421.

NOTE ON THE SOLVIBILITY OF EQUATIONS BY MEANS OF RADICALS.

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IN regard to the theorem that the general quintic equation of the nth order is not solvible by radicals, I believe that the proofs which have been given depend, or at any rate that a proof may be given that shall depend, on the following two lemmas:

I. A one-valued (or symmetrical) function of n letters is a perfect kth power, only when the kth root is a one-valued function of the n letters.

There is an exception in the case $k=2$, whatever be the value of n: viz. the product of the squares of the differences is a one-valued function, a perfect square; but its square root, or the product of the simple differences, is a two-valued function. It is in virtue of this exception that a quadric equation is solvible by radicals; we have the one-valued function $(x_1-x_2)^2$, the square of a two-valued function x_1-x_2, and thence the two roots are each expressible in the form

$$\tfrac{1}{2}\{x_1+x_2+\sqrt{(x_1-x_2)^2}\}.$$

II. A two-valued function of n letters is a perfect kth power, only when the kth root is a two-valued function of the n letters.

There is an exception in the case $k=3$, when $n=3$ or 4: viz. for $n=3$ we have $(x_1+\omega x_2+\omega^2 x_3)^3$ (ω an imaginary cube root of unity) a two-valued function, and a perfect cube; whereas its cube root is the six-valued function $x_1+\omega x_2+\omega^2 x_3$. And similarly for $n=4$ we have, for instance,

$$\{x_1x_2+x_3x_4+\omega(x_1x_3+x_2x_4)+\omega^2(x_1x_4+x_2x_3)\}^3$$

a two-valued function, and a perfect cube, whereas its cube root is a six-valued function. And it is in virtue of this exception that a cubic or a quartic equation is solvible by radicals. But I assume that for $n>4$ the lemma is true without exception.

The course of demonstration would be something as follows: Imagine, if possible, the root of an equation expressed, by means of radicals, in terms of the coefficients; the expression cannot contain any radical such as $\sqrt[p]{X}$, $p>2$, where X is a one-valued (or rational) function of the coefficients, not a perfect pth power, for the reason that, expressing the coefficients in terms of the roots, such function $\sqrt[p]{X}$ is not a rational function of the roots; if it were so, by lemma I. it would be a one-valued (that is, a symmetrical) function of the roots; consequently a rational function of the coefficients, or X expressed in terms of the coefficients, would be a perfect pth power.

The expression may however contain a radical $\sqrt{X}$, X a one-valued (or rational) function of the coefficients, not a perfect square: viz. X may be any square function multiplied into that function of the coefficients which is equal to the product of the squared differences of the roots, or, say, multiplied into the discriminant; that is, we may have $X=Q^2\nabla$, or $\sqrt{X}=Q\sqrt{\nabla}$.

We have next to consider whether the expression can contain any radical $\sqrt[p]{X}$, where X, not being a rational function of the coefficients, is a function expressible by radicals. But the foregoing reasoning shows that if this be so, X cannot contain any radical other than the radical $\sqrt{Q^2\nabla}$ or $Q\sqrt{\nabla}$, as above; that is, X must be $=P+Q\sqrt{\nabla}$, where P and Q are rational functions of the coefficients, and where we may assume that $P+Q\sqrt{\nabla}$ is not a perfect pth power of a function of the like form $P'+Q'\sqrt{\nabla}$. But then, expressing the coefficients in terms of the roots, we have $P+Q\sqrt{\nabla}$, a (rational) two-valued function of the roots; and there is no radical $\sqrt[p]{P+Q\sqrt{\nabla}}$, which is a rational function of the roots; for by lemma II., if such radical existed we should have $\sqrt[p]{P+Q\sqrt{\nabla}}$ a (rational) two-valued function of the roots; that is, it would be $=P'+Q'\sqrt{\nabla}$, P' and Q' one-valued (symmetrical) functions of the roots, consequently rational functions of the coefficients; or $P+Q\sqrt{\nabla}$ would be a perfect pth power $(P'+Q'\sqrt[p]{\nabla})^p$.

The conclusion is that for $n>4$ there is not (besides the function $P+Q\sqrt{\nabla}$) any function of the coefficients, expressible by means of radicals, which, when the coefficients are expressed in terms of the roots, will be a rational function of the roots, and consequently there is no possibility of expressing the roots in terms of the coefficients by means of radicals.

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