688.

GEOMETRICAL CONSIDERATIONS ON A SOLAR ECLIPSE.

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I CONSIDER, from a geometrical point of view, the phenomena of a solar eclipse over the earth generally; attending at present only to the penumbral cone, the vertex of which I denote by V. It is convenient to regard the earth as fixed, and the sun and moon as moving each of them with its proper motion, and also with the diurnal motion. The penumbral cone meets the earth's surface in a curve which may be called the penumbral curve; viz. when the cone is not completely traversed by the earth's surface, (that is, when only some of the generating lines of the cone meet the earth's surface), the penumbral curve is a single (convex or hour-glass-shaped) oval; separated, as afterwards mentioned, into two parts, one of them lying away from the sun, and having no astronomical significance; but when the cone is completely traversed by the earth's surface, then the penumbral curve consists of two separate (convex) ovals; one of them lying away from the sun and having no astronomical significance, the other lying towards the sun. The intermediate case is when the cone just traverses the earth's surface, or is touched internally by the earth's surface; the penumbral curve is then a figure of eight, one portion of which lies away from the sun, and has no astronomical significance: there is another limiting case when the cone is touched externally by the earth's surface, the penumbral curve being then a mere point.

It is necessary to consider on the earth's surface a curve which may for shortness be termed the horizon; viz. this is the curve of contact of the cone, vertex V, circumscribed about the earth; it is a small circle nearly coincident with the great circle, which is the intersection by a plane through the centre of the earth at right angles to the line from this point to the centre of the sun.

Regarding V as a point in the heavens, capable of being viewed notwithstanding the interposition of the moon; the horizon, as above defined, is the curve separating the portions of the earth's surface for which V is visible and invisible respectively. The horizon does or does not meet the penumbral curve, according as this last consists of a single oval or of two distinct ovals; viz. in the latter case the horizon lies between the two ovals, in the former case the horizon traverses the area of the oval (separating this area into two parts), thus meeting the oval, or penumbral curve, in two points, or say these points separate the oval into two parts; from any point of the one part V is visible, from any point of the other part V is invisible; and from each of the two points themselves V is visible as a point on the horizon in the ordinary sense of the word; that is, there is an exterior contact of the sun and moon visible on the horizon. It is to be observed that, in the limiting cases where the penumbral curve is a mere point and a figure of eight respectively, the horizon passes through the mere point and through the node of the figure of eight respectively.

The two points of intersection of the penumbral curve with the horizon may for shortness be termed critic points. The lines which present themselves in a diagram of a solar eclipse, (see *Nautical Almanac*) are the "northern and south lines of simple contact," say for shortness the "limits"; viz. these are the envelope or, geometrically, a portion of the envelope of the penumbral curve; and the lines of "eclipse begins or ends at sunrise or sunset," say for shortness the critic lines; viz. these are the locus of the critic points.

The point V considered as a point in the heavens is a point occupying a position intermediate between those of the centres of the sun and moon; hence referring it to the surface of the earth by means of a line drawn from the centre, its position on the earth's surface is nearly coincident with that point to which the sun is then vertical; and its motion on the earth's surface is from east to west approximately along the parallel of latitude = sun's declination, and with a velocity of approximately 15° per hour. For any given position of V on the earth's surface, describing with a given angular radius nearly = 90° a small circle (nearly a great circle), this is the horizon; as V moves upon the surface of the earth, the horizon envelopes a curve which is very nearly a parallel, angular radius = sun's declination (there are two such curves in the northern and southern hemispheres respectively, but I attend only to one of them in the proper hemisphere, as will be explained), say this is the horizon-envelope; the horizon in each of its successive positions is thus a curvilinear tangent (nearly a great circle) to this horizon-envelope. If for a given position of V, and also for the consecutive position we consider the corresponding horizons, these intersect in a point K on the horizon-envelope, and the horizon for V is the circle centre V and angular radius VK; K is a point which is very nearly upon, and which may be taken to be upon, the meridian through V; the horizon may be regarded as a tangent which sweeps round the horizon-envelope; to each position thereof there corresponds a position of V, and consequently also a penumbral curve; and (when this is a single oval) the horizon meets it in two points, which are the critic points. It is to be added that, if for a given position of the horizon we consider as well K as the opposite point K_1, (viz., K_1 lies on the great circle KV), then the points K and K_1 divide the horizon into two portions; for any point on one of these portions

V (considered as a point in the heavens) is rising, for a point on the other of them it is setting; and for the points K and K_1 respectively it is moving horizontally; that is, first rising and then setting, or *vice versâ*.

A solar eclipse is of one of two classes; viz. either the penumbral cone completely traverses the earth, so that towards the middle of the eclipse the penumbral curve consists of two separate ovals: or the penumbral cone does not completely traverse the earth, so that throughout the eclipse the penumbral curve consists of a single oval only. In the former case, we have to consider the commencement, during which the penumbral curve passes from a mere point to a figure of eight: the middle, during which it passes from a figure of eight through two ovals to a figure of eight: and the termination, during which it passes from a figure of eight to a mere point. In the latter case, we consider the whole eclipse during which the penumbral curve passes from a mere point through a single oval to a mere point.

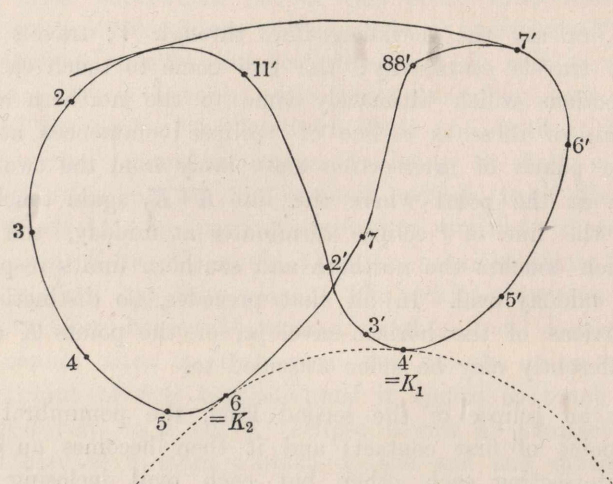
In an eclipse of the first class: for the commencement, the penumbral curve is at first a mere point (point of first contact); it then becomes a convex oval, each oval in the first instance inclosing the preceding ones, so that there is not any intersection of two consecutive ovals. We come at last to an oval which is touched north by its consecutive oval, and to an oval which is touched south by its consecutive oval (I presume that the contacts north and south do not take place on the same oval, but I am not sure); and after this, the ovals assume the hour-glass form, each oval intersecting the consecutive oval in two points north and two points south; the ovals thus beginning to form an envelope or limit. There are on each of the ovals two critic points, and we have thus a critic curve commencing at the mere point (point of first contact) and extending in each direction from this point. The point, where an oval is touched by the consecutive oval, is not so far as appears a critic point; that is, the critic curve does not at this point unite itself with the envelope or limit. But the critic curve comes subsequently to unite itself each way with the limit; and, since clearly it cannot intersect the limit, it will at each of these points touch the limit; that is, we have a critic curve extending each way from the point of first contact until it touches the northern limit and until it touches the southern limit. Observe that the penumbral curve, as being at first a mere point or an indefinitely small oval, does not at first contain within itself the point K or K_1: it can only come to do this by passing through a position where the curve passes through K or K_1; viz. K or K_1 would then be a critic point; and I assume for the present that this does not take place. The critic curve at the point of first contact is a curve "eclipse begins at sunrise," and as not coming to pass through a point or K_1, it cannot alter its character; that is, the critic curve, as extending each way from the point of first contact until it comes to touch the northern and southern limits respectively, is a curve "eclipse begins at sunrise"; at the terminal points in question, there is a mere contact of the sun and moon, so that they are points, where the eclipse begins and simultaneously ends at sunrise. Continuing the series of ovals until we arrive at the figure of eight, there are on each of them two critic points, which ultimately unite in the node of the figure of eight; these constitute a critic curve, extending each way from the node of the figure of eight to the contacts with the northern and southern limits respectively. There is, as before, no passage through a point K or K_1, the curve in question thus retains throughout the same character; and by consideration of the two terminal points it at once appears that it is a curve "eclipse ends at sunrise." The above-mentioned critic curves form together an oval touching the northern and southern limits respectively; say this is the sunrise oval.

The termination of the eclipse is similar to this, only the events happen in the reverse order; we have a critic line starting from the node of the figure of eight and extending each way until it comes to touch the northern and southern limits respectively, viz. this is the line "eclipse begins at sunset"; and then, extending each way from the points of contact to reunite itself at the point of last contact, this being the line "eclipse ends at sunset," and the two portions together form an oval touching the northern and southern limits respectively; say this is the sunset oval. It is to be noticed that certain portions of the two limits are generated as the envelope of the penumbral curve during the commencement and during the termination of the eclipse.

For the middle of the eclipse; the penumbral curve, in the first instance a figure of eight, breaks up into two ovals, but only one of these is attended to; and ultimately the oval unites itself with another oval so as to give rise to a new figure of eight. There is thus throughout the middle of the eclipse a single oval; this has, north and south, an envelope which joins itself on to the portions enveloped during the latter part of the commencement and the former part of the termination of the eclipse, and constitutes therewith the northern and southern limits respectively, viz. each of these is considered as extending from a point of contact with the sunrise oval to a point of contact with the sunset oval.

The line K_1VK, or say the meridian line through V, travels westwardly, while the penumbral curve travels eastwardly; the two come to touch each other, and there are then two intersections which ultimately come to the northern and southern limits respectively: the locus of these is a line of "eclipse commences at midday"; as the motion continues, the points of intersection move away from the two limits respectively and ultimately unite at the point where the line KVK_1 again touches the penumbral curve; the locus is the line of "eclipse terminates at midday," the two lines together forming an oval which touches the northern and southern limits respectively and which may be termed the midday oval. In all that precedes, no distinction has been made between the two portions of the horizon-envelope, or the points K and K_1, and either curve and point indifferently may be alone attended to.

Considering now an eclipse of the second kind, the penumbral curve is at first a mere point (the point of first contact) and it then becomes an oval, the successive ovals not at first intersecting each other, but each oval inclosing within itself the preceding ones. Any oval is met by the corresponding horizon in two points P and P', at first coinciding with each other at the point of first contact, and then separating from each other, one of them, say P, moving down towards and ultimately arriving at one of the horizon-envelopes, say to fix the ideas the southern one (which curve is henceforth selected as being, and is called, the horizon-envelope, and the points on this curve are taken to be the points K), viz. P is then a point K on the penumbral curve, I call it K_1. The successive ovals will in the meantime have begun to intersect each other so as to give rise to a northern limit; this will touch the critic line (locus of P, P'), and we have a portion of the critic line extending from the point of first contact, in one direction to the point of contact with the northern limit, and in the other direction to the point K_1 on the horizon-envelope; this is the line "eclipse begins at sunrise." As the horizon continues to sweep on, the other point P', which has not yet reached the horizon-envelope, will gradually approach and ultimately arrive at the horizon-envelope, say at the point K_2; we have thus a second portion of the critic line extending from the contact with the northern envelope to the point K_2; this is the line "eclipse ends at sunrise." The horizon continuing to sweep on, the point P beginning with the position K_1, which is now on the other side of the point of contact of the horizon with the horizon-envelope, will trace out a portion of the critic curve extending from K_1 to a second point of contact with the northern limit; this will be the line of "eclipse begins at sunset." And, finally, the point P from the last-mentioned point of contact, and the point P' from its position K_2, which is now on the other side of the point of contact of the horizon with the horizon-envelope, (that is, P, P' have now each passed through the point of contact of the horizon with the horizon-envelope, and are both of them on the same side thereof, viz. the side opposite to their original side), will come to unite at the point of the last contact; we have thus a fourth portion of the critic curve extending from K_2 to the second point of contact with the northern limit, viz. this is the line "eclipse ends at sunset." The description will be more intelligible by means of the figure, in which 1, 1′, 2, 2′, ..., 8, 8′ represent successive corresponding



positions of the points P, P', the successive positions of the horizon being given by the right lines 11′, 22′, &c., all of them tangents to the dotted circle or horizon-envelope.

The entire critic line is thus a figure of eight, twice touching the horizon-envelope and also twice touching the limit. If we consider, as before, the intersections of KV with the corresponding penumbral curve, this will be a curve extending from K_1 so as to touch the limit, and thence onward to K_2, the portion from K_1 to the contact with the limit being the line "eclipse begins at transit," and the portion from the limit to K_2 the line "eclipse ends at transit." I say "transit" instead of midday, since for a circumpolar place the phenomenon may happen at one or the other transit of the sun over the meridian. It is to be remarked, that the node of the figure of eight is a point, such that the eclipse there begins at sunrise and ends at sunset; this point does not appear to be an important one in the geometrical theory.

The two loops of the critic line may be of very unequal magnitudes, and in particular one of them may actually vanish; viz. the points K_1 and K_2 then coincide together, and the critic curve is a closed cuspidal curve touching the horizon-envelope at the cusp; moreover, instead of two contacts with the limit there is one proper contact, and an improper contact at the cusp, that is, the limit simply passes through the cusp. And through this special separating case, we pass to the case where, instead of the figure of eight, we have a single oval, not touching the horizon-envelope (viz. the points K_1, K_2 have become imaginary), but still touching the limit twice; this is a distinct type for an eclipse of the second class.

And, similarly, in an eclipse of the first class, where the points K_1, K_2 do not in general exist (viz. geometrically they are imaginary), these points may present themselves in the first instance as two coincident points, viz. instead of the sunrise oval or the sunset oval (as the case may be), we have then a cuspidal curve; or they may be two real points, viz. instead of the same oval, we have then a figure of eight touching the horizon-envelope twice, and also touching each of the two limits. These are thus the several cases.

When the Earth traverses the penumbral cone, the critic curve is

1. A pair of ovals:
2. An oval and a cuspidate oval:
3. An oval and a figure of eight.

And when the Earth does not traverse the penumbral cone, the critic curve is

4. A figure of eight:
5. A cuspidate oval:
6. An oval.

To which may be added the transition case which separates 1 and 4, viz. here the Earth just has an internal contact with the penumbral cone, and the critic curve is

7. Two ovals touching each other.

But of course 2, 5, and 7 are so special that they may be disregarded altogether; and 3 and 6 are of rare occurrence. I have not sufficiently examined the conditions for the occurrence of these forms 3 and 6; my attention was called to them, and indeed to the whole theory, by a question proposed by Prof. Adams in the Cambridge Smith's Prize Examination for 1869.