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EXTRACT FROM A LETTER TO MR. C. W. MERRIFIELD.

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THE general integral of the equations

$$\frac{\alpha}{\beta}=\frac{\beta}{\gamma}=\frac{\gamma}{\delta},$$

$\left[\text{where } \alpha, \beta, \gamma, \delta=\frac{d^3z}{dx^3}, \frac{d^3z}{dx^2dy}, \frac{d^3z}{dx\,dy^2}, \frac{d^3z}{dy^3}\right]$, can, I think, be found, viz. $\frac{\alpha}{\beta}=\frac{\beta}{\gamma}$ gives $r=$ function s, and $\frac{\beta}{\gamma}=\frac{\gamma}{\delta}$ gives $s=$ function t. But $r=$ function s, is integrated as the equation of a developable surface (p instead of z), viz. we have

$$\left.\begin{aligned} p&=a x+hy+g\\ 0&=a'x+\ y+g'\end{aligned}\right\},$$

a and g functions of h, and

$$\left(a'=\frac{da}{dh},\quad g'=\frac{dg}{dh}\right);$$

similarly, $s=$ function t, gives

$$q=hx+by+f,$$

$$0=\ x+b'y+f',\quad \left(b'=\frac{db}{dh},\quad f'=\frac{df}{dh}\right).$$

Observe that the constants have been so taken, that $\frac{dp}{dy}=h$, $\frac{dq}{dx}=h$; but in order that h may, in the two pairs of equations, mean the same function of (x, y), we must have

$$a'=\frac{1}{b'}=\frac{g'}{f'},$$

that is,

$$b=\int\frac{dh}{a'},\quad f=\int\frac{g'dh}{a'},$$

or, writing $a=\phi h$, $g=\chi h$, we have

$$p=x\phi h+yh+\chi h,$$

$$q=hx+y\int\frac{dh}{\phi' h}+\int\frac{\chi' h\,.\,dh}{\phi' h},$$

where

$$x\phi' h+y+\chi' h=0.$$

The last equation gives h as a function of (x, y), and the values of p, q are then such that $dz=pdx+qdy$ is a complete differential, so that we obtain z by the integration of that equation.

A simple example is

$$p=\tfrac{1}{2}h^2x-hq,\quad q=-hx+y\log h,\quad hx-y=0,$$

that is,

$$p=-\tfrac{1}{2}\frac{y^2}{x},\quad q=-y+y\log\frac{y}{x},$$

whence

$$z=\tfrac{1}{2}y^2\log\frac{y}{x}-\tfrac{3}{4}y^2,$$

we have

$$r=\tfrac{1}{2}\frac{y^2}{x^2},\quad s=-\frac{y}{x},\quad t=\log\frac{y}{x},$$

$$\alpha=-\frac{y^2}{x^3},\quad \beta=\frac{y}{x^3},\quad \gamma=-\frac{1}{x},\quad \delta=\frac{1}{y},$$

or

$$\frac{\alpha}{\beta}=\frac{\beta}{\gamma}=\frac{\gamma}{\delta}\left(=-\frac{y}{x}\right),$$

as it should be.

Cambridge, 28 *July*, 1871.