694.

DESIDERATA AND SUGGESTIONS.

No. 1. The theory of groups.

[From the *American Journal of Mathematics*, t. I. (1878), pp. 50—52.]

SUBSTITUTIONS, and (in connexion therewith) groups, have been a good deal studied; but only a little has been done towards the solution of the general problem of groups. I give the theory so far as is necessary for the purpose of pointing out what appears to me to be wanting.

Let $\alpha, \beta, \ldots$ be functional symbols, each operating upon one and the same number of letters and producing as its result the same number of functions of these letters; for instance, $\alpha(x, y, z) = (X, Y, Z)$, where the capitals denote each of them a given function of (x, y, z).

Such symbols are susceptible of repetition and of combination;

$$\alpha^2(x, y, z) = \alpha(X, Y, Z), \text{ or } \beta\alpha(x, y, z) = \beta(X, Y, Z),$$

= in each case three given functions of (x, y, z); and similarly for α^3, $\alpha^2\beta$, &c.

The symbols are not in general commutative, $\alpha\beta$ not $=\beta\alpha$; but they are associative, $\alpha\beta\,.\,\gamma = \alpha\,.\,\beta\gamma$, each $=\alpha\beta\gamma$, which has thus a determinate signification.

The associativeness of such symbols arises from the circumstance that the definitions of $\alpha, \beta, \gamma, \ldots$ determine the meanings of $\alpha\beta$, $\alpha\gamma$, &c.: if $\alpha, \beta, \gamma, \ldots$ were quasi-quantitative symbols such as the quaternion imaginaries i, j, k, then $\alpha\beta$ and $\beta\gamma$ might have by definition values δ and ϵ such that $\alpha\beta\,.\,\gamma$ and $\alpha\,.\,\beta\gamma$ ($=\delta\gamma$ and $\alpha\epsilon$ respectively) have unequal values.

Unity as a functional symbol denotes that the letters are unaltered, $1(x, y, z) = (x, y, z)$; whence $1\alpha = \alpha 1 = \alpha$.

The functional symbols *may* be substitutions; $\alpha(x, y, z) = (y, z, x)$, the same letters in a different order: substitutions can be represented by the notation $\alpha = \frac{yzx}{xyz}$, the substitution which changes xyz into yzx, or as products of cyclical substitutions, $\alpha = \frac{yzx\,wu}{xyz\,uw}$, $= (xyz)(uw)$, the product of the cyclical interchanges x into y, y into z, and z into x; and u into w, w into u.

A set of symbols $\alpha, \beta, \gamma, \ldots$, such that the product $\alpha\beta$ of each two of them (in each order, $\alpha\beta$ or $\beta\alpha$), is a symbol of the set, is a group. It is easily seen that 1 is a symbol of every group, and we may therefore give the definition in the form that a set of symbols, $1, \alpha, \beta, \gamma, \ldots$ satisfying the foregoing condition is a group. When the number of the symbols (or terms) is $= n$, then the group is of the nth order; and each symbol α is such that $\alpha^n = 1$, so that a group of the order n is, in fact, a group of symbolical nth roots of unity.

A group is defined by means of the laws of combination of its symbols: for the statement of these we may either (by the introduction of powers and products) diminish as much as may be the number of independent functional symbols, or else, using distinct letters for the several terms of the group, employ a square diagram as presently mentioned.

Thus, in the first mode, a group is $1, \beta, \beta^2, \alpha, \alpha\beta, \alpha\beta^2$ $(\alpha^2 = 1,\ \beta^3 = 1,\ \alpha\beta = \beta^2\alpha)$; where observe that these conditions imply also $\alpha\beta^2 = \beta\alpha$.

Or, in the second mode, calling the same group $(1, \alpha, \beta, \gamma, \delta, \epsilon)$, the laws of combination are given by the square diagram

	1	α	β	γ	δ	ϵ
1	1	α	β	γ	δ	ϵ
α	α	1	γ	β	ϵ	δ
β	β	ϵ	δ	α	1	γ
γ	γ	δ	ϵ	1	α	β
δ	δ	γ	1	ϵ	β	α
ϵ	ϵ	β	α	δ	γ	1

,

for the symbols $(1, \alpha, \beta, \gamma, \delta, \epsilon)$ are in fact $= (1, \alpha, \beta, \alpha\beta, \beta^2, \alpha\beta^2)$.

The general problem is to find all the groups of a given order n; thus if $n = 2$, the only group is $1, \alpha$ $(\alpha^2 = 1)$; if $n = 3$, the only group is $1, \alpha, \alpha^2$ $(\alpha^3 = 1)$; if $n = 4$, the groups are $1, \alpha, \alpha^2, \alpha^3$ $(\alpha^4 = 1)$, and $1, \alpha, \beta, \alpha\beta$ $(\alpha^2 = 1,\ \beta^2 = 1,\ \alpha\beta = \beta\alpha)$; if $n = 6$, there are three groups, a group $1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5$ $(\alpha^6 = 1)$, and two groups $1, \beta, \beta^2, \alpha, \alpha\beta,$

$\alpha\beta^2$ ($\alpha^2 = 1$, $\beta^3 = 1$); viz. in the first of these $\alpha\beta = \beta\alpha$, while in the other of them (that mentioned above) we have $\alpha\beta = \beta^2\alpha$, $\alpha\beta^2 = \beta\alpha$.

But although the theory as above stated is a general one, including as a particular case the theory of substitutions, yet the general problem of finding all the groups of a given order n, is really identical with the apparently less general problem of finding all the groups of the same order n, which can be formed with the substitutions upon n letters; in fact, referring to the diagram, it appears that 1, α, β, γ, δ, ϵ may be regarded as substitutions performed upon the six letters 1, α, β, γ, δ, ϵ, viz. 1 is the substitution unity which leaves the order unaltered, α the substitution which changes $1\alpha\beta\gamma\delta\epsilon$ into $\alpha 1\gamma\beta\epsilon\delta$, and so for β, γ, δ, ϵ. This, however, does not in any wise show that the best or the easiest mode of treating the general problem is thus to regard it as a problem of substitutions: and it seems clear that the better course is to consider the general problem in itself, and to deduce from it the theory of groups of substitutions.

Cambridge, 26th November, 1877.

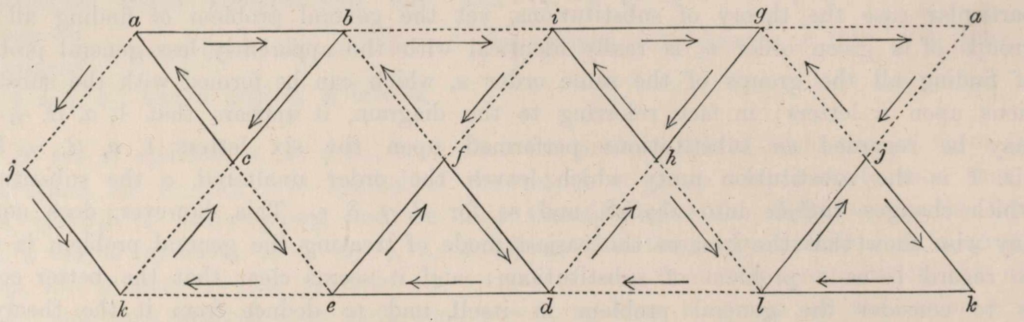
No. 2. The theory of groups; graphical representation.

[From the *American Journal of Mathematics*, t. I. (1878), pp. 174—176.]

In regard to a substitution-group of the order n upon the same number of letters, I omitted to mention the important theorem that every substitution is *regular* (that is, either cyclical or composed of a number of cycles each of them of the same order). Thus, in the group of 6 given in No. 1, writing a, b, c, d, e, f in place of 1, α, β, γ, δ, ϵ, the substitutions of the group are 1, $ace \,.\, bfd$, $aec \,.\, bdf$, $ab \,.\, cd \,.\, ef$, $ad \,.\, be \,.\, cf$, $af \,.\, bc \,.\, de$.

Let the letters be represented by points; a change a into b will be represented by a directed line (line with an arrow) joining the two points; and therefore a cycle abc, that is, a into b, b into c, c into a, by the three sides of the trilateral abc, with the three arrows pointing accordingly, and similarly for the cycles $abcd$, &c.: the cycle ab means a into b, b into a, and we have here the line ab with a two-headed arrow pointing both ways; such a line may be regarded as a bilateral. A substitution is thus represented by a multilateral or system of multilaterals, each side with its arrow; and in the case of a regular substitution the multilaterals (if more than one) have each of them the same number of sides. To represent two or more substitutions we require different colours, the multilaterals belonging to any one substitution being of the same colour.

In order to represent a group we need to represent only independent substitutions thereof; that is, substitutions such that no one of them can be obtained from the others by compounding them together in any manner. I take as an example a group of the order 12 upon 12 letters, where the number of independent substitutions is $=2$. See the diagram, wherein the continuous lines represent black lines, and the dotted lines, red lines.



The diagram is drawn, in the first instance, with the arrows but without the letters, which are then affixed *at pleasure;* viz. the *form of the group* is quite independent of the way in which this is done, though the group itself is of course dependent upon it. The diagram shows two substitutions, each of them of the third order: one is represented by the black triangles, and the other by the dotted triangles. It will be observed that there is *from* each point of the diagram (that is, in the direction of the arrow) one and only one black line, and one and only one dotted line; hence a symbol B, "move along a black line," B^2, "move successively along two black lines," BR (read always from right to left), "move first along a dotted line and then along a black line," has in every case a perfectly definite meaning and determines the path when the initial point is given; any such symbol may be spoken of as a "route."

a	b	c	d	e	f	g	h	i	j	k	l	1
b	c	a	e	f	d	h	i	g	k	l	j	abc . def . ghi . jkl (= B)
c	a	b	f	d	e	i	g	h	l	j	k	acb . dfe . gih . jlk
d	l	h	a	g	j	e	c	k	f	i	b	ad . bl . ch . eg . fj . ik
e	j	i	b	h	k	f	a	l	d	g	c	aeh . bjd . cil . fkg
f	k	g	c	i	l	d	b	j	e	h	a	afl . bkh . cgd . eij
g	f	k	l	c	i	j	d	b	a	e	h	agj . bfi . cke . dlh
h	d	l	j	a	g	k	e	c	b	f	i	ahe . bdj . cli . fgk
i	e	j	k	b	h	l	f	a	c	d	g	ai . be . cj . dk . fh . gl
j	i	e	h	k	b	a	l	f	g	c	d	ajg . bif . cek . dhl (= R)
k	g	f	i	l	c	b	j	d	h	a	e	ak . bg . cf . di . el . hj
l	h	d	g	j	a	c	k	e	i	b	f	alf . bhk . cdg . eji

The diagram has a remarkable property, *in virtue whereof it in fact represents a group.* It may be seen that any route leading from some one point a to itself, leads also from every other point to itself, or say from b to b, from c to $c, \ldots$, and from l to l. We hence see that a route, applied in succession to the whole series of initial points or letters *abcdefghijkl*, gives a new arrangement of these letters, wherein no one of them occupies its original place; a route is thus, in effect, a substitution. Moreover, we may regard as distinct routes, those which lead from a to a, to b, to $c, \ldots$, to l, respectively. We have thus 12 substitutions (the first of them, which leaves the arrangement unaltered, being the substitution unity), and these 12 substitutions form a group. I omit the details of the proof; it will be sufficient to give the square obtained by means of the several routes, or substitutions, performed upon the primitive arrangement *abcdefghijkl*, and the cyclical expressions of the substitutions themselves: it will be observed that the substitutions are unity, 3 substitutions of the order (or index) 2, and 8 substitutions of the order (or index) 3.

It may be remarked that the group of 12 is really the group of the 12 positive substitutions upon 4 letters *abcd*, viz. these are 1, *abc*, *acb*, *abd*, *adb*, *acd*, *adc*, *bcd*, *bdc*, *ab . cd*, *ac . bd*, *ad . bc*.

Cambridge, 16*th May*, 1878.

No. 3. The Newton-Fourier imaginary problem.

[From the *American Journal of Mathematics*, t. II. (1879), p. 97.]

The Newtonian method as completed by Fourier, or say the Newton-Fourier method, for the solution of a numerical equation by successive approximations, relates to an equation $f(x) = 0$, with real coefficients, and to the determination of a certain real root thereof a by means of an assumed approximate real value ξ satisfying prescribed conditions: we then, from ξ, derive a nearer approximate value ξ_1 by the formula $\xi_1 = \xi - \dfrac{f(\xi)}{f'(\xi)}$; and thence, in like manner, $\xi_1, \xi_2, \xi_3, \ldots$ approximating more and more nearly to the required root a.

In connexion herewith, throwing aside the restrictions as to reality, we have what I call the Newton-Fourier Imaginary Problem, as follows.

Take $f(u)$, a given rational and integral function of u, with real or imaginary coefficients; ξ, a given real or imaginary value, and from this derive ξ_1 by the formula $\xi_1 = \xi - \dfrac{f(\xi)}{f'(\xi)}$, and thence $\xi_1, \xi_2, \xi_3, \ldots$ each from the preceding one by the like formula.

A given imaginary quantity $x + iy$ may be represented by a point the coordinates of which are (x, y): the roots of the equation are thus represented by given points

A, B, C, ..., and the values ξ, ξ_1, ξ_2, ... by points P, P_1, P_2, ... the first of which is assumed at pleasure, and the others each from the preceding one by the like given geometrical construction. The problem is to determine the regions of the plane such that, P being taken at pleasure anywhere within one region, we arrive ultimately at the point A; anywhere within another region at the point B; and so for the several points representing the roots of the equation.

The solution is easy and elegant in the case of a quadric equation: but the next succeeding case of the cubic equation appears to present considerable difficulty.

Cambridge, March 3rd, 1879.

No. 4. The mechanical construction of conformable figures.

[From the *American Journal of Mathematics*, t. II. (1879), p. 186.]

Is it possible to devise an apparatus for the mechanical construction of conformable figures; that is, figures which are similar as regards corresponding infinitesimal areas? The problem is to connect mechanically two points P_1 and P_2 in such wise that P_1 (1) shall have two degrees of freedom (or be capable of moving over a plane area) its position always determining that of P_2: (2) that if P_1, P_2 describe the infinitesimal lengths P_1Q_1, P_2Q_2, then the ratio of these lengths, and their mutual inclination, shall depend upon the position of P_1, but be independent of the direction of P_1Q_1: or what is the same thing, that if P_1 describe uniformly an indefinitely small circle, then P_2 shall also describe uniformly an indefinitely small circle, the ratio of the radii, and the relative position of the starting points in the two circles respectively, depending on the position of P_1.

Of course a pentagraph is a solution, but the two figures are in this case similar; and this is not what is wanted. Any unadjustable apparatus would give one solution only: the complete solution would be by an apparatus containing, suppose, a flexible lamina, so that P_1 moving in a given right line, the path of P_2 could be made to be any given curve whatever.

Cambridge, July 9th, 1879.