690.

ON THE THEORY OF GROUPS.

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I RECAPITULATE the general theory so far as is necessary in order to render intelligible the quasi-geometrical representation of it which will be given.

Let α, β,.. be functional symbols each operating upon one and the same number of letters, and producing as its result the same number of functions of these letters. For instance, $\alpha(x, y, z) = (X, Y, Z)$, where the capitals denote each of them a given function of (x, y, z).

Such symbols are susceptible of repetition and combination;

$$\alpha^2(x, y, z) = \alpha(X, Y, Z),$$

or

$$\beta\alpha(x, y, z) = \beta(X, Y, Z),$$

in each case equal to three given functions of (x, y, z); and similarly for α^3, $\alpha^2\beta$, etc.

The symbols are not in general commutative, $\alpha\beta$ not $=\beta\alpha$; but they are associative, $\alpha\beta \, . \, \gamma = \alpha \, . \, \beta\gamma$, each $=\alpha\beta\gamma$, which has thus a determinate meaning.

Unity as a functional symbol denotes that the letters are unaltered,

$$1(x, y, z) = (x, y, z);$$

whence

$$1\alpha = \alpha 1 = \alpha.$$

The functional symbols *may* be substitutions; $\alpha(x, y, z) = (y, z, x)$, the same letters in a different order. Substitutions can be represented by the notation $\dfrac{yzx}{zxy}$, the substitution which changes xyz into yzx, or, as products of cyclical substitutions, $\alpha = \dfrac{yzx\ wvu}{xyz\ uvw}$, $=(xyz)(uw)$, the product of the cyclical substitutions x into y, y into z, z into x, and u into w, w into u, the letter v being unaltered.

A set of symbols α, β, γ,.., such that the product $\alpha\beta$ of each two of them (in each order, $\alpha\beta$ and $\beta\alpha$) is a symbol of the set, is a group. It is easily seen that 1 is a symbol of every group, and we may therefore give the definition in the form that a set of symbols 1, α, β, γ,.. satisfying the foregoing condition is a group. When the number of symbols (or terms) is $=n$, then the group is of the order n; and each symbol α is such that $\alpha^n = 1$, so that a group of the order n is in fact a group of symbolical nth roots of unity.

A group is defined by means of the laws of combinations of its symbols. For the statement of these we may either (by the introduction of powers and products) diminish as much as may be the number of distinct functional symbols; or else, using distinct letters for the several terms of the group, employ a square diagram, as presently mentioned.

Thus, in the first mode, a group is 1, β, β^2, α, $\alpha\beta$, $\alpha\beta^2$, ($\alpha^2=1$, $\beta^3=1$, $\alpha\beta=\beta^2\alpha$), where observe that these conditions imply also $\alpha\beta^2=\beta\alpha$.

Or in the second mode, calling the symbols (1, α, β, $\alpha\beta$, β^2, $\alpha\beta^2$) of the same group (1, α, β, γ, δ, ϵ), or, if we please, (a, b, c, d, e, f), the laws of combination are given by one or other of the square diagrams:

	1	α	β	γ	δ	ϵ
1	1	α	β	γ	δ	ϵ
α	α	1	γ	β	ϵ	δ
β	β	ϵ	δ	α	1	γ
γ	γ	δ	ϵ	1	α	β
δ	δ	γ	1	ϵ	β	α
ϵ	ϵ	β	α	δ	γ	1

,

a	b	c	d	e	f
b	a	d	c	f	e
c	f	e	b	a	d
d	e	f	a	b	c
e	d	a	f	c	b
f	c	b	e	d	a

where, taking for greater symmetry the second form of the square, observe that the square is such that no letter occurs twice in the same line, or in the same column (or what is the same thing, each of the lines and of the columns contains all the letters). But this is not sufficient in order that the square may represent a group; the square must be such that the substitutions by means of which its several lines are derived from any line thereof are (in a different order) the same substitutions by which the lines are derived from a particular line, or say from the top line. These, in fact, are:

$$
\begin{array}{l}
1 \\
ab\,.\,cd\,.\,ef, \\
ace\;\;.\,bfd, \\
ad\,.\,be\,.\,cf, \\
aec\;\;.\,bdf, \\
af\,.\,bc\,.\,de,
\end{array}
$$

where, for shortness, *ab*, *ace*, &c., are written instead of (*ab*), (*ace*), &c., to denote the cyclical substitutions *a* into *b*, *b* into *a*; and *a* into *c*, *c* into *e*, *e* into *a*, &c.; and it is at once seen that by the same substitutions the lines may be derived from any other line.

It will be noticed that in the foregoing substitution-group each substitution is *regular*, that is, composed of cyclical substitutions each of the same number of letters; and it is easy to see that this property is a general one; each substitution of the substitution-group must be regular.

By what precedes, the group of any order composed of the functional symbols is replaced by a substitution-group upon a set of letters the number of which is equal to the order of the group, and wherein all the substitutions are regular.

The general theory being thus explained, I endeavour to form a substitution-group with the twelve letters *abcdefghijkl*; and I assume that there is one substitution, such as *abc.def.ghi.jkl*, and another substitution, such as *agj.bfi.cek.dhl*. Observe that, if the twelve letters are to be thus arranged in two different ways as a set of four triads, without repetition of any duad, all the ways in which this can be done are essentially similar, and there is no loss of generality in taking the two sets of triads to be those just written down. But the substitution to be formed with either set of triads will be different according as any triad thereof, for instance *agj*, is written in this form or in the reversed form *ajg*. There are thus in all sixteen substitutions which can be formed with the first set of triads, and sixteen substitutions which can be formed with the second set of triads; and the relation of a triad of the first set to a triad of the second set is by no means independent of the selection of the triads out of the two sets respectively. To show this, take the two substitutions quite at random; suppose they are those written down above, say

$$\alpha = abc\,.\,def\,.\,ghi\,.\,jkl, \quad \beta = agj\,.\,bfi\,.\,cek\,.\,dhl;$$

and perform these in succession on the primitive arrangement $\Omega = abcdefghijkl$. The operation stands thus:

$$\beta\alpha\Omega = fegkihlbjcda,$$

$$\alpha\Omega = bcaefdhigklj,$$

$$\Omega = abcdefghijkl,$$

whence

$$\beta\alpha, = afhbeijcgl\,.\,dk,$$

is not a regular substitution; and, by what precedes, α, β cannot belong to a group.

But take the substitutions to be

$$\alpha \text{ (as before)} = abc\,.\,def\,.\,ghi\,.\,jkl, \quad \beta = ajg\,.\,bif\,.\,cek\,.\,dhl,$$

then we have

$$\beta\alpha\Omega = iejkbhlfacdg,$$

$$\alpha\Omega = bcaefdhigklj,$$

$$\Omega = abcdefghijkl,$$

whence

$$\beta\alpha = ai\,.\,be\,.\,cj\,.\,dk\,.\,fh\,.\,gl,$$

a regular substitution; and, for anything that appears to the contrary, α, β may belong to a group. It is convenient to mention at once that these two substitutions do, in fact, give rise to a group; viz. the square diagram is

a	b	c	d	e	f	g	h	i	j	k	l
b	c	a	e	f	d	h	i	g	k	l	j
c	a	b	f	d	e	i	g	h	l	j	k
d	l	h	a	g	j	e	c	k	f	i	b
e	j	i	b	h	k	f	a	l	d	g	c
f	k	g	c	i	l	d	b	j	e	h	a
g	f	k	l	c	i	j	d	b	a	e	h
h	d	l	j	a	g	k	e	c	b	f	i
i	e	j	k	b	h	l	f	a	c	d	g
j	i	e	h	k	b	a	l	f	g	c	d
k	g	f	i	l	c	b	j	d	h	a	e
l	h	d	g	j	a	c	k	e	i	b	f

and the substitutions, obtained therefrom by writing successively each line over the top line, are

$$\begin{array}{ll}
1 & =1, \\
abc\,.\,def\,.\,ghi\,.\,jkl & \alpha, \\
acb\,.\,dfe\,.\,gih\,.\,jlk & \alpha^2, \\
ad\,.\,bl\,.\,ch\,.\,eg\,.\,fj\,.\,ik & \beta^2\alpha\beta^2, \\
aeh\,.\,bjd\,.\,cil\,.\,fkg & \beta\alpha^2, \\
afl\,.\,bkh\,.\,cgd\,.\,eij & \beta^2\alpha, \\
agj\,.\,bfi\,.\,cke\,.\,dlh & \beta^2, \\
ahe\,.\,bdj\,.\,cli\,.\,fgk & \beta\alpha^2\beta\alpha^2, \\
ai\,.\,be\,.\,cj\,.\,dk\,.\,fh\,.\,gl & \beta\alpha, \\
ajg\,.\,bif\,.\,cek\,.\,dhl & \beta, \\
ak\,.\,bg\,.\,cf\,.\,di\,.\,el\,.\,hj & \beta^2\alpha^2, \\
alf\,.\,bkh\,.\,cdg\,.\,eji & \beta^2\alpha\beta^2\alpha.
\end{array}$$

To explain the theory, I introduce the notion of a hemipolyhedron, or say a hemihedron, viz. this is a figure obtained from a polyhedron by the removal of certain faces. In a polyhedron each edge occurs twice (more properly it occurs in the two forms *ab* and *ba*), as belonging to two faces; but in a hemihedron one of these faces must always be removed, so that the edge may occur once only; and again (what is apparently, although not really, a different thing), we may remove two intersecting faces, leaving their edge of intersection; this edge is, in fact, then considered as a bilateral face $ab = ab \,.\, ba$, just as *abc* is a trilateral face $abc = ab \,.\, bc \,.\, ca$. Thus, if in a prism we remove the lateral faces, leaving the lateral edges, and leaving also the terminal faces, we have a hemihedron: thus, the prism being trilateral, say the faces of the hemihedron are *abc*, *def*, *ad*, *be*, *cf*, where *ad*, *be*, *cf* are the edges regarded as bilateral faces. And, for the present purpose, *abc* denotes the cyclical substitution *a* into *b*, *b* into *c*, *c* into *a*; and *ad* denotes in like manner the cyclical substitution (or interchange) *a* into *d*, *d* into *a*.

But the hemihedron about to be considered has no bilateral faces; it is, in fact, the figure composed of the 8 triangular faces of the octo-hexahedron or figure obtained by truncating the summits of a hexahedron (or of an octahedron) so as to obtain a polyhedron of 8 triangular faces and 6 square faces, representing the faces of the octahedron and the hexahedron respectively. The faces of the octo-hexahedron may be taken to be

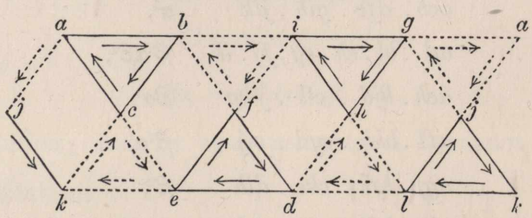
abc, *def*, *ghi*, *jkl*,

ajg, *bif*, *cek*, *dhl*,

cbfe, *fihd*, *hgjl*, *jack*, *agib*, *klde*,

(where I observe in passing that the symbols are written in such manner that each edge *ab* occurs under the two opposite forms *ab* in *abc* and *ba* in *agib*). And then, omitting the square faces, represented by the third line, we have the hemihedron, wherein as before *abc* denotes the cyclical substitution *a* into *b*, *b* into *c*, *c* into *a*; and so for the other faces.

I represent this by a diagram, the lines of which were red and black, and they



will be thus spoken of, but the black lines are in the woodcut continuous lines, and the red lines broken lines: each face indicates a cyclical substitution, as shown by the arrows. The figure should be in the first instance drawn with the arrows, but without the letters, and these may then be affixed to the several points in a perfectly arbitrary manner; but I have in fact affixed them in such wise that the group given by the diagram, as presently appearing, may (instead of being any other equivalent group) be that group which contains the before-mentioned substitution

$$\alpha = abc \, . \, def \, . \, ghi \, . \, jkl, \text{ and } \beta = ajg \, . \, bif \, . \, cek \, . \, dhl.$$

Observe that in the diagram, considering the lines to be drawn as shown by the arrows, there is *from* any given point whatever only one black line, and only one red line. Let B denote motion along a black line, R motion along a red line (always from a point to the next point); then B^2 will denote motion along two black lines successively, BR (any such symbol being read always from right to left) will denote motion first along a red line, and then along a black line, and so in other cases; a symbol or "route" $\ldots R^\beta B^\alpha$ has thus a perfectly definite signification, determining the path when the initial point is given.

The diagram has the property that every route, leading from any one letter to itself, leads also from every other letter to itself; or say a route leading from a to a, leads also from b to b, from c to $c, \ldots,$ from l to l; and we can thus in the diagram speak absolutely (that is, without restriction as to the initial point) of a route as leading from a point to itself, or say as being equal to unity; *it is in virtue of this property* that the diagram gives a group.

For, assuming the property, it at once follows (1) that two routes, each leading say from the point a to the same point f, lead also from any other point b to one and the same point g. Such routes are said to be equivalent, or equal to each other; and the number of distinct routes (including the route unity) is thus equal to the numbers of the letters, viz. we have only the routes from a to a, to $b, \ldots,$ to l, respectively; (2) a route, leading from a point a to a point f, leads from any other point b to a different point g; and (3) two routes, leading from the same point a to different points b and c, lead also from any other point f to different points k and l. Hence a given route leads from the several points $abc \ldots l$ successively to the same series of points taken in a different order, or we thus obtain a new arrangement of the points; and dealing in this manner successively with the routes from a to a, to $b, \ldots,$ to l, we obtain so many distinct arrangements, beginning with the letters $a, b, c, \ldots, l$ respectively, such that in no two of them does the same letter occupy the same place; we thus obtain a square of 12 such as that already written down, and which is, in fact, the same square, the several routes of course corresponding to the substitutions of the square. The hemihedron thus gives the foregoing group of 12.

Observe that the diagram is composed of the four black triangles representing the substitution $abc \, . \, def \, . \, ghi \, . \, jkl$, and of the four red triangles representing the substitution $ajg \, . \, bif \, . \, cek \, . \, dhl$; viz. these are independent substitutions which by their powers and products serve to express all the substitutions of the group; that they are sufficient appears by the diagram itself, in that every point thereof is (by black and red lines) connected with every other point thereof. The group might have contained three or more independent substitutions, and the diagram would then have contained the like number of differently coloured sets of lines. The essential characters are that the lines of any given colour shall form polygons of the same number of sides (but for different colours the polygons may have different numbers of sides; in particular, for any given colour or colours, the polygons may be bilaterals, represented each by a line with a double arrow pointing opposite ways); that there shall be *from* each point only one line of the same colour; that every point shall be connected with every other point; and finally, that every route leading from one point to itself shall lead also from every other point to itself. When these conditions are satisfied the foregoing investigation in fact shows that the diagram, or say the hemihedron, gives rise to a group.

It may be remarked that we can, if we please, introduce into the diagram a set of lines of a new colour to represent any dependent substitution of the group; thus, in the example considered, a substitution is $aeh . bjd . cil . fkg$, and if we draw these triangles in green (the arrows being from a to e, e to h, h to a, &c.), then there will be *from* each point one black line, one red line, and one green line; any route $\ldots G^{\gamma}R^{\beta}B^{\alpha}$ will thus be perfectly definite, and will have the same properties as a route composed of black and red lines only; and the theory thus subsists without alteration.

I remark, in conclusion, that the group of 12 considered above is, in fact, the group of 12 positive substitutions upon 4 letters $abcd$; viz. the substitutions are 1, abc, acb, abd, adb, acd, adc, bcd, bdc, $ab . cd$, $ac . bd$, $ad . bc$; the groups each contain unity, three substitutions of the order (or index) 2, and 8 substitutions of the order (or index) 3, and their identity can be easily verified.