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NOTE ON THE MAXIMA OF CERTAIN FACTORIAL FUNCTIONS.

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I CONSIDER the functions

$$\begin{aligned} y_1 &= x(x-1),\\ y_2 &= x(x-\tfrac{1}{2})(x-1),\\ y_3 &= x(x-\tfrac{1}{3})(x-\tfrac{2}{3})(x-1),\\ &\vdots\\ y_n &= x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\dots\left(x-\frac{n-1}{n}\right)(x-1). \end{aligned}$$

Attending only to the absolute values, disregarding the signs, y_n has n maxima, viz. if n be odd, $=2p+1$ suppose, these are

$$Y_1,\ Y_2, \dots,\ Y_p,\ Y_{p+1},\ Y_p, \dots\ Y_1,$$

where Y_{p+1} corresponds to the value $x=\tfrac{1}{2}$, and $Y_1, Y_2, \dots, Y_p$ to values of x between

$$0 \text{ and } \frac{1}{2p+1},\ \frac{1}{2p+1} \text{ and } \frac{2}{2p+1}, \dots,\ \frac{p-1}{2p+1} \text{ and } \frac{p}{2p+1}.$$

But if n be even, $=2p$ suppose, then the maxima are

$$Y_1,\ Y_2, \dots,\ Y_p,\ Y_p, \dots,\ Y_1,$$

where $Y_1, Y_2, \dots, Y_p$ correspond to values of x between

$$0 \text{ and } \frac{1}{2p},\ \frac{1}{2p} \text{ and } \frac{2}{2p}, \dots,\ \frac{p-1}{2p} \text{ and } \tfrac{1}{2}.$$

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In every case the maxima decrease from Y_1 which is the greatest, to Y_p or Y_{p+1} which is the least; in particular, $n = 2p+1$, then

$$Y_{p+1} = \tfrac{1}{2}\left(\tfrac{1}{2} - \frac{1}{2p+1}\right) \dots \left(\tfrac{1}{2} - 1\right)$$

$$= \left(\tfrac{1}{2}\,\frac{2p-1}{2\,.\,2p+1} \dots \frac{1}{2\,.\,2p+1}\right)^2$$

$$= \frac{\{1\,.\,3 \dots (2p-1)\}^2}{2^{2p+2}\,.\,(2p+1)^{2p}} = \tfrac{1}{4}\,\frac{(\frac{1}{2}\,.\,\frac{3}{2} \dots p - \frac{1}{2})^2}{(2p+1)^{2p}},$$

which is

$$= \tfrac{1}{4}\,\frac{\{\Gamma(p+\frac{1}{2}) \div \Gamma\frac{1}{2}\}^2}{(2p+1)^{2p}} = \frac{\Gamma^2(p+\frac{1}{2})}{4\pi(2p+1)^{2p}}.$$

Suppose p is large; then, as for large values of x,

$$\Gamma x = \sqrt{(2\pi)}\, x^{x-\frac{1}{2}} e^{-x},$$

we have

$$\Gamma(p+\tfrac{1}{2}) = \sqrt{(2\pi)}\,(p+\tfrac{1}{2})^p\, e^{-p-\frac{1}{2}}$$

$$= \sqrt{(2\pi)}\, p^p e^{p\log\left(1+\frac{1}{2p}\right)} e^{-p-\frac{1}{2}} = \sqrt{(2\pi)}\, p^p\, e^{-p},$$

$$(2p+1)^{2p} = (2p)^{2p}\,.\,e^{2p\log\left(1+\frac{1}{2p}\right)} = 2^{2p}\, p^{2p}\, e,$$

and so

$$Y_{p+1} = \frac{2\pi p^{2p} e^{-2p}}{4\pi 2^{2p} p^{2p} e} = \frac{p^2 e^{-2p-1}}{2^{2p-1}} = p^2\left(\frac{1}{2e}\right)^{2p+1}.$$

Also Y_1 corresponds *approximately* to

$$x = \tfrac{1}{2}\,\frac{1}{2p+1} = \frac{1}{2n},$$

$$Y_1 = \frac{1}{2n}\,.\,\frac{1}{2n}\,.\,\frac{3}{2n} \dots \frac{2n-1}{2n} = \frac{1}{n^{n+1}}\,\tfrac{1}{2}\,.\,\tfrac{1}{2}\,.\,\tfrac{3}{2} \dots (n-\tfrac{1}{2})$$

$$= \frac{1}{n^{n+1}}\,\tfrac{1}{2}\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma\frac{1}{2}} = \frac{1}{2\,(2p+1)^{2p+1}}\,\frac{\Gamma(2p+\frac{3}{2})}{\sqrt{(\pi)}}.$$

Now

$$\Gamma(2p+\tfrac{3}{2}) = \sqrt{(2\pi)}\,(2p+\tfrac{3}{2})\, e^{-2p-\frac{3}{2}} = \sqrt{(2\pi)}\,(2p)^{2p+2} e^{(2p+\frac{3}{2})\log\left(1+\frac{3}{4p}\right)}\, e^{-2p-\frac{3}{2}}$$

$$= \sqrt{(2\pi)}\, 2^{2p+2} p^{2p+2} e^{-2p},$$

and

$$(2p+1)^{2p+1} = (2p)^{2p+1}\, e^{(2p+1)\log\left(1+\frac{1}{2p}\right)} = (2p)^{2p+1} e;$$

so that

$$Y_1 = \frac{1}{2^{2p+2} p^{2p+1} e}\,.\,\frac{\sqrt{(2\pi)}\,.\,2^{2p+2}\,.\,p^{2p+2}\, e^{-2p}}{\sqrt{(\pi)}}$$

$$= \frac{p\sqrt{(2)}}{e^{2p+1}},$$

so that, p being large, Y_1 is far larger than Y_{p+1}.

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