly the curve U' is a m-tuple line, and the complete section by the plane Π' is made up of this curve taken m times and of mm' generating lines; there are two singular tangent planes such that the section by each of them is made up of mm' generating lines and of a curve of the order mm'; the planes in question are obviously those through the lines $\Omega\Omega'$ and the coincident points of the two ranges on the line I, say the points A, B respectively.

8. The foregoing results will be modified in special cases. Suppose, for instance, that the curve U passes ω times, α times, and β times through the points Ω, A, B, respectively, and that the curve U' passes ω' times, α' times, and β' times through the points Ω', A, B respectively. Then to each point on the curve U there correspond the $m' - \omega'$ intersections (other than the point Ω') on a line through Ω', so that U' is a $(m' - \omega')$tuple line on the surface. The curve U' meets the line I in m' points and corresponding to each of them we have a line through Ω meeting the curve U in $(m - \omega)$ points, exclusive of the point Ω; this would give $m'(m - \omega)$ generating lines in the plane Π; but among the m' points are included the point $A\alpha'$ times, and the point $B\beta'$ times; the $(m - \omega)$ points corresponding to A include the point $A\alpha$ times, and we have thus the point A corresponding to itself $\alpha\alpha'$ times, and giving a reduction $= \alpha\alpha'$ in the number $m'(m - \omega)$ of generating lines: similarly the $m - \omega$ points corresponding to B include the point $B\beta$ times, and we have thus the point B corresponding to itself $\beta\beta'$ times and giving a reduction $= \beta\beta'$ in the number $m'(m - \omega)$ of generating lines; the number of generating lines in the plane Π is thus $= m'(m - \omega) - \alpha\alpha' - \beta\beta'$. The complete section by the plane Π is made up of the

curve $U\,(m-\omega')$ times (order $m\,(m'-\omega')$) and of the $m'(m-\omega)-\alpha\alpha'-\beta\beta'$ generating lines; the order of the section, and consequently also the order of the scroll, is thus $=2mm'-m\omega'-m'\omega-\alpha\alpha'-\beta\beta'$. It is clear that in like manner the curve U' is a $(m-\omega)$tuple line on the surface, and that the complete section by the plane Π' is made up of this curve taken $(m-\omega)$ times, order $m'(m-\omega)$, and of $m\,(m'-\omega')-\alpha\alpha'-\beta\beta'$ generating lines.

9. The section by the tangent plane through A is made up of $(m-\omega)(m'-\omega')-\alpha\alpha'$ generating lines (viz. these are, the line ΩA $\alpha'(m-\omega-\alpha)$ times, the line $\Omega' A$ $\alpha\,(m'-\omega'-\alpha')$ times, and $(m-\omega-\alpha)\,(m'-\omega'-\alpha')$ other generating lines) and of a curve of the order $mm'-\omega\omega'-\beta\beta'$: similarly the section by the tangent plane through B is made up of $(m-\omega)\,(m'-\omega')-\beta\beta'$ generating lines (viz. these are, the line ΩB $\beta'(m-\omega-\beta)$ times, the line $\Omega' B$ $\beta\,(m'-\omega'-\beta')$ times, and $(m-\omega-\beta)\,(m'-\omega'-\beta')$ other generating lines), and of a curve of the order $mm'-\omega\omega'-\alpha\alpha'$.

10. A very interesting case is when (m, m' being each even) we have

$$\omega=\alpha=\beta=\frac{1}{2m},\quad \omega'=\alpha'=\beta'=\tfrac{1}{2}\,m'.$$

Here the curve U is a $\frac{1}{2}m'$-tuple line on the scroll, and the complete section by the plane Π is this curve taken $\frac{1}{2}m'$ times; the order of the section, and therefore of the scroll is thus $=\frac{1}{2}mm'$; of course in like manner the curve U' is a $\frac{1}{2}m$-tuple line on the scroll, and the complete section by this plane is the curve U' taken $\frac{1}{2}m$ times: the section by each of the planes $\Omega\Omega' A$, $\Omega\Omega' B$ is a curve of the order $\frac{1}{2}mm'$, the planes in question being in the present case no longer singular tangent planes, or even tangent planes at all, of the scroll.

Article Nos. 11 to 14. *Analytical Theory.*

11. It will be convenient to denote by D, C respectively the points heretofore called Ω, Ω' respectively: this being so, we have a tetrahedron $ABCD$, of which the faces ABD, ABC are the planes heretofore called Π, Π' respectively, and the other two faces CDA, CDB are the singular tangent planes $\Omega\Omega' A$, $\Omega\Omega' B$ respectively. And then, taking

$$x=0,\quad y=0,\quad z=0,\quad w=0$$

for the equations of the faces BCD, CDA, DAB, ABC of the tetrahedron, we may write for the equations of the curve U, $z=0$, $f_3\,(x,\,y,\,w)=0$, for those of the curve U', $w=0$, $f_4\,(x,\,y,\,z)=0$; and take the homographic ranges on the line $I\,(z=0,\ w=0)$ to be given as the intersections of this line with the pencils of planes $x-\theta y$, $x-k\theta y=0$ respectively (θ a variable parameter, k a constant). The points K are therefore given by

$$x-\theta y=0,\quad z=0,\quad f_3\,(x,\,y,\,w)=0,$$

the points K' by

$$x-k\theta y=0,\quad w=0,\quad f_4\,(x,\,y,\,z)=0;$$

and then the lines KK' belonging to the different values of the parameter θ generate the scroll.

12. Or, what is the same thing, taking (X, Y, Z, W) as the coordinates of K, (X', Y', Z', W') as the coordinates of K', we have

$$X - \theta Y = 0, \quad Z = 0, \quad f_3(X, Y, W) = 0,$$
$$X' - k\theta Y = 0, \quad W' = 0, \quad f_4(X', Y', Z') = 0,$$

and then the equations of the line KK' are

$$\left\| \begin{matrix} x, & y, & z, & w \\ X, & Y, & 0, & W \\ X', & Y', & Z', & 0 \end{matrix} \right\| = 0:$$

or, as these may be written,

$$\begin{array}{rrrr} \cdot & -WZ'y + & WY'z + & YZ'w = 0, \\ WZ'x + & \cdot & -WX'z - & XZ'w = 0, \\ -W'Yx + & WX'y & \cdot & +(XY'-X'Y)w = 0, \\ -YZ'x + & XZ'y & -(XY'-X'Y)z & \cdot \quad = 0, \end{array}$$

equivalent of course to two equations. The elimination of $X, Y, W, X', Y', Z', \theta$ from all the equations gives the equation of the scroll.

13. Substituting the values $X = \theta Y$, $X' = k\theta Y'$, we have

$$f_3(\theta Y, Y, W) = 0, \quad f_4(k\theta Y', Y', Z') = 0,$$

$$\begin{array}{rrrr} \cdot & -WZ'y + & WY'z + & YZ'w = 0, \\ WZ'x & \cdot & -k\theta WY'z - & \theta YZ'w = 0, \\ -WY'x + & k\theta WY'y & \cdot & +\theta(1-k)YY'w = 0, \\ -YZ'x + & \theta YZ'y & -\theta(1-k)YY'z & \cdot \quad = 0; \end{array}$$

or, what is the same thing, writing $\frac{W}{Y} = \omega$, and $\frac{Z'}{Y'} = \zeta$, we have

$$f_3(\theta, 1, \omega) = 0, \quad f_4(k\theta, 1, \zeta) = 0,$$

$$\begin{array}{rrrr} \cdot & -\omega\zeta y + & \omega z + & \zeta w = 0, \\ \omega\zeta x & \cdot & -k\theta\omega z - & \theta\zeta w = 0, \\ -\omega x + & k\theta\omega y & \cdot & +\theta(1-k)w = 0, \\ -\zeta x + & \theta\zeta y - & \theta(1-k)z & \cdot \quad = 0. \end{array}$$

Recollecting that the last four equations are equivalent to two equations only, and substituting for ω, ζ their values in terms of θ, we have in effect two equations, which by the elimination of θ lead to a relation in (x, y, z, w), the equation of the scroll.

14. We may find the sections of the scroll by the planes $x = 0$, $y = 0$ respectively. Writing first $x = 0$, we have

$$\omega y = \frac{k-1}{k} w, \quad \zeta y = -(k-1)z.$$

Hence taking the other two equations in the form

$$f_3(\theta y,\ y,\ \omega y)=0,\quad f_4\left(\theta y,\ \frac{y}{k},\ \frac{\zeta y}{k}\right)=0,$$

and putting $\theta y=u$, we have

$$f_3\left(u,\ y,\ \frac{(k-1)\,w}{k}\right)=0,\quad f_4\left(u,\ \frac{y}{k},\ \frac{-(k-1)\,z}{k}\right)=0,$$

from which eliminating u we obtain an equation $f_1(y,\ z,\ w)=0$, the equation of the section by the plane $x=0$.

Similarly, writing $y=0$, we have

$$\frac{x\omega}{\theta}=-(k-1)\,w,\quad \frac{x\zeta}{\theta}=(k-1)\,z,$$

whence taking the equations in the form

$$f_3\left(x,\ \frac{x}{\theta},\ \frac{x\omega}{\theta}\right)=0,\quad f_4\left(kx,\ \frac{x}{\theta},\ \frac{x\zeta}{\theta}\right)=0,$$

and writing $\frac{x}{\theta}=v$, we have

$$f_3(x,\ v,\ -(k-1)\,w)=0,\quad f_4(kx,\ v,\ (k-1)\,z)=0,$$

from which, eliminating v, we obtain an equation $f_2(x,\ z,\ w)=0$, the equation of the section by the plane $y=0$.

Article Nos. 15 to 29. *The Curves U, U' are henceforward "triangular" curves.*

15. Let $r=\pm\frac{p}{q}$, where p, q are positive integers prime to each other, and let the given sections be

$$z=0,\quad A\,x^r+B\,y^r\quad .\quad +Dw^r=0,$$

$$w=0,\quad A'x^r+B'y^r+C'z^r\quad .\quad =0,$$

where it is to be observed that r being $=+\frac{p}{q}$, the two given sections are of the order pq, the order of the scroll is $=2p^2q^2$, each of the given sections is a pq-tuple line on the scroll, and the plane thereof meets the scroll in the section taken pq times, and in the pq generating lines: but r being $=-\frac{p}{q}$, the two given sections are each of the order $2pq$, with three pq-tuple points ($\omega=\alpha=\beta=pq$, $\omega'=\alpha'=\beta'=pq$), and thence the order of the scroll is $\frac{1}{2}(2pq)^2$, $=2p^2q^2$; each of the sections is a pq-tuple line on the scroll, and the plane meets the scroll only in the section taken pq times. But in either case, if q be >1, that is, if r be fractional, it will presently appear that the scroll of the order $2p^2q^2$ breaks up into q scrolls each of the order $2p^2q$.

16. To find the section by the plane $x=0$, we have

$$A\,u^r + B\;y^r \qquad . \qquad + D\left(\frac{k-1}{k}\,w\right)^r = 0,$$

$$A'u^r + B'\;\left(\frac{y}{k}\right)^r + C'\;\left(-\frac{k-1}{k}\,z\right)^r \qquad . \qquad = 0,$$

and eliminating u we obtain

$$\left(AB'\frac{1}{k^r} - A'B\right)y^r + AC'\left(-\frac{k-1}{k}\right)^r z^r - A'D\left(\frac{k-1}{k}\right)^r w^r = 0:$$

writing $\left(\frac{k-1}{k}\right)^r = (-)^r\left(\frac{1-k}{k}\right)^r$, this is

$$\left(AB'\frac{1}{k^r} - A'B\right)y^r + AC' \quad \left(\frac{1-k}{k}\right)^r z^r - (-)^r\left(\frac{1-k}{k}\right)^r w^r = 0,$$

or, what is the same thing, it is

$$-(-)^r\frac{AB' - A'Bk^r}{(1-k)^r}\,y^r - (-)^{-r}AC' \quad z^r + \quad A'Dw^r = 0.$$

And in regard to this and the other equations which contain $(-)^{-r}$, it is to be observed that r being integral we have $(-)^{-r}=(-)^r$, and that r being fractional, every value of $(-)^{-r}$ is also a value of $(-)^r$; so that we may in every case write $(-)^r$ in place of $(-)^{-r}$.

Similarly for the section by the plane $y=0$, we have

$$Ax^r + Bv^r \qquad . \qquad + D\left(-(k-1)\;w\right)^r = 0,$$

$$A'(kx)^r + B'v^r + C\left((k-1)\,z\right)^r \qquad . \qquad = 0,$$

and eliminating v, we have

$$(AB' - A'Bk^r)\,x^r \qquad - BC'(k-1)^r z^r + B'D\left(-(k-1)\right)^r w^r = 0\,;$$

or, what is the same thing,

$$\frac{AB' - A'Bk^r}{(1-k)^r}\,x^r \qquad -(-)^r BC'z^r + B'D \qquad w^r = 0.$$

17. The four sections thus are

$$x=0, \qquad . \qquad -(-)^r\frac{AB'-A'Bk^r}{(1-k)^r}\,y^r - (-)^r AC'z^r + A'Dw^r = 0,$$

$$y=0, \quad \frac{AB'-A'Bk^r}{(1-k)^r}\,x^r \qquad . \qquad -(-)^r BC'z^r + B'Dw^r = 0,$$

$$z=0, \qquad Ax^r \qquad + By^r \qquad . \qquad + Dw^r = 0,$$

$$w=0, \qquad A'x^r \qquad + B'y^r + \qquad C'z^r \qquad . \qquad = 0.$$

It will be convenient to speak of these four curves as *directrices* of the scroll.

18. Suppose for a moment that r is integral; as either of the given equations may be multiplied by a constant, we may assume that $D = -(-)^r C'$; substituting this value and dividing the first and second equations each by C', we have

$$\begin{array}{llllll} x=0, & . & -(-)^r \dfrac{AB'-A'Bk^r}{(1-k)^r C'} y^r & -(-)^r Az^r & -(-)^r A'w^r & =0, \\ y=0, & \dfrac{AB'-A'Bk^r}{(1-k)^r C'} x^r & . & -(-)^r Bz^r & -(-)^r B'w^r & =0, \\ z=0, & A x^r + & By^r & . & -(-)^r C'w^r & =0, \\ w=0, & A'x^r + & B'y^r + & C'z^r & . & =0, \end{array}$$

so that the diagonally opposite coefficients differ only by the factor $-(-)^r$; viz. the matrix is symmetrical or skew symmetrical according as r is odd or even.

19. If r be fractional, it is to be observed that, although the three symbols $(-)^r$ and the two symbols $(1-k)^r$ which enter into the first and second equations of No. 17, do not in the first instance represent of necessity the same values of $(-)^r$ and $(1-k)^r$ respectively, yet there is no loss of generality in assuming that they do so—the irrational equations are mere symbols for the rational equations to which they respectively give rise—and the irrationalities $(-)^r$ and $(1-k)^r$ will on the rationalisation of the equations disappear along with the irrationalities x^r, y^r, z^r, to which they are attached. But the case is otherwise with the irrationality k^r involved in the expression $AB' - A'Bk^r$; writing as before $r = \pm \dfrac{p}{q}$ (p and q positive integers prime to each other), the symbol k^r has q different values; and there is not in the first instance any relation between the k^r of the first equation and the k^r of the second equation: for each of these equations the rationalised equation (that is, the equation rationalised in regard to the coordinates) will contain the irrationality k^r, and will thus for each of the q values of k^r represent a distinct curve. The given equations (viz. the first and second equations) represent each of them a single curve of the order pq or $2pq$, according as r is positive or negative; the first and second equations represent each of them q such curves.

20. Hence, starting from the two given curves in the planes $z=0$ and $w=0$, respectively, and with a given value of k, the section of the scroll by the plane $y=0$ is made up of q curves, viz. the curves obtained from the second equation of No. 17, by assigning to the radical k^r each of its q different values; the scroll consequently breaks up into q different scrolls, viz. the lines passing through the two given curves, and any one of the q curves in the plane $y=0$, constitute a distinct scroll. The lines in question meet the plane $x=0$, not indifferently in any one of the q curves in that plane, but in a certain one of these curves, viz. in that curve for which the radical k^r has the same value as for the curve in question in the plane $y=0$. Hence we may in the first and second equations regard the radicals k^r as having the same meaning, and the system of four equations in effect breaks up into q systems, viz. the systems obtained by giving to the radical k^r its q different values; each of these systems gives a scroll, and the scroll derived from the two given curves with a given

value of k is made up of these q scrolls. And hence, attaching a unique value to each of the symbols $(-)^r$, $(1-k)^r$, and k^r, we may, as before, write $D=(-)^r C'$, and so reduce the original equations as in the case r integral, to the form No. 18, in which the diagonally opposite coefficients differ only by the factor $-(-)^r$.

21. Let the two given equations be taken to be

$$\begin{array}{ll} z=0, & bx^r-(-)^r ay^r \quad . \quad +hw^r=0, \\ w=0, & -(-)^r fx^r-(-)^r gy^r-(-)^r hz^r \quad . \quad =0\,; \end{array}$$

we have then

$$\frac{AB'-A'Bk^r}{(1-k)^r C'}=\frac{(-)^r bg+afk^r}{(-)^r h\,(1-k)^r},$$

or, putting this $=-(-)^r c$, that is, writing

$$afk^r+bg\,(-1)^r+ch\,(1-k)^r=0,$$

the four equations become

$$\begin{array}{ll} x=0, & \quad . \quad cy^r-(-)^r bz^r+fw^r=0, \\ y=0, & -(-)^r cx^r \quad . \quad + \quad az^r+gw^r=0, \\ z=0, & bx^r-(-)^r ay^r \quad . \quad +hw^r=0, \\ w=0, & -(-)^r fx^r-(-)^r gy^r-(-)^r hz^r \quad . \quad =0\,; \end{array}$$

where c being considered as given, k is determined as mentioned above, or, what is the same thing, $k:-1:1-k=\lambda:\mu:\nu$, we have $\lambda:\mu:\nu$, and thence k, determined by the equations

$$\lambda+\ \mu+\ \nu=0,$$

$$af\lambda^r+bg\mu^r+ch\nu^r=0.$$

22. Consider for a moment λ, μ, ν, as the coordinates of a point in a plane, then $\left(r=\pm\frac{p}{q}\text{ as before}\right)$, the equation $af\lambda^r+bg\mu^r+ch\nu^r=0$, is that of a curve of the order pq or $2pq$, according as r is positive or negative: and this curve is met by the line $\lambda+\mu+\nu=0$, in pq or $2pq$ points, that is, k has this number pq or $2pq$, of values: but to each of these values of k there corresponds (not q values but) only a single value of k^r, viz. that value for which $afk^r+bg\,(-1)^r+ch\,(1-k)^r=0$; that is, starting from the two directrices in the planes $z=0$, $w=0$, respectively, and a given third directrix in the plane $y=0$ (or in the plane $x=0$), we may by means of each of the pq or $2pq$ values of k construct a scroll passing through the three directrices, and which will also pass through the fourth directrix in the plane $x=0$ (or in the plane $y=0$), but such scroll is only one (not each) of the q scrolls which can be constructed from the two given sections in the planes $z=0$, $w=0$, respectively, and from the assumed value of k. It has been mentioned that whether r is $=+\frac{p}{q}$, or $=-\frac{p}{q}$, the total scroll constructed from the two given directrices in the planes $z=0$, $w=0$, and from a given value of k is of the order

$2p^2q^2$, and that such scroll breaks up into q distinct scrolls, hence the order of each of the distinct scrolls is $= 2p^2q$. Whence, starting with the given directrices in the planes $z = 0$, $w = 0$, and a given third directrix in the plane $y = 0$ (or in the plane $x = 0$), we have pq or $2pq$ scrolls each of the order $2p^2q$, and passing through these three directrices, and through the given fourth directrix in the plane $x = 0$ (or in the plane $y = 0$).

23. It is to be observed that when q is > 1, then considering the three directrices as given, the pq or $2pq$ scrolls each of the order $2p^2q$, do not make up the total scroll generated by the lines which pass through the three given directrices. I call to mind that for three given directrices the orders of which are m, n, p, respectively, and which meet, the second and third, the third and first, and the first and second, in α points, β points, and γ points respectively, the order of the scroll generated by the lines which meet the three directrices is $= 2mnp - \alpha m - \beta n - \gamma p$. Suppose first, that $r = +\frac{p}{q}$, then the directrices are each of the order pq, and they do not any two of them meet; the order of the scroll is $= 2p^3q^3$. Suppose secondly, $r = -\frac{p}{q}$, then the directrices are each of the order $2pq$, but each two of them have in common two pq-tuple points counting as $2p^2q^2$ intersections; the order of the scroll is thus $(16 - 3 \, . \, 4)\, p^3q^3 = 4p^3q^3$. In the first case the lines which meet the three directrices generate a residuary scroll of the order $2p^3(q^2 - q^2)$, and the pq scrolls each of the order $2p^2q$; in the second case they generate a residuary scroll of the order $4p^3(q^3 - q^2)$, and the $2pq$ scrolls each of the order $2pq$.

24. In the case $r = +\frac{p}{q}$, by way of illustration of the origin of the pq scrolls each of the order $2p^2q$, I consider the particular case $p = 1$, that is, $r = \frac{1}{q}$, the reciprocal of a positive integer q, and where it is to be shown that we have q scrolls each of the order $2q$. The given directrices are here

$$z = 0, \quad Ax^{\frac{1}{q}} + By^{\frac{1}{q}} \quad . \; + Dw^{\frac{1}{q}} = 0,$$

$$w = 0, \quad A'x^{\frac{1}{q}} + B'y^{\frac{1}{q}} + C'z^{\frac{1}{q}} \quad . \quad = 0,$$

each of them a unicursal curve; we may in fact satisfy the two equations respectively, by writing in the first of them

$$x : y : w = a(\phi + \alpha)^q : b(\phi + \beta)^q : d(\phi + \delta)^q;$$

and in the second

$$x : y : z = a'(\phi' + \alpha')^q : b'(\phi' + \beta')^q : c'(\phi' + \gamma')^q,$$

where a, b, d, α, β, δ, a', b', c', α', β', γ' are properly determined constants, ϕ, ϕ' are variable parameters. It follows that, considering the points K, K' which are the intersections of the first curve by the line $x - \theta y = 0$, and of the second curve by the corresponding line $x - k\theta y = 0$, we have not only a correspondence of q points K with

k points K', but we may establish, and that in q different manners, a correspondence between single points K and K'. For, substituting the foregoing values of $x : y$ in the equations $x - \theta y = 0$ and $x - k\theta y = 0$ respectively, we have

$$\theta = \frac{a}{b}\frac{(\phi+\alpha)^q}{(\phi+\beta)^q},\quad k\theta = \frac{a'}{b'}\frac{(\phi'+\alpha')^q}{(\phi'+\beta')^q},$$

and thence

$$\frac{a}{b}\frac{(\phi+\alpha)^q}{(\phi+\beta)^q} = \frac{1}{k}\frac{a'}{b'}\frac{(\phi'+\alpha')^q}{(\phi'+\beta')^q};$$

so that, extracting the qth root of each side, we have, in q different ways corresponding to the q values of the radical $\left(\frac{1}{k}\frac{ba'}{b'a}\right)^{\frac{1}{q}}$, a relation of the form $\phi' = \frac{l\phi + m}{n\phi + r}$; and considering ϕ' as having this value, the points K, K' as given by the equations

$$z = 0,\quad x : y : w = a(\phi+\alpha)^q : b(\phi+\beta)^q : d(\phi+\delta)^q,$$

and

$$w = 0,\quad x : y : z = a'(\phi'+\alpha')^q : b'(\phi'+\beta')^q : c'(\phi'+\gamma')^q,$$

respectively, correspond as single points to each other. We have thus in q different ways a series of corresponding points K, K', and consequently q series of lines KK' each of them generating a scroll which (as the order of the scroll generated by all the q series is $= 2q^2$), must be each of them of the order $2q$; and the decomposition in question is thus explained.

25. In the scroll of the order $2q^2$, each directrix is a q-tuple line, and the complete section by the plane of the directrix is made up of the directrix q times (order q^2), and of q^2 generating lines, in fact, of q q-fold generating lines: to show that this is so, consider the directrix in the plane $z = 0$, viz. the equation of this is $Ax^{\frac{1}{q}} + By^{\frac{1}{q}} + Dw^{\frac{1}{q}} = 0$. Writing herein $w = 0$, we have $Ax^{\frac{1}{q}} + By^{\frac{1}{q}} = 0$, that is, $A^q x - (-)^q B^q y = 0$; it is clear that the rationalised equation must reduce itself to $\{A^q x - (-)^q B^q y\}^q = 0$, and that the line $w = 0$, is thus a tangent of q-pointic intersection at the point $w = 0$, $A^q x - (-)^q B^q y = 0$. Taking K at this point we have, in each of the scrolls of the order $2q$, q coincident positions of K', that is, a q-fold line KK' in the plane $w = 0$; and the like for the plane $z = 0$, so that the total section by the plane $z = 0$ is made up of the directrix q times and of q q-fold generating lines; and it follows that for each of the scrolls of the order $2q$, the section by the plane $z = 0$ is made up of the directrix once, and of a q-fold generating line.

26. It is easy to see that in the general case $r = +\frac{p}{q}$, the like conclusion holds; for the scroll of the order $2p^2q^2$, the section by the plane of the directrix consists of the directrix pq times (order p^2q^2), and of p^2q q-fold generating lines: whence for each of the q component scrolls of the order $2p^2q$, the section is made up of the directrix p times (that is, the directrix is a p-tuple line on the scroll) and of p^2 q-fold generating lines.

27. In the case $r=-\frac{p}{q}$, where the order of the directrix is $=2pq$, then in the scroll of the order $2p^2q^2$, the directrix is a pq-tuple line on the scroll, and taken pq times it constitutes the complete section by the plane of the directrix; whence in each of the q component scrolls of the order $2p^2q$, the directrix is a p-tuple line; and taken p times it constitutes the complete section by the plane of the directrix.

28. It is convenient to exhibit the foregoing results in a tabular form as follows:

$r=+\frac{p}{q}$.	$r=-\frac{p}{q}$.
Each directrix is of order pq.	Each directrix is of order $2pq$, with three pq-tuple points.

Scroll belonging to two directrices, and a given value of k, is of the order

$2p^2q^2$, breaking up into q scrolls each of order $2p^2q$, each which scroll of the order $2p^2q$ has each directrix for a p-tuple line and has besides p^2 q-fold generating lines in the plane of the directrix.	$2p^2q^2$, breaking up into q scrolls each of the order $2p^2q$, each which scroll of the order $2p^2q$ has each directrix for a p-tuple line, and consequently no generating line in the plane of the directrix.

Considering two directrices and a given third directrix,

k has pq values.	k has $2pq$ values.

Total scroll for the three directrices is made up of

pq scrolls each of order $2p^2q$ (viz. one for each value of k), and residuary scroll of order $2p^3(q^3-q^2)$.	$2pq$ scrolls each of order $2p^2q$ (viz. one for each value of k), and residuary scroll of order $4p^3(q^3-q^2)$.

29. The following are noticeable cases; $r=1$ gives the hyperboloid as derived from three directrix lines; $r=-1$ the hyperboloid as derived from three plane sections thereof; $r=2$, an octic surface, M. De la Gournerie's Quadrispinal; $r=-2$, an octic surface, his Quadricuspidal; $r=\frac{1}{3}$, a sextic surface which (as remarked by Dr Salmon), on writing therein (x^2, y^2, z^2, w^2), in place of (x, y, z, w), is converted into a surface of the twelfth order, locus of the centres of curvature of an ellipsoid.