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On the formulation of plasticity and viscoplasticity with internal variables (*)

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THE GENERAL structure of a plastic theory with internal variables is outlined, with particular emphasis on the choice of the loading-unloading criterium in relation to the principle of determinism. For the usual three-dimensional elastoviscoplasticity, the various concepts of plastic deformations to be introduced as internal variables are then discussed. Finally, conditions under which LEE's decomposition $F = F^e F^p$ and CLIFTON's $F = F^p F^e$ can be considered as equivalent are investigated.

Podano szkic ogólnej konstrukcji teorii plastyczności ze zmiennymi wewnętrznymi kładąc szczególny nacisk na wybór kryterium procesu obciążenie-odciążenie zgodnie z zasadą determinizmu. Dla zwykłej trójwymiarowej sprężysto-lepkoplastyczności wprowadzono oraz przedyskutowano różne koncepcje odkształceń plastycznych jako zmiennych wewnętrznych. W zakończeniu zbadano warunki, przy których rozkłady Lee $F = F^e F^p$ i Cliftona $F = F^p F^e$ mogą być uważane za równoważne.

Дается очерк общего построения теории пластичности с внутренними переменными, обращая особенно внимание на выбор критерия процесса нагрузка-разгрузка, согласно принципу причинности. Для обыкновенной трехмерной упруго-вязко-пластичности введены и обсуждены разные концепции пластических деформаций, как внутренних переменных. В заключении исследованы условия, при которых распределения Ли $F = F^e F^p$ и Клифтона $F = F^p F^e$ могут считаться эквивалентными.

1. Introduction

THERE has been in recent years a growing interest in the macroscopic description of finite deformations in elastoplastic viscoplastic materials. With the exception of few works by OWEN and by VALANIS, the functional approach has not been found very helpful. On the contrary, internal state variables have been used successfully. There are two main reasons for this: on one hand, from the theoretical point of view, the functional approach is unpleasant — if not questionable — for plastic materials which have no fading memory property; on the other hand, from the physical point of view, internal state variables are quite appropriate, since they can be given a precise physical meaning, at least for single crystals, thus allowing to construct the macroscopic theory from microscopic considerations [1].

Many elastoplastic or elastoviscoplastic theories with few relations between each other have been proposed. It is the purpose of the present work to draw out the common features of all these theories and thus to describe what may be called "the general structure of plasticity". This description is presented in Sec. 2 where a formal theory of plasticity is proposed which, at least in principle, could be applied to any kinematic and dynamic framework:

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three-dimensional continuum, shells and plates, medium with microstructure, ...The applications of this formal theory to the three-dimensional continuum are investigated in Sec. 3. Different concepts of plastic deformations are discussed and many theories are shown to be included as special cases of this formal theory. Section 4 is devoted to a comparison of two kinematic descriptions of elastic-plastic deformations.

For the sake of simplicity, we do not consider heat conduction, although it would be quite straightforward to take it into account. In the plastic case, our analysis is restricted to the case of a single yield condition, i.e. of a yield surface without corners. The extension to the case of multiple yield conditions is not obvious.

2. The formal theory

Using PERZYNA's notations [2], the evolution of the material is characterized by the evolution of two sets of variables: the *independent variables* $\boldsymbol{\Lambda}$ and the *dependent variables* $\boldsymbol{\pi}$. These variables depend on the continuum under consideration. For instance, in the usual three-dimensional continuum, we can choose for $\boldsymbol{\Lambda}$ the strain and temperature, and for $\boldsymbol{\pi}$ the specific free energy, stress and specific entropy. In more general cases these variables are obtained from the kinematic and dynamic analysis (or, more easily, from the elastic theory, which is characterized by the single constitutive equation $\boldsymbol{\pi} = \boldsymbol{\pi}(\boldsymbol{\Lambda})$).

These variables are chosen in view of the *principle of determinism* [3, § 26] which requires that $\boldsymbol{\pi}(t)$ be uniquely determined by the history of $\boldsymbol{\Lambda}$ up to time t. The internal variables hypothesis assumes that $\boldsymbol{\pi}(t)$ depends on the past history of $\boldsymbol{\Lambda}$ only through the present values of some *internal*, or hidden, *variables* $\boldsymbol{\omega}$ describing the internal state of the material.

We thus assume that the dependent variables are given as functions of the *state variables* $\boldsymbol{\Lambda}$ and $\boldsymbol{\omega}$

$$\boldsymbol{\pi} = \boldsymbol{\pi}(\boldsymbol{\Lambda}, \boldsymbol{\omega}), \tag{2.1}$$

while the rate $\dot{\boldsymbol{\omega}}$ of the internal variables is given by an evolution law which will be presently discussed. This law must be set up in such a manner that i) the principle of determinism, ii) the second law of thermodynamics are satisfied. With a correct choice of $\boldsymbol{\Lambda}$ and $\boldsymbol{\pi}$ the second law reduces to

$$-\frac{\partial \psi}{\partial \boldsymbol{\omega}} \cdot \dot{\boldsymbol{\omega}} \geqslant 0, \tag{2.2}$$

where ψ is the thermodynamic potential occuring in $\boldsymbol{\pi}$.

2.1. Viscoplasticity and plasticity

Viscoplasticity and plasticity can be characterized by the particular form of the evolution law. A viscoplastic law gives $\dot{\boldsymbol{\omega}}$ as a function of the *state variables* $\boldsymbol{\Lambda}$ and $\boldsymbol{\omega}$

$$\dot{\boldsymbol{\omega}}^{(v)} = \boldsymbol{\Omega}(\boldsymbol{\Lambda}, \boldsymbol{\omega}). \tag{2.3}$$

Plasticity is characterized by an admissible domain in the state space $(\boldsymbol{\Lambda}, \boldsymbol{\omega})$, which the state cannot leave:

$$Y(\boldsymbol{\Lambda}, \boldsymbol{\omega}) \leqslant 0. \tag{2.4}$$

The plastic law is defined according to the position and motion of the state $(\boldsymbol{\Lambda}, \boldsymbol{\omega})$ with respect to this domain (Fig. 1): $\dot{\boldsymbol{\omega}}$ disappears when the state lies inside this domain or during unloading (i.e. when the state lies on the yield surface but moves inwards), while the direction of $\dot{\boldsymbol{\omega}}$ is given during loading (i.e. when the state lies on the yield surface and remains on it). The plastic law can thus be written as:

$$\text{(2.5)} \qquad \begin{cases} \dot{\boldsymbol{\omega}}^{(p)} = 0 & \begin{cases} \text{if } Y(\boldsymbol{\Lambda}, \boldsymbol{\omega}) < 0 \\ \text{or if } Y(\boldsymbol{\Lambda}, \boldsymbol{\omega}) = 0 \text{ in unloading} \end{cases} \\ \dot{\boldsymbol{\omega}}^{(p)} = \lambda \boldsymbol{\Gamma}(\boldsymbol{\Lambda}, \boldsymbol{\omega}) & \text{if } Y(\boldsymbol{\Lambda}, \boldsymbol{\omega}) = 0 \text{ in loading,} \end{cases}$$

where λ is a scalar quantity which will be computed from the condition $\dot{Y} = 0$.

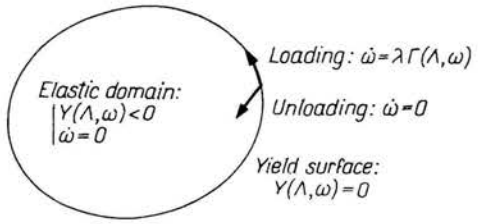

FIG. 1.

A comparison of (2.5) with (2.3) clearly shows the two main differences between viscoplastic and plastic behaviour:

i) a viscoplastic law gives $\dot{\boldsymbol{\omega}}$, while a plastic law gives its direction;

ii) usually the function $\boldsymbol{\Omega}$ disappears in some elastic domain but this is only a nice feature of this function. On the contrary, the plastic domain (2.4) is fundamental.

Following MANDEL [4], we shall assume that the behaviour is both plastic and viscoplastic, so that

$$\text{(2.6)} \qquad \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}^{(v)} + \dot{\boldsymbol{\omega}}^{(p)},$$

where $\dot{\boldsymbol{\omega}}^{(v)}$ is given by (2.3) and $\dot{\boldsymbol{\omega}}^{(p)}$ by (2.5).

2.2. The loading-unloading criterium

To complete the formulation of the theory, a loading-unloading criterium must be specified, allowing for a more precise definition of loading and unloading. Considering the fact that $\dot{Y}$ disappears during loading and is negative during unloading, this definition is usually provided by a *loading index*, a scalar quantity which is equal to $\dot{Y}$ during unloading and to some part of $\dot{Y}$ during loading. This loading index governs loading or unloading through its sign: loading or unloading occurs according to whether this loading index is positive or negative.

The most natural choice for the loading index is:

$$\text{(2.7)} \qquad \hat{Y} = \hat{Y}(\boldsymbol{\Lambda}, \boldsymbol{\omega}, \dot{\boldsymbol{\Lambda}}) = \frac{\partial Y}{\partial \boldsymbol{\Lambda}} \cdot \dot{\boldsymbol{\Lambda}} + \frac{\partial Y}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Omega}.$$

The evolution law for $\dot{\boldsymbol{\omega}}$ can then be written as:

$$(2.8)\qquad \begin{cases} \dot{\boldsymbol{\omega}} = \boldsymbol{\Omega}(\boldsymbol{\Lambda}, \boldsymbol{\omega}) \\ \dot{\boldsymbol{\omega}} = \boldsymbol{\Omega}(\boldsymbol{\Lambda}, \boldsymbol{\omega}) + \lambda \boldsymbol{\Gamma}(\boldsymbol{\Lambda}, \boldsymbol{\omega}) \end{cases} \quad \begin{cases} \text{if } Y(\boldsymbol{\Lambda}, \boldsymbol{\omega}) < 0 \\ \text{or if } Y(\boldsymbol{\Lambda}, \boldsymbol{\omega}) = 0 \text{ and } \hat{Y}(\boldsymbol{\Lambda}, \boldsymbol{\omega}, \dot{\boldsymbol{\Lambda}}) < 0 \\ \text{if } Y(\boldsymbol{\Lambda}, \boldsymbol{\omega}) = 0 \text{ and } \hat{Y}(\boldsymbol{\Lambda}, \boldsymbol{\omega}, \dot{\boldsymbol{\Lambda}}) \geqslant 0. \end{cases}$$

Obviously, $\dot{Y}$ is equal to $\hat{Y}$ during unloading, and to

$$(2.9)\qquad \dot{Y} = \hat{Y} + \lambda \left(\frac{\partial Y}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma} \right)$$

during loading. Since the state, during loading, must remain on the yield surface ($\dot{Y} = 0$), the Eq. (2.9) determines λ

$$(2.10)\qquad \lambda = -\left(\frac{\partial Y}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma} \right)^{-1} \hat{Y}.$$

From (2.8), the Clausius-Duhem inequality (2.2) requires that

$$(2.11)\qquad \frac{\partial \psi}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Omega} \leqslant 0 \quad \text{and} \quad \frac{\partial \psi}{\partial \boldsymbol{\omega}} \cdot \left[\boldsymbol{\Omega} - \left(\frac{\partial Y}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma} \right)^{-1} \hat{Y} \boldsymbol{\Gamma} \right] \leqslant 0.$$

Since these inequalities must hold for all positive $\hat{Y}$, they require

$$(2.12)\qquad \frac{\partial \psi}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Omega} \leqslant 0, \quad \left(\frac{\partial \psi}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma} \right) \left(\frac{\partial Y}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma} \right)^{-1} \geqslant 0.$$

Without loss of generality, we can assume

$$(2.13)\qquad \frac{\partial \psi}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Omega} \leqslant 0, \quad \frac{\partial \psi}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma} \leqslant 0, \quad \frac{\partial Y}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma} \leqslant 0.$$

The Eqs. (2.8) show that $\dot{\boldsymbol{\omega}}$ is a function of $(\boldsymbol{\Lambda}, \boldsymbol{\omega}, \dot{\boldsymbol{\Lambda}})$. This is a differential system allowing to compute $\boldsymbol{\omega}$ from the history of $\boldsymbol{\Lambda}$. The principle of determinism is automatically satisfied. For this reason, $\hat{Y}$ will be called the natural loading index.

2.3. Alternative formulation

In many theories, however, the yield condition and the loading-unloading criterium are not expressed in terms of strains, but in terms of stresses. In other words, the yield function Y is expressed as a function of some state functions $\boldsymbol{\zeta}(\boldsymbol{\Lambda}, \boldsymbol{\omega})$

$$(2.14)\qquad Y(\boldsymbol{\Lambda}, \boldsymbol{\omega}) = y(\boldsymbol{\zeta}, \boldsymbol{\omega}), \quad \text{where} \quad \boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\Lambda}, \boldsymbol{\omega}),$$

and the loading index is then taken as

$$(2.15)\qquad \hat{y} = \frac{\partial y}{\partial \boldsymbol{\zeta}} \cdot \dot{\boldsymbol{\zeta}} + \frac{\partial y}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Omega}.$$

The evolution law for $\dot{\boldsymbol{\omega}}$ becomes

$$(2.16)\qquad \begin{cases} \dot{\boldsymbol{\omega}} = \boldsymbol{\Omega}(\boldsymbol{\Lambda}, \boldsymbol{\omega}) \\ \dot{\boldsymbol{\omega}} = \boldsymbol{\Omega}(\boldsymbol{\Lambda}, \boldsymbol{\omega}) + \lambda \boldsymbol{\Gamma}(\boldsymbol{\Lambda}, \boldsymbol{\omega}) \end{cases} \quad \begin{cases} \text{if } y < 0, \\ \text{or if } y = 0 \text{ and } \hat{y} < 0, \\ \text{if } y = 0 \text{ and } \hat{y} \geqslant 0. \end{cases}$$

Similarly to the preceding subsection, it can be shown that λ is given by

$$\lambda = -\left(\frac{\partial y}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma}\right)^{-1} \hat{y} \tag{2.17}$$

and that the Clausius-Duhem inequality requires

$$\frac{\partial \psi}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Omega} \leqslant 0, \quad \frac{\partial \psi}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma} \leqslant 0, \quad \frac{\partial y}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma} \leqslant 0. \tag{2.18}$$

The Eqs. (2.16) show that $\dot{\boldsymbol{\omega}}$ depends on $\boldsymbol{\Lambda}$, $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\zeta}}$. The principle of determinism requires that $\dot{\boldsymbol{\omega}}$ be a function of $\boldsymbol{\Lambda}$, $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\Lambda}}$. If $y(\boldsymbol{\zeta}, \boldsymbol{\omega}) < 0$, $\dot{\boldsymbol{\omega}} = \boldsymbol{\Omega}(\boldsymbol{\Lambda}, \boldsymbol{\omega})$. If $y(\boldsymbol{\zeta}, \boldsymbol{\omega}) = 0$, $\boldsymbol{\Lambda}$, $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\Lambda}}$ being given, we must determine $\hat{y}$ (which in turn determines $\dot{\boldsymbol{\omega}}$ by (2.16) and (2.17)).

Assuming $\hat{y} < 0$ (unloading), (2.16) gives

$$\hat{y} = \frac{\partial y}{\partial \boldsymbol{\zeta}} \cdot \frac{\partial \boldsymbol{\zeta}}{\partial \boldsymbol{\Lambda}} \cdot \dot{\boldsymbol{\Lambda}} + \left(\frac{\partial y}{\partial \boldsymbol{\omega}} + \frac{\partial y}{\partial \boldsymbol{\zeta}} \cdot \frac{\partial \boldsymbol{\zeta}}{\partial \boldsymbol{\omega}}\right) \cdot \boldsymbol{\Omega} = \hat{Y}(\boldsymbol{\Lambda}, \boldsymbol{\omega}, \dot{\boldsymbol{\Lambda}}) < 0, \tag{2.19}$$

while assuming $\hat{y} > 0$,

$$\hat{y} = \hat{Y}(\boldsymbol{\Lambda}, \boldsymbol{\omega}, \dot{\boldsymbol{\Lambda}}) + \lambda \frac{\partial y}{\partial \boldsymbol{\zeta}} \cdot \frac{\partial \boldsymbol{\zeta}}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma}.$$

Taking the Eq. (2.17) into account, this equation can be written as

$$\hat{y} = \mathcal{N}(\boldsymbol{\Lambda}, \boldsymbol{\omega}) \hat{Y}(\boldsymbol{\Lambda}, \boldsymbol{\omega}, \dot{\boldsymbol{\Lambda}}) > 0, \tag{2.20}$$

where

$$\mathcal{N}(\boldsymbol{\Lambda}, \boldsymbol{\omega}) = \left[1 + \left(\frac{\partial y}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma}\right)^{-1} \frac{\partial y}{\partial \boldsymbol{\zeta}} \cdot \frac{\partial \boldsymbol{\zeta}}{\partial \boldsymbol{\omega}} \cdot \boldsymbol{\Gamma}\right]^{-1}. \tag{2.21}$$

It follows from (2.19) and (2.20) that the principle of determinism requires

$$\mathcal{N}(\boldsymbol{\Lambda}, \boldsymbol{\omega}) > 0. \tag{2.22}$$

Indeed, if $\mathcal{N}(\boldsymbol{\Lambda}, \boldsymbol{\omega}) < 0$, then both (2.19) and (2.20) or none of these inequalities hold according to the sign of $\hat{Y}(\boldsymbol{\Lambda}, \boldsymbol{\omega}, \dot{\boldsymbol{\Lambda}})$. From (2.16) it follows that $\dot{\boldsymbol{\omega}}$ can then be given two values if $\hat{Y} < 0$, and no value if $\hat{Y} > 0$. The principle of determinism cannot be satisfied.

On the contrary, if (2.22) holds, then $\hat{Y}$ and $\hat{y}$ have the same sign and $\hat{y}$ is given as a function of $\boldsymbol{\Lambda}$, $\boldsymbol{\omega}$, $\dot{\boldsymbol{\Lambda}}$ by (2.19) if $\hat{Y} < 0$ or by (2.20) if $\hat{Y} > 0$, and the principle of determinism is satisfied.

Now, it follows from (2.22) that the loading-unloading criterium defined from the loading index $\hat{y}$ is the same as the one defined from $\hat{Y}$, and the law (2.16) is only an alternative formulation of (2.8).

Since the choice of $\hat{y}$ as loading index requires (2.22), then the choice of $\hat{Y}$ as loading index is, from the mathematical point of view, preferable. The loading-unloading criterium should be expressed in terms of strains rather than in terms of stresses. Such a conclusion was reached by NGUYEN & BUI [5] in another context (softening materials); they also derived a similar inequality (2.22) and gave a nice interpretation for it in a one-dimensional case.

3. Three-dimensional plasticity

The formal theory described above can be applied, through an adequate choice of $\mathbf{\Lambda}$, $\boldsymbol{\pi}$ and $\boldsymbol{\omega}$, to any kind of plastic continuum. Attention will now be focussed on the usual, three-dimensional continuum. In this case, $\mathbf{\Lambda}$ and $\boldsymbol{\pi}$ can be chosen as

$$\mathbf{\Lambda}^{\dagger} = (\mathbf{F}, \theta), \quad \boldsymbol{\pi}^{\dagger} = (\psi, \mathbf{T}, \eta), \tag{3.1}$$

where $\mathbf{F}$ is the deformation gradient, θ the temperature, ψ the specific free energy, $\mathbf{T}$ the Cauchy stress tensor and η the specific entropy. As the theory must be objective, these variables can be replaced by

$$\mathbf{\Lambda} = (\mathbf{C}, \theta), \quad \boldsymbol{\pi} = (\psi, \mathbf{S}, \eta), \tag{3.2}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy-Green tensor and $\mathbf{S} = (\det\mathbf{F})\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-1T}$ the second Piola-Kirchhoff stress tensor in the reference configuration.

The Clausius-Duhem inequality

$$-\varrho_0(\dot{\psi}+\eta\dot{\theta}+\theta\dot{\eta})+1/2\mathbf{S}\cdot\dot{\mathbf{C}} \geqslant 0 \tag{3.3}$$

can then be written as:

$$\left(-\varrho_0\frac{\partial\psi}{\partial\mathbf{C}}+\frac{1}{2}\mathbf{S}\right)\cdot\dot{\mathbf{C}}-\varrho_0\left(\frac{\partial\psi}{\partial\theta}+\eta\right)\dot{\theta}-\varrho_0\frac{\partial\psi}{\partial\boldsymbol{\omega}}\cdot\dot{\boldsymbol{\omega}} \geqslant 0.$$

Since this inequality must hold for all $\mathbf{\Lambda}$, $\boldsymbol{\omega}$ and $\dot{\mathbf{\Lambda}}$,

$$\mathbf{S} = 2\varrho_0\frac{\partial\psi(\mathbf{C},\theta,\boldsymbol{\omega})}{\partial\mathbf{C}}, \quad \eta = -\frac{\partial\psi(\mathbf{C},\theta,\boldsymbol{\omega})}{\partial\theta}, \tag{3.4}$$

and the Clausius-Duhem inequality reduces to (2.2)

A specific theory will be characterized by the variables occuring in $\boldsymbol{\omega}$ and by the functions $\boldsymbol{\pi}$, $\boldsymbol{\Omega}$, Y and $\boldsymbol{\Gamma}$. When dealing with plasticity or viscoplasticity, some kinematic tensor describing the plastic deformation must occur in $\boldsymbol{\omega}$. Various kinematic descriptions of elastic-plastic finite strain have been proposed, the result of which are various theories of three-dimensional plasticity.

3.1. Dipolar displacement

In many cases, especially for the description of the behaviour of single crystals, the thermoelastic deformation is defined as the deformation of the material with respect to an ideal local natural configuration, whose orientation is somehow defined (through a director triad [4] or through the crystallographic directions [1] for instance). The deformation gradient $\mathbf{F}$ is then decomposed into

$$\mathbf{F} = \mathbf{AP}, \tag{3.5}$$

where $\mathbf{A}$ and $\mathbf{P}$ are the thermoelastic and plastic distortions, and $\mathbf{A}$ behaves like a dipolar displacement [6] in a change of frame, i.e. $\mathbf{F}$, $\mathbf{A}$ and $\mathbf{P}$ transform like

$$\mathbf{F} \to \mathbf{QF}, \quad \mathbf{A} \to \mathbf{QA}, \quad \mathbf{P} \to \mathbf{P}. \tag{3.6}$$

The internal variables $\boldsymbol{\omega}$ are taken as

$$\boldsymbol{\omega} = (\mathbf{P}, \boldsymbol{\alpha}), \tag{3.7}$$

where $\boldsymbol{\alpha}$ is a set of other internal state variables (including for instance the dislocation densities).

Generally, in this framework, the thermodynamic state is determined by $(\mathbf{A}, \theta, \boldsymbol{\alpha})$. If the elastic strain tensor $\mathbf{E}$ and the Piola-Kirchhoff stress tensor $\boldsymbol{\Pi}$ in the local natural configuration are now introduced

$$\begin{aligned} 2\mathbf{E} &= (\mathbf{A}^T\mathbf{A}-\mathbf{1}) = (\mathbf{P}^{-1T}\mathbf{C}\mathbf{P}^{-1}-\mathbf{1}), \\ \boldsymbol{\Pi} &= (\det\mathbf{A})\mathbf{A}^{-1}\mathbf{T}\mathbf{A}^{-1T} = (\det\mathbf{P}^{-1})\mathbf{P}\mathbf{S}\mathbf{P}^T. \end{aligned} \tag{3.8}$$

Since ψ, function of $\mathbf{A}$, θ, $\boldsymbol{\alpha}$, must be invariant in (3.6), it depends on $\mathbf{A}$ only through $\mathbf{E}$. (3.8_1) allows us to write (3.4_1) as

$$\mathbf{S} = 2\varrho_0\frac{\partial\psi}{\partial\mathbf{C}} = \varrho_0\mathbf{P}^{-1}\frac{\partial\psi}{\partial\mathbf{E}}\mathbf{P}^{-1T}. \tag{3.9}$$

A comparison of this equation with (3.8_2) gives

$$\boldsymbol{\Pi} = \tilde{\varrho}\,\frac{\partial\psi(\mathbf{E}, \theta, \boldsymbol{\alpha})}{\partial\mathbf{E}}, \qquad \eta = -\frac{\partial\psi(\mathbf{E}, \theta, \boldsymbol{\alpha})}{\partial\theta}, \tag{3.10}$$

where $\tilde{\varrho}$ is the mass density in the local natural configuration.

The Clausius-Duhem inequality (2.2) can be written as:

$$-\tilde{\varrho}\frac{\partial\psi}{\partial\boldsymbol{\omega}}\cdot\dot{\boldsymbol{\omega}} = -\tilde{\varrho}\frac{\partial\psi}{\partial\mathbf{P}}\cdot\dot{\mathbf{P}} - \tilde{\varrho}\frac{\partial\psi}{\partial\boldsymbol{\alpha}}\cdot\dot{\boldsymbol{\alpha}} \geqslant 0. \tag{3.11}$$

Using (3.8_1), (3.10) and the symmetry of $\boldsymbol{\Pi}$,

$$\begin{aligned} -\tilde{\varrho}\frac{\partial\psi}{\partial\mathbf{P}}\cdot\dot{\mathbf{P}} &= -\tilde{\varrho}\frac{\partial\psi}{\partial\mathbf{E}}\cdot\left[\frac{1}{2}\dot{(\mathbf{P}^{-1T})}\mathbf{C}\mathbf{P}^{-1} + \frac{1}{2}\mathbf{P}^{-1T}\mathbf{C}\dot{(\mathbf{P}^{-1})}\right] \\ &= -\boldsymbol{\Pi}\cdot\mathbf{P}^{-1T}\mathbf{C}\dot{(\mathbf{P}^{-1})} = \boldsymbol{\Pi}\cdot\mathbf{P}^{-1T}\mathbf{C}\mathbf{P}^{-1}\dot{\mathbf{P}}\mathbf{P}^{-1} = (\mathbf{1}+2\mathbf{E})\boldsymbol{\Pi}\cdot\dot{\mathbf{P}}\mathbf{P}^{-1} \end{aligned}$$

and substituting this result in (3.11), we obtain

$$(\mathbf{1}+2\mathbf{E})\boldsymbol{\Pi}\cdot\dot{\mathbf{P}}\mathbf{P}^{-1} - \tilde{\varrho}\frac{\partial\psi}{\partial\boldsymbol{\alpha}}\cdot\dot{\boldsymbol{\alpha}} \geqslant 0. \tag{3.12}$$

The Eqs. (3.10) and (3.12) are the fundamental thermodynamic equations in all theories based on (3.5). The evolution laws are taken in agreement with the theory of Sec. 2. For instance, MANDEL [4] takes Y, $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ as

$$\begin{aligned} Y &= y(\boldsymbol{\Pi}, \theta, \boldsymbol{\alpha}), \\ \boldsymbol{\Omega} &= \big(\mathbf{a}(\boldsymbol{\Pi}, \theta, \boldsymbol{\alpha})\mathbf{P}, \mathbf{h}(\boldsymbol{\Pi}, \theta, \boldsymbol{\alpha})\big), \\ \boldsymbol{\Gamma} &= \big(\mathbf{B}(\boldsymbol{\Pi}, \theta, \boldsymbol{\alpha})\mathbf{P}, \mathbf{1}(\boldsymbol{\Pi}, \theta, \boldsymbol{\alpha})\big), \end{aligned} \tag{3.13}$$

and the loading index is taken as in subsection 2.3 with $\boldsymbol{\zeta} = (\boldsymbol{\Pi}, \theta)$. The conditions resulting from (2.22) have been discussed in [7]. Moreover, MANDEL assumes that $\mathbf{a}$ and $\mathbf{h}$ disappear inside some elastic domain $G(\boldsymbol{\Pi}, \theta, \boldsymbol{\alpha}) \leqslant 0$, but this is not fundamental.

Most theories for single crystals (cf. [1] where many other references will be found) are special cases of the preceding theory; they are mostly restricted to viscoplasticity and often deal with microscopic considerations which lead to special forms for the functions y, $\mathbf{a}$ an $\mathbf{h}$ occuring in (3.13). It must also be noted that LEE's theory [8] can be obtained from the theory presented above by restricting it to plasticity and to isotropic materials.

3.2. Intermediate configuration

Another concept of plastic deformation issues from an intermediate configuration which leads to a decomposition

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{3.14}$$

formally identical to (3.5). However, there exists no preferred orientation in this configuration, so that it is only defined to within an arbitrary rotation. It has been shown in [9] that, in a change of frame, $\mathbf{F}$, $\mathbf{F}^e$ and $\mathbf{F}^p$ can be considered as transforming like

$$\mathbf{F} \to \mathbf{Q}\mathbf{F}, \quad \mathbf{F}^e \to \mathbf{Q}\mathbf{F}^e\mathbf{Q}_1^T, \quad \mathbf{F}^p \to \mathbf{Q}_1\mathbf{F}^p, \tag{3.15}$$

$\mathbf{Q}$ and $\mathbf{Q}_1$ being two orthogonal tensors.

If we take $\boldsymbol{\omega}^\dagger = (\mathbf{F}^p, \boldsymbol{\alpha})$, then the evolution law, which gives $\dot{\boldsymbol{\omega}}^\dagger$, cannot be invariant in (3.15). More precisely, it can be shown [9] that only the symmetric part $\mathbf{D}^p$ of $\mathbf{L}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p-1}$ is objective under (3.15) and that $\boldsymbol{\omega}$ must be taken as

$$\boldsymbol{\omega} = (\mathbf{E}^p, \boldsymbol{\alpha}), \tag{3.16}$$

where $\mathbf{E}^p = 1/2(\mathbf{F}^p\mathbf{F}^{pT} - \mathbf{1})$ is the Lagrangian plastic strain tensor. This approach is equivalent to assuming a symmetric plastic strain tensor [2, 10].

Many macroscopic phenomenological theories are obtained as special cases of the theory based on (3.4), (2.2) and (3.16) with the evolution law described in Sec. 2. Since this theory is invariant under (3.15), it includes some kind of structural isotropy. On the other hand, since $\mathbf{C}$ and $\mathbf{C}^p$ appear as state variables, it can account for a macroscopic anisotropy induced by plastic deformations.

PERZYNA's [1] and GREEN & NAGHDI's [10] theories can be obtained by restricting the previous theory to viscoplasticity and plasticity, respectively. Moreover, it has been shown in [9] that LEE's theory [8] can be obtained as a special case of [10] when assuming that the state variables are only $(\mathbf{F}^e, \theta)$.

3.3. Other descriptions

Similarly, any description of elastic-plastic deformation can be dealt with in the same framework once a proper $\boldsymbol{\omega}$ is found and the invariance condition is taken into account. For instance, KRATOCHVIL's suggestion to write $\mathbf{F} = \mathbf{EMP}$ [11], with $\mathbf{M}$ and $\mathbf{P}$ being invariant in a change of frame, leads to the choice $\boldsymbol{\omega} = (\mathbf{M}, \mathbf{P}, \boldsymbol{\alpha})$ and our theory reduces, in the viscoplastic case, to [11].

We shall now investigate a far less obvious example. Dealing with wave propagation [12], CLIFTON introduced instead of (3.14)

$$\mathbf{F} = \bar{\mathbf{F}}^p\bar{\mathbf{F}}^e \tag{3.17}$$

with an auxiliary configuration defined to within an arbitrary rotation, so that, following [13], (3.15) is replaced by

$$\mathbf{F} \to \mathbf{QF}, \quad \bar{\mathbf{F}}^p \to \mathbf{Q}\bar{\mathbf{F}}^p\mathbf{Q}_2^T, \quad \bar{\mathbf{F}}^e \to \mathbf{Q}_2\bar{\mathbf{F}}^e. \tag{3.18}$$

The thermodynamic state is characterized by $\boldsymbol{\Lambda}^\dagger$, as defined by (3.1) and $\boldsymbol{\omega}^\dagger = \bar{\mathbf{F}}^p$. (For the sake of simplicity, the other internal variables $\boldsymbol{\alpha}$ are omitted). Invariance of the constitutive equation giving $\boldsymbol{\pi}^\dagger$ as a function of $\boldsymbol{\Lambda}^\dagger$, $\boldsymbol{\omega}^\dagger$ under (3.18) shows by the usual procedure

$$\boldsymbol{\pi} = \boldsymbol{\pi}(\mathbf{C}, \theta, \bar{\mathbf{C}}^e), \quad \bar{\mathbf{C}}^e = \bar{\mathbf{F}}^{eT}\bar{\mathbf{F}}^e. \tag{3.19}$$

Now, the evolution law will give some measure of the plastic rate of deformation, which must be objective under (3.18). It follows from [13] that the convected time derivative $\bar{\mathbf{A}}^p_{\mathcal{C}}$ of the Eulerian plastic strain tensor $\bar{\mathbf{A}}^p$,

$$\bar{\mathbf{A}}^p_{\mathcal{C}} = \dot{\bar{\mathbf{A}}}^p + \mathbf{L}^T\bar{\mathbf{A}}^p + \bar{\mathbf{A}}^p\mathbf{L}, \quad \mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \quad 2\bar{\mathbf{A}}^p = \mathbf{1} - (\bar{\mathbf{F}}^p\bar{\mathbf{F}}^{pT})^{-1}$$

is a possible choice. The evolution law

$$\bar{\mathbf{A}}^p_{\mathcal{C}} = \bar{\mathbf{A}}^p_{\mathcal{C}}(\boldsymbol{\Lambda}^\dagger, \boldsymbol{\omega}^\dagger, \dot{\boldsymbol{\Lambda}}^\dagger)$$

can then be reduced to the following form:

$$\dot{\bar{\mathbf{E}}}^p = \dot{\bar{\mathbf{E}}}^p(\mathbf{C}, \theta, \bar{\mathbf{C}}^e, \dot{\mathbf{C}}, \dot{\theta}), \tag{3.20}$$

where $\bar{\mathbf{E}}^p = 1/2\,(\bar{\mathbf{C}} - \bar{\mathbf{C}}^e) = \mathbf{F}^T\bar{\mathbf{A}}^p\mathbf{F}$ is a plastic strain tensor in the reference configuration. Since $\bar{\mathbf{C}}^e = \bar{\mathbf{C}} - 2\bar{\mathbf{E}}^p$, (3.19) and (3.20) show that we can take

$$\boldsymbol{\omega} = \bar{\mathbf{E}}^p. \tag{3.21}$$

But this is not the only possible choice. In fact, others will correspond to other objective plastic strain rate tensors, and in particular to other invariant time derivatives of $\bar{\mathbf{A}}^p$. For instance, let us define for any symmetric tensor $\mathbf{T}$ the invariant time derivative

$$\mathbf{T}_R = \dot{\mathbf{T}} + \mathbf{T}(\dot{\mathbf{R}}\mathbf{R}^T) - (\dot{\mathbf{R}}\mathbf{R}^T)\mathbf{T}, \tag{3.22}$$

where $\mathbf{R}$ is the rotation tensor (obtained from the polar decomposition $\mathbf{F} = \mathbf{RU}$). Assuming an evolution law giving $\bar{\mathbf{A}}^p_R$, a somewhat lengthy calculation shows that we can take

$$\boldsymbol{\omega} = \bar{\bar{\mathbf{E}}}^p, \tag{3.23}$$

where $\bar{\bar{\mathbf{E}}}^p$ is a plastic strain tensor defined by

$$2\bar{\bar{\mathbf{E}}}^p = \mathbf{R}^T\bar{\mathbf{F}}^p\bar{\mathbf{F}}^{pT}\mathbf{R} - \mathbf{1} = \mathbf{U}\bar{\mathbf{C}}^{e-1}\mathbf{U} - \mathbf{1}. \tag{3.24}$$

This case will be important further on.

This shows that (3.17) leads to some difficulties: there is no natural plastic strain or plastic strain rate tensor. As shown by (3.21) or (3.24), they all include some coupling with the elastic deformation.

4. Equivalence of $\bar{\mathbf{F}}^p\bar{\mathbf{F}}^e$ and $\mathbf{F}^e\mathbf{F}^p$

According to CLIFTON, its decomposition (3.17) is more convenient than (3.14), when dealing with some problems. On the other hand, it is far less convenient to build a special theory when physical considerations are used. In particular, some coupling between elastic and plastic deformations cannot be avoided both in the constitutive equation and in the evolution law.

It may be interesting to discuss conditions under which (3.17) and (3.14) may be considered as equivalent. From the formal point of view, they are obviously equivalent, since (3.16) and (3.21) show that both can be described in terms of a theory with a symmetric plastic strain tensor in the reference configuration as the internal variable. In particular, GREEN & NAGHDI's theory can be considered as being based on (3.17) as well as on (3.14).

We are interested here in some stronger equivalence, namely in the requirement that plastic and elastic deformations associated to (3.14) and (3.17) be related in some sense. Let us consider the same deformation gradient tensor $\mathbf{F}$ decomposed according to (3.14) and (3.17) (Fig. 2)

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p = \bar{\mathbf{F}}^p\bar{\mathbf{F}}^e. \tag{4.1}$$

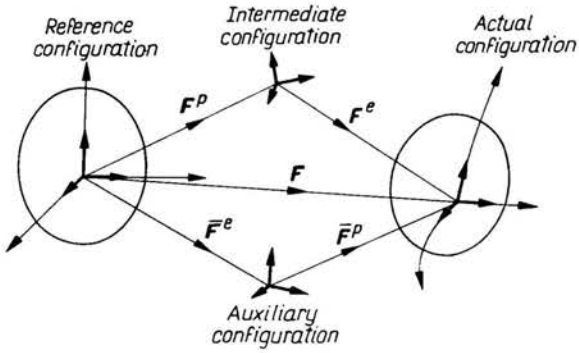

FIG. 2.

From $\mathbf{F}$, $\mathbf{F}^e$, $\mathbf{F}^p$, $\bar{\mathbf{F}}^e$ and $\bar{\mathbf{F}}^p$ are defined the corresponding left and right stretch tensors, Cauchy-Green tensors and the rotation tensors, for instance:

$$\begin{gathered}\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \quad \mathbf{B} = \mathbf{FF}^T = \mathbf{V}^2, \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \\ \mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e = \mathbf{V}^e\mathbf{R}^e, \quad \mathbf{B}^e = \mathbf{F}^e\mathbf{F}^{eT} = \mathbf{V}^{e2}, \quad \ldots\end{gathered} \tag{4.2}$$

DEFINITION. *The two decompositions in* (4.1) *will be kinematically equivalent if they lead to the same elastic and plastic principal stretches.*

Since $\mathbf{U}^e$ and $\bar{\mathbf{U}}^e$ as well as $\mathbf{U}^p$ and $\bar{\mathbf{U}}^p$ have the same eigenvalues, there exist two orthogonal tensors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ such as

$$\mathbf{Q}_1\mathbf{U}^e\mathbf{Q}_1^T = \bar{\mathbf{U}}^e, \quad \mathbf{Q}_2\bar{\mathbf{U}}^p\mathbf{Q}_2^T = \mathbf{U}^p \tag{4.3}$$

and (remembering that the intermediate and auxiliary configurations in Fig. 2 are defined to within an arbitrary rotation) there exist, among all the possible ones, one intermediate configuration and one auxiliary configuration, both denoted by asterisks and such that

$$\bar{\mathbf{U}}^{e*} = \mathbf{U}^{e*}, \quad \bar{\mathbf{U}}^{p*} = \mathbf{U}^{p*}. \tag{4.4}$$

Using (3.14), (3.17) and (4.2) we obtain

$$\mathbf{C}^p = \mathbf{F}^T\mathbf{B}^{e-1}\mathbf{F} = \mathbf{U}\mathbf{R}^T\mathbf{R}^e\mathbf{C}^{e-1}\mathbf{R}^{eT}\mathbf{R}\mathbf{U},$$

$$\bar{\mathbf{C}}^p = \bar{\mathbf{R}}^{pT}\bar{\mathbf{B}}^p\bar{\mathbf{R}}^p = \bar{\mathbf{R}}^{pT}\mathbf{R}\mathbf{U}\bar{\mathbf{C}}^{e-1}\mathbf{U}\mathbf{R}^T\bar{\mathbf{R}}^p,$$

which shows that if (4.3_1) holds and if

$$\mathbf{U}\mathbf{R}^T\mathbf{R}^e\mathbf{Q}_1^T\bar{\mathbf{C}}^{e-1}\mathbf{Q}_1\mathbf{R}^{eT}\mathbf{R}\mathbf{U} = \mathbf{Q}_2\bar{\mathbf{R}}^{pT}\mathbf{R}\mathbf{U}\bar{\mathbf{C}}^{e-1}\mathbf{U}\mathbf{R}^T\bar{\mathbf{R}}^p\mathbf{Q}_2^T, \tag{4.5}$$

then (4.3_2) holds too. An obvious solution of (4.5) is

$$\mathbf{Q}_1 = \mathbf{R}^T\mathbf{R}^e, \quad \mathbf{Q}_2 = \mathbf{R}^T\bar{\mathbf{R}}^p. \tag{4.6}$$

Substituting (4.6_1) into (4.3_1) shows that if

$$\bar{\mathbf{U}}^e = \mathbf{R}^T\mathbf{R}^e\mathbf{U}^e\mathbf{R}^{eT}\mathbf{R} = \mathbf{R}^T\mathbf{V}^e\mathbf{R}, \tag{4.7}$$

then (4.5) and thus (4.3_2) hold with $\mathbf{Q}_1$ and $\mathbf{Q}_2$ given by (4.6). A straightforward calculation shows that (4.7) is equivalent to

$$\mathbf{R}^T\bar{\mathbf{V}}^p\mathbf{R} = \mathbf{U}^p. \tag{4.8}$$

The condition (4.7) or (4.8), which has already been proposed by CLIFTON [12], is a sufficient (but not necessary) condition for the kinematic equivalence of the two decompositions in (4.1).

Similar calculations starting from the left stretch tensors $\mathbf{V}^e$, $\mathbf{V}^p$, $\bar{\mathbf{V}}^e$, $\bar{\mathbf{V}}^p$ also leads to the same conditions (4.7) and (4.8) and the results can be summarized in what follows:

THEOREM. *If one of the two equivalent conditions*

$$\bar{\mathbf{U}}^e = \mathbf{R}^T\mathbf{V}^e\mathbf{R} \quad \text{or} \quad \mathbf{R}^T\bar{\mathbf{V}}^p\mathbf{R} = \mathbf{U}^p$$

holds, then the two decompositions in (4.1) *are kinematically equivalent. Moreover, there exist one intermediate and one auxiliary configurations defined by*

$$\mathbf{Q}_1^* = \mathbf{R}^T\mathbf{R}^e, \quad \mathbf{Q}_2^* = \mathbf{R}^T\bar{\mathbf{R}}^p$$

such that the plastic and elastic right stretch tensors associated to both decompositions are equal

$$\mathbf{U}^{e*} = \bar{\mathbf{U}}^{e*}, \quad \mathbf{U}^{p*} = \bar{\mathbf{U}}^{p*}$$

and two other configurations defined by

$$^*\mathbf{Q}_1 = \mathbf{R}\mathbf{R}^{pT}, \quad ^*\mathbf{Q}_2 = \mathbf{R}\bar{\mathbf{R}}^{eT}$$

such that the left stretch tensors have the same property

$$^*\mathbf{V}^e = {}^*\bar{\mathbf{V}}^e, \quad ^*\mathbf{V}^p = {}^*\bar{\mathbf{V}}^p.$$

Let us now come back to the plastic theories based on (3.14) and (3.17). As it has been shown in subsection 3.3, a theory based on (3.17) is by no means unique. However, the theory based on (3.22) or (3.23), cumbersome as it may seem, plays an important part, since it is in some sense, equivalent to the theory based on (3.14). Indeed, let us consider two theories based on (3.16) and (3.23) respectively, with the same response functions $\boldsymbol{\pi}$, $\boldsymbol{\Omega}$, Y, $\boldsymbol{\Gamma}$. Then, obviously for any process, $\mathbf{E}^p$ in the first theory will be equal to $\bar{\bar{\mathbf{E}}}^p$ in the second theory. But it follows from (3.24) and the definition of $\bar{\bar{\mathbf{E}}}^p$ that

$$\bar{\bar{\mathbf{E}}}^p = \mathbf{E}^p \Rightarrow \mathbf{R}^T \bar{\mathbf{B}}^p \mathbf{R} = \mathbf{C}^p \tag{4.9}$$

and (4.8) holds. For all processes the decompositions $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p = \bar{\mathbf{F}}^p\bar{\mathbf{F}}^e$ are kinematically equivalent.

More generally, given any theory based on (3.14), we may expect to find a theory based on (3.17) which will be kinematically equivalent, but it seems doubtful whether this theory can be useful.

5. Conclusion

We have provided a general formalism allowing to deal with all three-dimensional plasticity theories in a unified framework. From the theoretical point of view, this formalism is very useful to exhibit the fundamental structure which is essentially the same for all theories. It is not very convenient, however, for practical purposes: once the appropriate kinematic description is chosen, according to the kind of material and the kind of problem considered, the constitutive equations must be elaborated further taking into account the physical assumptions that will be made. It remains that the essential structure of the theory is entirely characterized by our formal theory and by the choice of the internal variables.

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