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On certain representations of solutions of non-linear boundary value problems in gas dynamics

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TO OBTAIN the solutions of the non-linear equation for the irrotational three-dimensional gas flow special series in the hodograph space are used. Applications of the obtained series for the solution of problems for flow a round smooth are investigated. The local converqence of the series obtained for the problems stated is proved.

W celu otrzymania rozwiązań nieliniowego równania dla bezwirowego, trójwymiarowego przepływu gazu posłużono się specjalnymi szeregami w przestrzeni hodografu. Zbadano zastosowania tych szeregów do rozwiązania gładkich przepływów. Udowodniono ich lokalną zbieżność w sformułowanych problemach.

В пространстве годографа в виде специальных рядов построены классы точных решений нелинейных уравнений для потенциала скоростей пространственных движений газа. Исследованы приложения построенных рядов к решению задач об обтекании гладких и заостренных осесимметричных тел. Для этих задач доказана локальная сходимость рядов.

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THE SOLUTION of mixed Cauchy problems for linear hyperbolic partial differential systems may be presented as [1]

$$u(x, t) = \sum_{\nu=0}^{\infty} g_\nu(x, t) S_\nu(\varphi). \tag{1.1}$$

Here $x = (x_1, \dots x_m)$, $\frac{d}{d\varphi} S_\nu(\varphi) = S_{\nu-1}(\varphi)$, $S_0(\varphi)$ is an arbitrary generalized function, u — vector function determining the solution; equation $\varphi(x, t) = \text{const}$ is the equation of characteristic surfaces, while smooth coefficients $g_\nu(x, t)$ are defined from the system of ordinary differential equations which are realized along the arbitrary bicharacteristics.

If the value of the solution u is known on a particular surface which intersects the characteristic surfaces $\varphi = \text{const}$, the coefficients g_ν are defined in one way.

Utilization of the representations of the type (1.1) for the solution of quasi-linear systems of equations (viz. such systems describe gas-dynamic processes) confronts the following difficulties:

1. The form of characteristic surfaces and bicharacteristics covering them are unknown (usually only one characteristic is known, the one separating the regions of known background solutions and perturbed motion).

2. One should not make use of the system of functions $S_\nu(\varphi)$ having arbitrary $S_0(\varphi)$. For what classes of quasi-linear equations we succeeded to obtain the representations of the type (1.1) will be mentioned below and gas-dynamic applications of the solutions constructed will be described.

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1. Let us enumerate classes of the equations which will be dealt with further (possible generalizations will be mentioned in the last paragraph).

a. The equation for the velocity potential $\Phi(x_1, x_2, x_3, t)$ of space unsteady motion of polytropic gas

$$\Phi_{tt}+2\sum_i \Phi_{x_i}\Phi_{x_i t}+\sum_{ik}(1-\delta_{ik})\Phi_{x_i}\Phi_{x_k}\Phi_{x_i x_k}-\sum_i (c^2-\Phi_{x_i}^2)\Phi_{x_i x_i}=0, \tag{2.1}$$

$$c^2=(\gamma-1)\Big(K-\Phi_t-\frac{1}{2}\sum_i \Phi_{x_i}^2\Big), \qquad K=\text{const},$$

γ is an adiabate index, c — sound velocity, δ_{ik} — Kronecker symbol.

b. The equation for velocity potential $\Phi(x_1, x_2, x_3)$ of supersonic steady gas flows ($\Phi_{x_3} > c$) is readily obtained from (2.1).

2. The equation for smooth (flat) non-stationary double waves in the plane of velocity hodograph u_1, u_2 [2]

$$\frac{2}{\gamma-1}\theta[(1-\theta_1^2)\theta_{22}+2\theta_1\theta_2\theta_{12}+(1-\theta_2^2)\theta_{11}]+\frac{\gamma-3}{2}(\theta_1^2+\theta_2^2)+2=0, \tag{2.2}$$

$$\theta=\frac{2}{\gamma-1}c(u_1, u_2), \qquad \theta_i=\frac{\partial\theta}{\partial u_i}.$$

3. Quasi-linear equation of general type for $u(x, t)$ function of two independent variables

$$u_t+a(x, t, u)u_x=b(x, t, u), \tag{2.3}$$

where a and b are analytical functions of their arguments.

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1. Let us consider in more details the question of representations of some classes of solutions of the equation (2.1) in the form of special series [3]. Previously, let us pass from variables x_1, x_2, x_3, t over to variables u_1, u_2, u_3, t ($u_i=\Phi_{x_i}$) by using Legendre transformation

$$\Phi=\sum_k x_k u_k-\nabla+Kt. \tag{3.1}$$

Instead of (2.1) for the function $\nabla(u_1, u_2, u_3, t)$ we obtain a new equation, non-linear, relative to the second derivatives, in which spherical coordinates r, φ, θ are introduced instead of u_1, u_2, u_3.

Finally, we come to

$$\begin{aligned}(3.2)\qquad & \frac{\nabla_{tt}}{r^2}[2RST+\nabla_{rr}(PQ-S^2)-PT^2-QR^2]-(PQ-S^2)+\frac{2}{r}[\nabla_{rt}(PQ-S^2) \\ & +\nabla_{\varphi t}(RS-PT)+\nabla_{\theta t}(ST-QR)]+(\gamma-1)\left(\nabla_t-\frac{r^2}{2}\right)\left[\frac{1}{r^2}(PQ-S^2)\right. \\ & \left.+\nabla_{rr}(P\sin^2\theta+Q)-R^2\sin^2\theta-T^2\right]-\frac{1}{r^2}[\nabla_{rt}^2(PQ-S^2)+\nabla_{\varphi t}^2(\nabla_{rr}P-R^2) \\ & +\nabla_{\theta t}^2(\nabla_{rr}Q-T^2)+2\nabla_{rt}\nabla_{\varphi t}(RS-PT)+2\nabla_{rt}\nabla_{\theta t}(ST-QR) \\ & +2\nabla_{\varphi t}\nabla_{\theta t}(RT-\nabla_{rr}S)] = 0,\end{aligned}$$

where

$$P = \nabla_{\theta\theta}+r\nabla_r, \qquad Q = \nabla_{\varphi\varphi}+r\nabla_r\sin^2\theta+\nabla_\theta\sin\theta\cos\theta,$$

$$R = \nabla_{r\theta}-\frac{\nabla_\theta}{r}, \qquad S = \nabla_{\varphi\theta}-\nabla\frac{\cos\theta}{\sin\theta}, \qquad T = \nabla_{r\varphi}-\frac{\nabla_\varphi}{r}.$$

Despite the cumbersome construction of the equation obtained it appears that in the variables r, φ, θ, assuming that $r = 0$ is the characteristic surface of the Eq. (3.2), one may succeed in getting in quadratures all the coefficients of expansions being utilized.

The class of solutions of the equation obtained may be presented as

$$(3.3)\qquad \nabla(r,\varphi,\theta,t) = \sum_{k=0}^{\infty}\Psi^{(k)}(\varphi,\theta,t)r^k, \qquad r = |\bar{u}|,$$

the functions $\Psi^{(k)}$ successively for $k \geqslant 2$ being defined from linear differential equations of the form:

$$(3.4)\qquad \Psi_t^{(k)}-\frac{k-1}{2}\,\frac{P_0\sin^2\theta+Q_0}{R_0}\Psi^{(k)} = F_k,$$

$$\begin{aligned} & \Psi^{(0)} = (\gamma-1)t, \qquad \Psi^{(1)} = t+\Psi(\varphi,\theta), \\ (3.5)\qquad & P_0 = \Psi_{\theta\theta}+\Psi^{(1)}, \qquad Q_0 = \Psi_{\varphi\varphi}+\Psi_\theta\sin\theta\cos\theta+\Psi^{(1)}\sin^2\theta, \\ & R_0 = P_0Q_0-\left(\Psi_{\varphi\theta}-\Psi\frac{\cos\theta}{\sin\theta}\right)^2.\end{aligned}$$

The functions F_k depend on $\Psi^{(k-1)}$ $(i \geqslant 1)$ and their derivatives and may be written in a general form.

Thus, each of $\Psi^{(k)}$ contains an arbitrary function of two arguments φ and θ. Functions r^k are present in expansion (3.3) as $S_\nu(\varphi)$ functions used in (1.1), though absolute analogy cannot be drawn. In (1.1) the surfaces $\varphi = c$, where c is an arbitrary constant, are characteristic as well along bicharacteristics covering them, common differential equations are held, the ones from which coefficients can be defined, then in (3.3) the planes $r = \text{const} \neq 0$ are not in general characteristic surfaces, $r = 0$ being the only characteristic plane. Function $\Psi(\varphi, \theta)$ in (3.5) is arbitrary; by aid of it, for instance, at $t = 0$ one may prescribe the form of a weak discontinuities surface S_0 when passing to the limit at $r \to 0$ in the formulae for reestablishing the flow in physical space

$$\bar{R} = \nabla_r \bar{l}_1 + \frac{\nabla_\varphi}{r \sin\theta} \bar{l}_2 + \frac{\nabla_\theta}{r} \bar{l}_3,$$

(3.6) $$\bar{R} = (x_1, x_2, x_3), \quad \bar{l}_1 = (\cos\varphi\cos\theta, \sin\varphi\sin\theta, \cos\theta),$$

$$\bar{l}_2 = (-\sin\varphi, \cos\varphi, 0), \quad \bar{l}_3 = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, -\sin\theta).$$

In this case, solutions of the type (3.3) may be used for solving the following problem. Let at the moment of $t = 0$ the stationary homogeneous polytropic gas be outside or inside a sufficiently smooth convex closed surface Q_0. Since the moment $t = 0$, the piston Q_t with zero initial velocity and non-zero acceleration (at $t = 0$ Q_t coincides with Q_0) starts to move into (or out of) the gas, according to arbitrary space region x_1, x_2, x_3 enclosed between the piston Q_t and the weak discontinuity sucrce R_t, breaking off the piston at the moment of $t = 0$ and propagating in rest gas. The series of the type (3.3), in whose coefficients the arbitrary functions contained in $\Psi^{(k)}$ have been defined from a present law for the movement of piston Q_t, solve the given problem. Convergence of such series for small t has been proved in [4]. Transition from space r, φ, θ, t to physical space x_1, x_2, x_3, t for small t and convex surface Q_0 is possible.

For supersonic steady potential motions of gas, the series of the type (3.3) (x_3 plays the role of t variables and $r^* = \sqrt{u_1^2 + u_2^2}$ is taken instead of $r = |\bar{u}|$ can also be constructed in a general form [5].

Here are the reasons on the ground of which all considerations have realized in a special hodograph space u_1, u_2, u_3, t but not directly in physical space x_1, x_2, x_3, t.

1. If the representations of the type (1.1) are used for velocity potential $\left(\Phi = \sum_{\nu=0}^{\infty} g_\nu \varphi^\nu\right)$, then for the determination of the functions g_ν even in the case of rest region, there will be obtained the partial differential equations of the first order (one of them is non-linear) which are not integrable in quadratures, (3.4) being, in fact, linear ordinary differential equations and their solution being readily wirtten in a general form.

2. In the hodograph space it is convenient to consider the flow zones of large gradients of gas-dynamic values, no particular features being in the coefficients of the series (3.3) (in physical space the features are available).

3. In some of gas-dynamic problems in large regions of space x_1, x_2, x_3, t the ratios $|\bar{u}|/c$ or r^*/c are much less than unity and the series of the type (3.3) converge readily, so the availability of a small number of their members remaining allows to describe the picture of motion in large regions of physical space. But in physical space one may usually consider only the neighbourhood of the characteristic surface $\varphi = 0$.

The main drawback of considerations in a hodograph space is the necessity to check up each time the possibility of reestablishing the flow in physical space by means of the formulae (3.6). It is necessary to watch whether or not the limit multiformities with infinite gradients of gas-dynamic values appear.

4

It appears that for the equations of selfsimilar flows with the variables $\mathcal{G}_i = x_i/t$ or $\eta_i = \frac{x_i}{x_3}$ $(i = 1, 2)$ they are obtained after passing over to the corresponding variables in the equations a and b — representations of the type (3.3) do not exist. In particular, in a two-dimensional non-stationary case after passing over to the unknown function $\theta(u_1, u_2)$ (it is described by the Eq. (2.2)) only coefficients $\Psi^{(0)}$, $\Psi^{(1)}$ and $\Psi^{(2)}$ can be defined, the conditions of compatibility in defining $\Psi^{(3)}$ being not realized. Therefore, to correctly present the solutions of two-wave Eq. (2.2) in the neighbourhood of the weak discontinuity $r = |\bar{u}| = 0$, it is necessary to utilize special functions $S_\nu(r)$ in the series of the type (3.3) instead of $S_\nu(r) = r^\nu$. Thus, when we consider the case of the flow with symmetrical hodograph, and when in (2.2) $\theta = \theta(r)$, we can prove that solution in the neighbourhood of the point $r = 0$ in case when a self-similar double wave is adjoining the region of rest with sound velocity $c = 1$, is representable in the form:

$$\theta(r) = \frac{2}{\gamma-1} + \sum_{\substack{n=0\\ m=1}}^{\infty} a_{mn}(\gamma, c) r^{m+2n} \ln^n r, \tag{4.1}$$

where

$$a_{10} = \pm 1, \quad a_{20} = \frac{\gamma+1}{4}, \quad a_{11} = \frac{(\gamma+1)(\gamma+4)}{6}, \quad a_{30} = c, \ldots$$

c — arbitrary constant.

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Let us consider conformably to the Eq. (2.3) the question of some possibilities in choosing the functions $S_\nu(r)$ in representations of the type (3.3). The Eq. (2.3) is a pattern for considering the choice of functions S_ν. Similar results could be obtained, for instance, for a quasi-linear second-order equation of general type with n-independent variables, when the application of Legendre transformation (though it usually makes the equation more complicated) appears to result in the fact that in a newly-obtained space, where the conjugation with the region of rest is established, we know the characteristic surface $r = 0$ (in multi-dimensional case $r = \sqrt{\sum u_{x_k}^2}$), expansion coefficients being defined from common linear differential equations. But when the problem is considered directly in original coordinates $t, x_1, \ldots, x_n$, then even in the case of conjugation of the perturbed solution area [where expansions of (3.3) type are being constructed] with the area of the solution $u = u_0 = \text{const}$, the shape of characteristic surface is usually unknown, while the expansion coefficients are being defined from cumbersome equations in partial derivatives of the first order. It is with these considerations in mind that in the case of patterns (2.3) as well the passing over to the Eq. (5.2) has been realized. Before that let us introduce a new unknown function $\Phi(r, t)(u_x = r)$ instead of the function $u(x, t)$ assuming

$$u = xu_x - \Phi. \tag{5.1}$$

Then by utilization of the analyticity of coefficients $a(x, t, u)$ and $b(x, t, u)$ in (2.3) we shall obtain for Φ the equation

$$(5.2) \qquad -\Phi_t + r\sum_{k,l=0}^{\infty} a_{kl}(t)\,\Phi_r^k(r\Phi_r-\Phi)^l = \sum_{k,l=0}^{\infty} b_{kl}(t)\,\Phi_r^k(r\Phi_r-\Phi)^l.$$

In (5.2) a_{kl} and b_{kl} are the coefficients in the expansions of functions $a(x, t, u)$ and $b(x, t, u)$ into series according to powers x and u. Besides we shall assume, not depreciating the generality, that $u = u_0 = \text{const}$ is the solution to the Eq. (2.3), while the initial and boundary conditions for (2.3) are of the form:

$$(5.3) \qquad \begin{aligned} &u(0, x) = u_0, && x \geqslant 0,\\ &u(t, 0) = f(t), && t \geqslant 0 (f(0) = u_0), \end{aligned}$$

and $f(t)$ is analytical function of t in a certain neighbourhood of zero.

Let us introduce into consideration the system of functions $S_\nu(r)$ with the following properties (ν may be negative too)

$$(5.4) \qquad \begin{aligned} &1.\ S_0 \equiv 1;\\ &2.\ S_i \cdot S_j = S_{p(i,j)}, \quad p(i,j) \geqslant i+j;\\ &3.\ \frac{d}{dr}S_i = \sum_{\lambda=0}^{\infty} \alpha_{i,i-m}S_{i-m+\lambda}, \end{aligned}$$

where $m \geqslant 1$ is a natural number, α_{ik} — prescribed coefficients;

$$4.\ r = \sum_{\lambda=0}^{\infty} c_{m+\lambda}S_{m+\lambda}, \quad (c_k \text{ — prescribed numbers});$$

$$5.\ \lim_{r\to 0} S_{\nu+1}/S_\nu = 0.$$

The above conditions are satisfied, for instance, by the systems of equations

$$(5.5) \qquad \begin{aligned} &S_\nu = r^\nu, \quad S_\nu = r^{\frac{\nu}{N}},\\ &S_\nu = (e^{\alpha r}-1)^{\frac{\nu}{N}} \quad (\alpha = \text{const}) \quad (N \text{ is a natural number}) \quad \text{etc.} \end{aligned}$$

We shall search for the solution of the mixed Cauchy problem (5.3) in the form:

$$(5.6) \qquad \Phi(r, t) = \sum_{\nu=0}^{\infty} g_\nu(t)S_\nu(r).$$

It turns out that on realizing the conditions (5.4) the formal solution (5.6) can be constructed. Coefficients g_ν are determined successively from ordinary differential equations ($g_0 = -u_0$), while the arbitrary constants resulting after each integrating of the equation for $g_\nu(t)$ are successively defined from the boundary condition (5.3). If function $S_\nu(r)$ is one of the above functions (5.5), then from the results [4] for small r in the neighbourhood of the point $(0, 0)$ of the plane r, t there follows the convergence of the series obtained.

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Let us describe some possible gas-dynamic applications of the series (3.2). First of all, representation of the velocity potential analogue (3.3) is suitable to be used for describing the flows behind the weak shock waves. In this case an entropy jump on the wave, as is known, can be ignored and then by virtue of (3.3) the propagation of the weak shock waves of arbitrary form over the homogeneus background can be described. Analogous application of the series of the type (3.3) is also possible for an approximate description of supersonic steady flows dragging the attached shock waves when supersonic space flow is flowing round the sharp space bodies [5].

The fact that behind weak shock waves r is small compared to sound velocity seems to be very essential and in expansions of (3.3) type one may restrict oneself to 2–3 terms and obtain rather good approximate representation for the solutions in a large area of physical space following the weak shock wave.

From other applications of the representations obtained for a velocity potential one can mention the calculation of steady flows which are formed at a supersonic discharge of a gas homogeneous flow from a rectilinear tube either into a widening (according to a certain law) part of the tube or into vacuum [5]. The problem of determining time and locality of the breaking of non-stationary potential flow resulting from plunging the piston into the gas (the problem is described in Ses. 3) occurs in [8].

In conclusion, considering the question of broader classes of equations and systems of equations in partial derivatives for which there exist the representations of solutions of the type (3.3), one can say that difficulties in constructing the solutions of the type (3.3) for sufficiently general systems of quasi-linear equations of hyperbolic type are likely to be surmountable (naturally, in this case one will have to deal with physical space).

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