Archives of Mechanics • Archiwum Mechaniki Stosowanej • 27, 4, pp. 605-615, Warszawa 1975

Some problems of optimum design problems of vibrating systems

B. OLSZOWSKI (KRAKÓW)

ON THE EXAMPLE of a cantilever system with two degrees of freedom the formulation and discussion of some optimization problems concerning rational forming of the whole eigen frequency spectrum of vibrations are given. As the decision variables the values of both concentrated masses with fixed mass and bending rigidity of the rods of the system are assumed. A particular attention is called to the new design possibilities of avoiding resonance phenomena which are essential from practical point of view. The paper constitutes an introductory study of the problem considered and therefore the discussion of the general methods proper for the solution of the more complicated problems for the systems with a large number of degrees of freedom is not included here.

Na przykładzie układu wspornikowego o dwóch stopniach swobody podano sformułowanie i omówienie kilku problemów optymalizacyjnych, dotyczących racjonalnego kształtowania całego widma częstości drgań własnych. Jako zmienne decyzyjne przyjęto wartości obydwu mas skupionych przy ustalonej masie i sztywności zginania prętów układu. Szczególną uwagę zwrócono na pojawiające się przy projektowaniu możliwości omijania danych stref rezonansowych, istotne z praktycznego inżynierskiego punktu widzenia. Praca stanowi wstępne studium zagadnienia i nie zawiera omówienia metod ogólnych, nadających się do rozwiązywania problemów bardziej złożonych dla układów o większej liczbie stopni swobody.

На примере консольной системы с двумя степенями свободы даются формулировка и обсуждение нескольких оптимизационных задач, касающихся рационального формирования целого спектра частот собственных колебаний. Как децизионные переменные приняты значения обоих сосредоточенных масс при установленной массе и жесткости изгиба стержней системы. Особенное внимание обращено на появляющиеся при проектировании возможности обхода данных резонансных зон, которые существенны из практической инженерской точки зрения. Работа составляет вступительное исследование вопроса и не содержит обсуждения общих методов пригодных для решения более сложных задач для систем с большим количеством степеней свободы.

1. Introduction

THE NECESSITY of designing systems having prescribed dynamic properties is a consequence of the high quality requirements imposed on modern structures.

Dynamic problems of the same type may occur in the designing process of structures with different appropriations. As an example let us mention the determining of the properties of vibration isolation [13, 14, 15] with the aim of protecting sensitive objects against shock, and the selection of the dimensions of structural elements of space-crafts in order to avoid undesirable couplings in their control systems [2, 5]. Because of that it is possible to use across the results obtained by specialists in various branches of technics.

Optimum design of vibrating structures has become possible owing to the achievements in the fields of mechanics (the finite elements method, for instance), modern computer technics and the theory of control and optimization. The interest of many specialists has been focused for some ten years on this desing.

Without attempting to give a detailed survey of publications in the domain of optimum design of vibrating structures it can be stated that the works in that domain can be divided into the following general groups of problems.

Group A [1–12]. The design of systems of maximum dynamic rigidity as determined by the value of the fundamental frequency of natural vibrations. In this group we can distinguish two approaches. The first of these consists in minimizing the weight of the system with a limitation from below of the fundamental frequency. In the second approach the value of that frequency is maximized under a prescribed weight of the structure.

Group B [13–19]. Minimization of the response of the system to definite excitations. Two cases are considered: single short-duration loads of the impulse type and long-duration harmonic excitations.

Group C [20–25]. Analysis of the dependence of the natural frequencies and modes on definite parameters of the system. In this group we can include works devoted to the analysis of extremum values of the fundamental frequency.

The above classification is limited to those works, which appear to the present author to be the most representative of the works now available. The classification itself is based, however, on a much more detailed survey of the literature, therefore it gives a rather good image of the new development trends in the domain of optimization of vibrating systems. As is seen, there is not much variety in these trends.

Each particular work classified in the same group differs considerably in the concepts and the computation methods used and also in the types of systems subject to optimization. The discussion of these aspects of the problem exceeds the scope of this paper, however.

The principal object of the present paper is to state and discuss some new optimization problems arising in the design process of vibrating systems, even the simplest ones. These problems have to the author's knowledge not yet been formulated, and they are interesting and important in both theory and practice. They concern the design of structures having definite properties of the entire natural frequency spectrum and have also been discussed and solved for the simplest system with two degrees of freedom. A problem which is still open is that of generalizing and solving the above problems for more complicated cases.

2. Derivation of fundamental relations

Let us consider a system with two degrees of freedom as shown in Fig. 1, the matrices of rigidity and inertia being as follows:

$$K = \frac{6EI}{7l^3}\begin{pmatrix} 16 & -5 \\ -5 & 2 \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{2.1}$$

Such a system may be used as the model of a structure in which the influence of the mass of the bars on the natural frequencies is small and may be taken into account in an approximate manner by correcting the values of the two concentrated masses m_1 and m_2.

The characteristic equation of the problem

$$|K-\omega^2 M| = \left|\frac{6EI}{7l^3}\begin{pmatrix} 16 & -5 \\ -5 & 2 \end{pmatrix} - \omega^2\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\right| = 0,$$

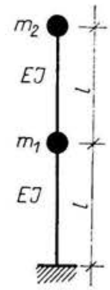

FIG. 1.

where ω is the natural angular frequency, can be rewritten in the simpler form

$$\begin{vmatrix} 16-\lambda m_1 & -5 \\ -5 & 2-\lambda m_2 \end{vmatrix} = \lambda^2 m_1 m_2 - 2\lambda(m_1+8m_2)+7 = 0, \tag{2.2}$$

where $\lambda = 7l^3\omega^2/6EI$.

If the quantity λ is treated as a parameter it is found that the Eq. (2.2) establishes a functional relationship between two variables m_1 and m_2

$$m_2 = f(m_1;\lambda) = \frac{2\lambda m_1 - 7}{\lambda(\lambda m_1 - 16)} \tag{2.3}$$

representing a family of hyperbolas with λ as a parameter. The variables m_1 and m_2 cannot become negative, therefore we are interested only in those parts of the branches of the hyperbola that lie in the positive quadrant of the plane of coordinates $0m_1m_2$ (Fig. 2).

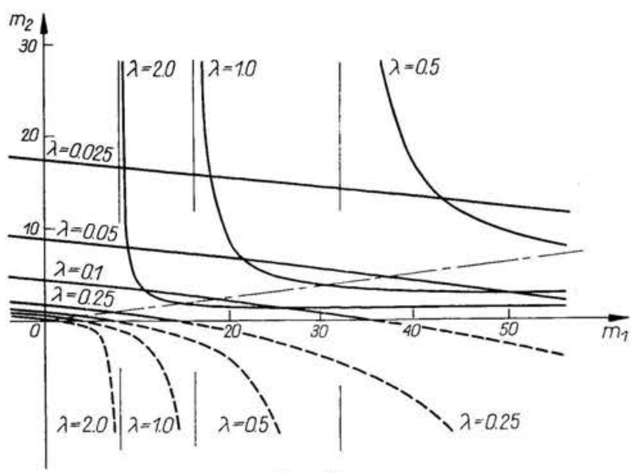

FIG. 2.

Each of the hyperbolas represents a set of solutions to the Eq. (2.2) corresponding to a fixed value of the parameter λ. In agreement with the definition of this parameter each hyperbola describes, therefore, a set of systems of Fig. 1, characterized by constant natural angular frequency ω and differing by the values of the masses m_1 and m_2.

It is worth while observing an essential fact that through every point of the first quadrant of the plane $0m_1m_2$ pass exactly two branches of hyperbolas with various values of the parameter λ. Thus, there exists a one-valued correspondence between each one pair of numbers (m_1, m_2) and exactly one pair of numbers (λ_1, λ_2). This correspondence is not one-to-one, however, because there exist pairs (λ_1, λ_2) corresponding to two, one or none pair (m_1, m_2).

3. Formulation of design problems

For the sake of accuracy it should be stressed right from the beginning that our considerations will be based on the assumption of constant rigidity and mass of the bars of the system as shown in Fig. 1, and the variables are exclusively the values of the concentrated masses. Such assumptions differ from the usual assumption in literature that the variables of the problem are the masses and the rigidities of the bars (or their finite elements) with fixed values of the non-constructive masses usually concentrated at certain points of the system.

The treatment used in this paper suits the situation which occurs in practice in the design of frame buildings, for instance, for which the concentrated masses of Fig. 1 are modelling the masses of the floors and their variable loads which can vary within certain limits. This treatment has been chosen chiefly for the simplicity of statement of the problem, the generality of the phenomena involved remaining unaffected.

Within the frames of our assumptions each design of system as in Fig. 1 is defined in an unambiguous manner by four numbers $(m_1, m_2; l, EI)$, the task of optimum design reducing therefore to the best selection (from the point of view of the criterion assumed) of the variables m_1 and m_2 for the fixed values of l and EI. This choice cannot be arbitrary because, for practical reasons, each admissible variant of the solution (admissible design) must satisfy certain conditions which will be referred to in what follows, as constraints. The form of the constraints and the criterion for the optimum (the object function) depend on the design problem under consideration. In design problems of vibrating systems and, in particular, in the case of optimum design we can distinguish two essentially different types of constraints.

Constraints of the first type directly concern the values of the concentrated masses m_1 and m_2 and are usually the results of structural and technological requirements, therefore they may be termed in a conventional manner as structural constraints. They determine admissible variability intervals of m_1 and m_2 and have usually the form

$$m_{1d} \leqslant m_1 \leqslant m_{1g}, \quad m_{2d} \leqslant m_2 \leqslant m_{2g}, \tag{3.1}$$

where m_{1d}, m_{1g}, m_{2d}, m_{2g} are prescribed numerical values. The set of solutions which are admissible from the point of view of (3.1) therefore a set of structurally admissible solutions, is a rectangle $D = B \cup C \cup E \cup F$ in the $0m_1m_2$ plane (Fig. 4). Theoretically possible cases of other structural constraints such as those of $m_{1d} \leqslant m_1$ or $m_2 \leqslant m_{2g}$ do not appear to be of practical importance, and therefore they will not be considered here.

Constraints of the second kind concern the eigenvalues λ_1 and λ_2 and are usually termed frequency constraints [10, 11]. They have an implicit form on account of the variables m_1, m_2, because the eigenvalues $\lambda_1(m_1m_2)$ and $\lambda_2(m_1, m_2)$ are their functions as the roots of the Eq. (2.2).

The introduction of frequency constraints is essential from both the practical and the theoretical point of view, because it enables the formulation and solution of problems of rational design of the natural frequency spectrum of the system, which cannot be solved by classical design methods.

Among thes eproblems the possibility of avoiding prescribed resonance zones should be mentioned above all.

As an example let us consider the case of a single zone, determined by a segment $[\lambda_d, \lambda_g]$ of the frequency axis λ. To prevent resonance phenomena we must introduce frequency constraints in the form $\lambda_1, \lambda_2 \notin [\lambda_d, \lambda_g]$. Such a form is not convenient for computation and must be written in the form of three sets of inequalities mutually excluding each other

$$
\begin{array}{lll}
1 & \lambda_2 \leqslant \lambda_d, & (\lambda_1 \leqslant \lambda_d), \\
2 & \lambda_1 \leqslant \lambda_d, & \lambda_2 \geqslant \lambda_g, \\
3 & \lambda_1 \geqslant \lambda_g, & (\lambda_2 \geqslant \lambda_g),
\end{array}
\tag{3.2}
$$

and corresponding to three different ways of avoiding the resonance. The region admissible in the sense of the constraints (3.2) is shown in Fig. 3. It is seen to be composed

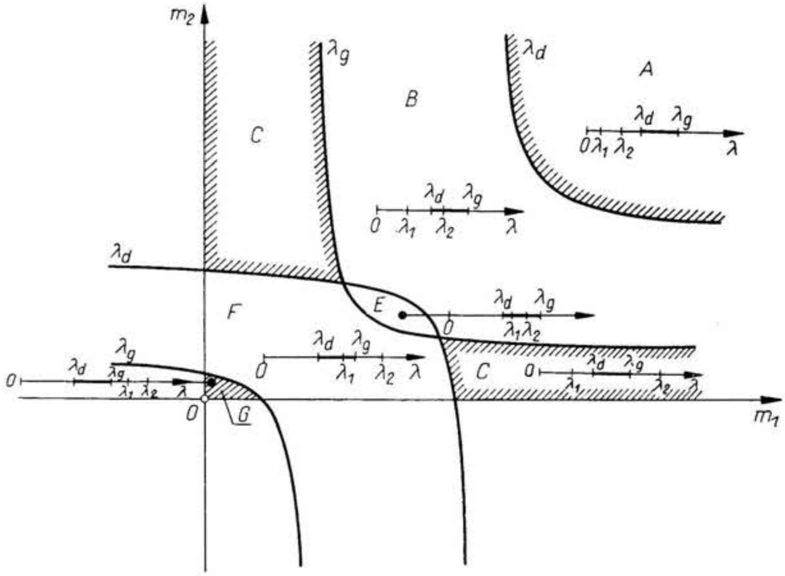

FIG. 3.

of separate regions A, C and G. It is interesting to observe that, depending on the values of the parameters λ_d and λ_g, the region C may be composed either of two separate parts (Fig. 3) or constitute a connected region (if the region E is an empty set).

Separate consideration of structural and frequency constraints (3.1) and (3.2), respectively has, of course, an auxiliary character. For design purposes, a set of solutions admissible as regards all the constraints is the only set that may be of importance. Figure 4 shows one of the possible cases in which the admissible set C is a concave set C.

The above considerations of the possibility of avoiding a single zone of resonance can easily be generalized in the case of any number n of zones. This requires only the introduction of a frequency constraints in the form $\lambda_1, \lambda_2 \notin \{[\lambda_{1d}, \lambda_{1g}], [\lambda_{2d}, \lambda_{2g}], \ldots, [\lambda_{nd}, \lambda_{ng}]\}$. Effective solution of the problem thus stated is, however, now much more complicated.

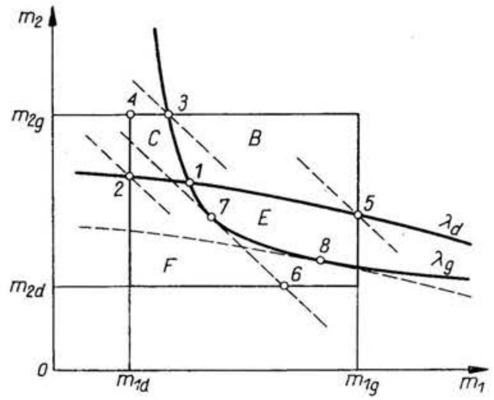

FIG. 4.

The necessity of avoiding certain frequency zones may also be the result of causes having no direct connection with resonance type phenomena. The simplest of such situations is the case (most frequently met in the literature), in which we are interested in ensuring a required dynamic rigidity of the system. If, as a measure of that rigidity, we take the value of the fundamental frequency of natural vibration, the problem is reduced to the introduction of the constraint $\lambda_1 \geqslant \lambda_0$, the value of λ_0 being selected in a suitable manner. The set of admissible solutions is, in this case $D = E \cup F$ of Fig. 4 (if we assume that $\lambda_0 = \lambda_d$). We can also imagine more general situations in which the designer is interested in limiting from below not only the fundamental frequency. In the case of structure shown in Fig. 1 he will be obliged to require that $\lambda_i \geqslant \lambda_{0i}$ $(i = 1, 2)$, as a result of which a concave admissible region $D = F$ will be determined Fig. 4 (assuming that $\lambda_{01} = \lambda_d$ and $\lambda_{02} = \lambda_g$).

Owing to the frequency constraints we can also formulate design problems of systems with given values of natural frequencies. If, for instance, we require for the system of Fig. 1 that $\lambda_1 = \lambda_d$, $\lambda_2 = \lambda_g$, the solution of the problem [taking into consideration (3.1)] will be represented in this particular case, by point 1 in Fig. 4.

In the above considerations we were concerned with the problems of the constraints only. If, however, the set of solutions admissible on account of these constraints is composed of two elements at least, there arises in a natural way the problem of selecting a solution which is the best from a certain point of view.

Thus we arrive at the general formulation of the problem of optimum design of the vibrating system as represented in Fig. 1

(3.3)
$$\min_D (\text{or max})\, D f(m_1, m_2),$$
$$D = \{(m_1, m_2): \varphi_i(m_1, m_2) \leqslant 0, \quad i = 1, 2, \ldots, n\}.$$

Among the constraints, the general form of which is $\varphi_i(m_1, m_2) \leqslant 0$ $i = 1, 2, \ldots, n$ we shall distinguish structural constraints

$$\varphi_i(m_1, m_2) \leqslant 0, \quad i = 1, 2, \ldots, r \tag{3.4}$$

and frequency constraints

$$\psi_k(\lambda_1, \lambda_2) = \varphi_k(m_1, m_2) \leqslant 0, \quad k = r+1, r+2, \ldots, n. \tag{3.5}$$

Let us reconsider, on the basis of (3.3), the problem of avoiding a single resonance zone. Since the condition of avoiding this zone is satisfied by any admissible solution, therefore by assuming $f = m_1 + m_2$, for instance, we can seek for solution for which the value of the total mass of the system is extremum. It is worth while observing that, for practical reasons the lightest and the heaviest system may both be of interest (see numerical example). These two cases are represented by points 2 and 3, respectively, Fig. 4.

It may also be of interest to assume that $f = m_1$ or $f = m_2$. This means that the extremum value is sought for the mass of only one structural level. Such problems occur often in the activity of experts in structural dynamics, for instance. If we seek for maximum m_1 in the case shown in Fig. 4, we obtain exactly one solution (point *1*). If m_1 is minimized the solution is not unique because, as is seen, it may be represented by any point of the segment *2–4*. The situation of $f = m_2$ is similar.

An additional possibility of avoiding a single resonance zone is created by formulating the problem of maximizing the function $f = \lambda_2(m_1, m_2) - \lambda_1(m_1, m_2)$. By solving it, the "forbidden" zone is not only avoided but also enlarged as much as possible.

In the domain of design of systems with required dynamic rigidity let us mention two problems — those of $\max(m_1 + m_2)$ with $\lambda_1 \geqslant \lambda_0$ and $\max \lambda_1$ with $m_1 + m_2 = m_0$, which are directly related to problems of stability loss of a system (for $\lambda_1 = 0$). Assuming that $\lambda_0 = \lambda_d$, the solutions of these problems are represented by the points *5* and *6* in Fig. 4, respectively.

Finally let us mention other possibilities of formulating problems of optimum design, of less practical importance, however. Thus, for instance, we can seek for $\max \lambda_2$ with $m_1 + m_2 = m_0$ (point *7*, Fig. 4), $\min \lambda_1$ with $\lambda_2 = \lambda_g$(point *8*), etc.

Solution of all the above optimization problems of the system in Fig. 1 is possible owing to the space of decision variables being two-dimensional. This enables the relevant diagrams to be plotted in the $0m_1m_2$ plane and the optimum solutions to be found in a very simple manner. With more decision variables we shall be obliged to seek for solutions by methods and algorithms such that the concavity and separateness of the admissible regions are taken account.

4. Numerical example

Let us consider a single-naved two-floor reinforced concrete frame constituting the transverse load-carrying system of a store building constructed with a modulus of 6.0 m. The columns are of a rectangular cross-section, the dimensions of which are 30×40 cm and are constant over the entire height of builiding, which is 2×3.50 m. The extensions of both transoms of the frame constitute cantilevers of 3.0 m in length so that the dimensions of the horizontal projection of the building are 12.0×18.00 m. Thus, each column carries a load stored on an area of $36.0\ m^2$. The mean weight of floor is assumed to be about 400 kG/m^2 and their useful load 500 kG/m^2. The weight of the roof structure and the protecting walls which are light, are neglected for simplicity.

Fans are to be installed in the building, there being a possibility of choosing between two types differing from the nominal speed: $N_1 = 150$ r.p.m and $N_2 = 200$ r.p.m. Assuming the minimum width of the resonance zone to be $N \pm 20$ per cent, our task is to make a proper choice of the fan type and the admissible loads on the floors.

As a dynamic model of the structure considered let us assume the system of Fig. 1.

$l = 3.50$ m,

$E = 1.6 \times 10^5\ kG/cm^2$,

$I = 30 \times 40^3/12 = 1.6 \times 10^5\ cm^4$,

$EI = 2.56 \times 10^{10}\ kGcm^2 = 2.56 \times 10^3\ Tm^2$.

The limit values of the concentrated masses are

$m_{1d} = m_{2d} = 6.0 \times 6.0 \times 400/9.81 = 1468\ kGsec^2/m = 1.468\ Tsec^2/m$,

$m_{1g} = m_{2g} = 6.0 \times 6.0 \times 900/9.81 = 3.305\ Tsec^2/m$.

Since $\lambda = 7l^3\omega^2/6EI = 77l^3/6EI \times (2\pi N/60)^2 = 21.45 \times 10^{-5} \times N^2$; we obtan the following limitis of the resonance zone

for the excitation with frequency $N_1 = 150$ r.p.m.

$$\lambda_{1d} = 0.8 \times 21.45 \times 10^{-5} \times 150^2 = 0.8 \times 4.83 = 3.86,$$
$$\lambda_{1g} = 1.2 \times 4.83 = 5.80;$$

for the excitation with frequency $N_2 = 200$ r.p.m.

$$\lambda_{2d} = 0.8 \times 21.45 \times 10^{-5} \times 200^2 = 0.8 \times 8.58 = 6.87;$$
$$\lambda_{2g} = 1.2 \times 8.58 = 10.3.$$

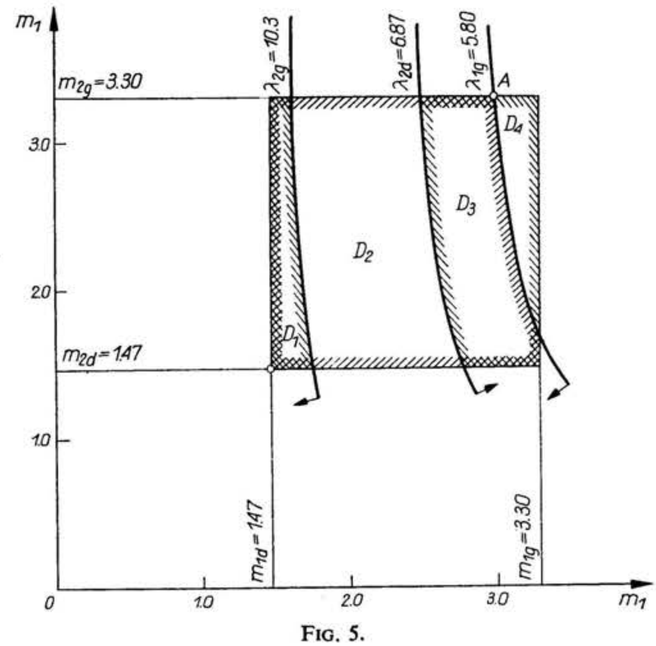

FIG. 5.

As is seen from Fig. 5 a set of admissible solutions in the case of $N = 150$ r.p.m. is $D = D_1 \cup D_2 \cup D_3$, therefore, in this case we cannot make full use of the storing capacity of the building, because (it being necessary to avoid the zone D_4) the admissible maximum mass of the system is attained at point A, the coordinates of which are (3.00, 3.30). This maximum is about 83.6 per cent of the full storing capacity under static conditions. It is essential that it is the first floor that must be partially relieved, while the load on the second floor remains complete. It is seen also that with an insignificant decrease in the excitation frequency, we could move the resonance zone outside the region of structurally admissible solutions.

For practical reasons the case of excitation at a frequency of $N = 200$ r.p.m. is the more disadvantageous, because it is the set $D = D_1 \cup D_3 \cup D_4$ which is admissible this time. The resonance zone D_2 divides the region of structurally admissible solutions into two parts. Although full use of the storing capacity of the system is possible, there are many combinations of intermediate loads for which resonance phenomena may occur.

Thus, we should use the fan of lower r.p.m. and reduce to an appropriate degree the storing capacity of the first floor. This reduction does not exceed 16.4 per cent of the full storing capacity of the building.

5. Conclusions

The above is only a brief survey of optimization problems, which can be formulated for the simplest vibrating system (Fig. 1) with the object of forming its natural frequency spectrum. The variety and complexity of these problems increase rapidly with the increasing number of degrees of freedom, thus presenting a wide field for various analytic and numerical investigations. The latter appears to be particularly promising by the finite element method [26] and if use is made of advanced methods of computation eigenvalues and eigenvectors.

It is still difficult to foresee the results of the efforts tending towards rationally forming of the natural frequency spectrum in general cases. There are known cases in which, for instance, the maximum increase in the fundamental frequency is no more than a few per cent (6.6 per cent for instance [1]). This requires therefore some caution in the assesment of the possibilities offered by the optimum design of vibrating systems. Further study of these possibilities will be interesting especially in view of the fact that the information available on this subject is scarce.

References 1

1. F. I. NIORDSON, *On the optimal design of a vibrating beam*, Quart. Appl. Math., **23**, 1, 47–53, 1965.
2. M. J. TURNER, *Design of minimum mass structures with specified natural frequencies*, AIAA J., **5**, 3, 406–412, 1967.

3. J. E. TAYLOR, *Minimum mass bar for axial vibration at specified natural frequency*, AIAA J., **5**, 10, 1911–1913, 1967.
4. J. E. TAYLOR, *Optimum design of a vibrating bar with specified minimum cross-section*, AIAA J., **6**, 7, 1379–1381, 1968.
5. M. S. ZARGHAMEE, *Optimum frequency of structures*. AIAA J., **6**, 4, 749–750, 1968.
6. C. Y. SHEU, W. PRAGER, *Minimum-weight design with piecewise constant specific stiffness*, JOTA, **2**, 3, 179–186, 1968.
7. C. Y. SHEU, *Elastic minimum-weight design for specified fundamental frequency*, Int. J. Solids Str., **4**, 10, 953–958, 1968.
8. W. PRAGER, J. E. TAYLOR, *Problems of optimal structural design*, Trans. ASME, J. Appl. Mech., **35**, 3, 102–106, 1968.
9. B. R. MCCART, E. J. HAUG, T. D. STREETER, *Optimal design of structures with constraints on natural frequency*, AIAA J., **8**, 6, 1012–1019, 1970.
10. C. P. RUBIN, *Minimum weight design of complex structures subject to a frequency constraint*, AIAA J., **8**, 5, 923–927, 1970.
11. D. L. SIPPEL, W. H. WARNER, *Minimum-mass design of multielement structures under a frequency constraint*, AIAA J., **11**, 4, 483–489, 1973.
12. A. CARDOU, *Piecewise uniform optimum design for axial vibration requirement*, AIAA J., **11**, 12, 1760–1761, 1973.
13. В. В. ГУРЕЦКИЙ, *Предельные возможности защиты оборудования от воздействия ударов*, Изв. АН СССР, ОТН, Механика и машиностроение, № 2, 76–81, 1965.
14. М. З. КОЛОВСКИЙ, *Об оптимальной амортизации*, Машиноведение, № 5, 21–28, 1966.
15. B. OLSZOWSKI, *Efficiency of the optimal shock isolation*, Rev. Roum. Sci. Techn.-Mec. Appl., **13**, 5, 973–980, 1968.
16. B. OLSZOWSKI, *Kształtowanie dynamiczne elementów konstrukcji*, Mech. Teor. Stos., **7**, 3, 299–309, 1969.
17. L. J. ICERMAN, *Optimal structural design for given dynamic deflection*, Int. J. Solids Str., **5**, 473–490, 1969.
18. B. OLSZOWSKI, *Optymalizacja parametryczna układu drgającego o dwóch stopniach swobody*, Rozpr. Inż., **18**, 2, 171–183, 1970.
19. J. M. CHERN, J. F. DAFALIAS, J. B. MARTIN, *Structural design for bounds on dynamic response*, Proc. ASCE, EM2, **99**, 261–270, 1973.
20. J. HALLQUIST, V. SNYDER, *Synthesis of two discrete vibratory systems using eigenvalue modification*, AIAA J., **11**, 2, 247–249, 1973.
21. R. M. BRACH, *On the extremal fundamental frequencies of vibrating beams*, Int. J. Solids Str., **4**, 667–674, 1968.
22. W. H. WITTRICK, *Rates of change of eigenvalues with reference to buckling and vibration problems*, J. Royal Aeronaut. Soc., **66**, 621, 590–591, 1962.
23. R. L. FOX, H. P. KAPOOR, *Rates of change of eigenvalues and eigenvectors*, AIAA J., **6**, 12, 2426–2429, 1968.
24. L. C. ROGERS, *Derivatives of eigenvalues and eigenvectors*, AIAA J., **8**, 5, 943–944, 1970.
25. R. H. PLAUT, K. HUSEYIN, *Derivatives of eigenvalues and eigenvectors in nonself-adjoint systems*, AIAA J., **11**, 2, 250–251, 1973.
26. M. P. KAMAT, G. J. SIMITSES, *Optimal beam frequencies by the finite element displacement method*, Int. J. Solids Str., **9**, 3, 415–429, 1973.

Additional references

1. B. SCHWARZ, *On the extrema of frequencies of nonhomogeneous strings with equimeasurable density*, J. Math. Mech., **10**, 401–422, 1961.

2. B. SCHWARZ, *Some results on the frequencies of nonhomogeneous rods*, J. Math. Anal. Appl., **5**, 169–175, 1962.
3. B. SCHWARZ, *Bounds for the principal frequencies of nonuniformly loaded strings*, Israel J. Math., **1**, 11–21, 1963.
4. J. B. MARTIN, *A displacement bound principle for elastic continua subjected to certain classes of dynamic loading*, J. Mech. Phys. of Solids, **12**, 165–175, 1964.
5. H. J. FRENCH, *The design of rotors to have the minimum weight consistent with a required first critical speed*, J. Mech. Engg. Sci., **1**, 99–100, 1964.
6. P. LANCASTER, *On eigenvalues of matrices dependent on a parameter*, Numerische Mathematik, **6**, 377–387, 1964.
7. P. LANCASTER, *Algorithms for lambda-matrices*, Numerische Mathematik, **6**, 388–394, 1964.
8. W. H. YONKIN, *Optimal control of two distributed parameter systems*, Univ. of California at Los Angeles, Dept. of Engg., Report No 66–64, 1966.
9. J. W. YOUNG, H. N. CHRISTIANSEN, *Synthesis of a space truss based on dynamic criteria*, Proc. ASCE, ST6, **92**, 425–442, 1966.
10. J. T. WEISSENBURGER, *The effect of local modifications on the eigenvalues and eigenvectors of linear systems*, D. Sc. dissertation, 1966, Sever Inst., Washington Univ., St. Louis, Mo.
11. W. PRAGER, R. T. SHIELD, *Optimal design of multi-purpose structures*, Int. J. Solids Struct., **4**, 469–475, 1968.
12. J. B. MARTIN, *Displacement bounds for dynamically loaded elastic structures*, J. Mech. Engg. Sci., **10**, 213–218, 1968.
13. S. C. MCINTOSH, F. E. EASTEP, *Design of minimum-mass structures with specified stiffness properties* AIAA J., **6**, 962–964, 1968.
14. R. M. BRACH, *Minimum dynamic response for a class of simply supported beam shapes*, Int. J. Mech. Sci., **10**, 429–439, 1968.
15. V. KOMKOV, *The optimal control of a transverse vibration of a beam*, SIAM J. on Control, **6**, 1968.
16. B. R. MCCART, E. J. HAUG, T. D. STREETER, *Optimal design of structures with constraints on natural frequency*, Volume on structural dynamics, AIAA structural dynamics and aeroelasticity specialist conference, AIAA, New York, April 1969.
17. R. J. POMAZAL, *The effect of local modifications on the eigenvalues and eigenvectors of damped linear systems*, Ph. D. dissertation, 1969, Dept. of Mechanical Engineering — Engineering Mechanics, Michigan Technological Univ., Houghton, Mich.
18. N. OLHOFF, *Optimal design of vibrating circular plates*, Int. J. Solids Struct., **6**, 139–156, 1970.
19. R. L. FOX, M. P. KAPOOR, *Structural optimization in the dynamics response regime: A computational approach*, AIAA J., **8**, 1798–1804, 1970.
20. R. H. PLAUT, *On minimizing the response of structures to dynamic loadings*, J. Appl. Math. Phys., **21**, 1004–1010, 1970.
21. B. L. KARIHALOO, F. I. NIORDSON, *Optimum design of vibrating cantilevers*, Danish Center for Appl. Math. Mech., **15**, 1971.
22. R. J. POMAZAL, V. W. SNYDER, *Local modifications of damped linear systems*, AIAA J., **9**, 2216–2221, 1971.
23. B. L. KARIHALOO, F. I. NIORDSON, *Optimum design of vibrating beam under axial compression*, Arch. Mech. Stos., **24**, 1029–1037, 1972.
24. J. B. MARTIN, *On the application of the bounding theorems of plasticity to impulsively loaded structures*, G. Herrmann and N. Perrone, Pergamon Press, New York, 73–93, 1972.
25. M. P. KAMAT, *Optimization of structural elements for stability and vibration*, Ph. D. dissertation, Georgia Institute of Technology, 1972.
26. A. CARDOU, *Minimum mass structural design with a frequency requirement*, Ph. D. thesis, 1972, Univ. of Minnesota, Minneapolis, Minn.

27. D. L. SIPPEL, *Minimum-mass design of structural elements and multi-element systems with specified natural frequencies*, Ph. D. thesis, 1970, Univ. of Minnesota, Minneapolis, Minn.
28. N. OLHOFF, *Optimal design of vibrating rectangular plates*, Int. J. Solids Struct., **10**, 93–109, 1974.

TECHNICAL UNIVERSITY OF KRAKÓW.

Received September 5, 1974.